§2. Free subgroups with strongly regular elements

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We need only to prove that $\mathbf{SL}_2(\Omega)$ contains a non-commutative free subgroup F. If Ω has characteristic zero, we may take any torsion-free subgroup of $\mathbf{SL}_2(\mathbf{Z})$. Let now $p = \operatorname{char} \Omega$ be >0. Then, by the arithmetic method, using division quaternion algebras over global fields, we can construct a discrete cocompact subgroup of $\mathbf{SL}_2(L)$, where L is a local field of characteristic p (cf. A. Borel-G. Harder, *Crelle J. 298* (1978), 53-74). The latter has a torsion-free subgroup F of finite index (H. Garland, *Annals of Math.* 97 (1973), 375-423) which is then free, since it acts freely on a tree, namely the Bruhat-Tits building of $\mathbf{SL}_2(L)$.

2) For any non-zero $n \in \mathbf{Z}$, the power map $g \mapsto g^n$ is dominant (because it is surjective on any maximal torus [1 : 8.9]), hence Theorem 1 is obvious if the sum of the exponents of one letter in the word w is not zero. (See [11] for a similar remark in the context of compact groups.)

3) If U and V are non-empty open subsets in a connected algebraic group H, then $H = U \cdot V$ [1: 1.3]. It follows then from Theorem 1 that if w, w' are two words in two letters, say, then the map $G^4 \to G$ defined by

$$f(g_1, g_2, g_3, g_4) = w(g_1, g_2) \cdot w'(g_3, g_4),$$

is surjective. For instance, every element of $G(\Omega)$ is the product of two commutators. However, the map f_w itself is not always surjective; for instance $x \mapsto x^2$ is not surjective in $\mathbf{SL}_2(\mathbf{C})$, as pointed out in [11].

4) If $K = \mathbf{C}$, then Theorem 1 implies that Im f_w contains a dense open set in the ordinary topology. If G is defined over $\mathbf{R}$, then Theorem 1 also shows that $f_w(G(\mathbf{R}))$ contains a non-empty subset of $G(\mathbf{R})$ which is open in the ordinary topology. However it may not be dense. For instance, it is pointed out in [11] that for $\mathbf{SU}_2$, the image of the map defined by $[x^2, yxy^{-1}]$ omits a neighborhood of -1; however this map is surjective in $\mathbf{SO}_3$.

It seems that little is known about the image of f_w, even over $\mathbf{R}$ or $\mathbf{C}$. A general fact however is that the commutator map is surjective in any compact connected semi-simple Lie group [9].

§2. Free subgroups with strongly regular elements

1. In the sequel, K is a field of infinite transcendence degree over its prime field. We shall need the following lemma:

LEMMA 2. *Let X be an irreducible unirational K-variety. Let L be a finitely generated subfield of K containing a field of definition of X, and $V_i (i \in \mathbf{N})$ a sequence of proper irreducible algebraic subsets of X defined over an* *algebraic closure* $\bar{L}$ *of* L. *Then* $X(K)$ *is not contained in the union of the* $V_i \cap X(K)$, $(i \in \mathbf{N})$.

By definition of unirationality, there exists for some $n \in \mathbf{N}$ a dominant K-morphism $\varphi : \mathbf{A}^n \to X$, where $\mathbf{A}^n$ denotes the affine n-dimensional space.

This map is already defined over some finitely generated extension of L. Replacing L by the former, we may assume φ to be defined over L, hence $\varphi^{-1}(V_i)$ to be defined over $\bar{L}$. It is a proper algebraic subset since φ is dominant. This reduces us to the case where $X = \mathbf{A}^n$. But then any point whose coordinates generate over $\bar{L}$ a field of transcendence degree n will do.

THEOREM 2. *Assume* G *to be defined over* K. *Let* $\mathscr{V} = \{V_i\}$ $(i \in \mathbf{N})$ *be a family of proper subvarieties of* G, *all defined over an algebraic closure* $\bar{L}$ *of a finitely generated subfield* L *of* K *over which* G *is also defined. Then* $G(K)$ *contains a non-commutative free subgroup* Γ *such that no element of* $\Gamma - \{1\}$ *is contained in any of the* V_i*'s. Given* $m \geqq 2$, *the set of* m*-tuples which freely generate a subgroup having this property is Zariski dense in* G^m.

We may (and do) assume that the identity element is contained in one of the V_i's.

Let w and f_w be as in §1. Then f_w is defined over L hence $f_w^{-1}(Z)$ is defined over $\bar{L}$ for every $Z \in \mathscr{V}$ and is a *proper* algebraic subset by Theorem 1. The sets $f_w^{-1}(Z)$, as w runs through all the non-trivial reduced words (in m letters and their inverses) and Z through $\mathscr{V}$, form then a countable collection of proper algebraic subsets, all defined over $\bar{L}$. But G, hence G^m, is a unirational variety over any field of definition of G [1: 18.2]. Lemma 2 implies therefore the existence of $g = (g_i) \in G(K)^m$ not belonging to any of these subsets. Then the g_i's are free generators of a subgroup which satisfies our conditions. In fact, we see that we can take for g any point of $G(K)^m$ which is generic over $\bar{L}$ and, since $\bar{L}$ has finite transcendence degree over the prime field, such points are Zariski-dense. This establishes the second assertion.

Remark. If $G = \mathbf{SO}_{2n}$ (resp. $\mathbf{SO}_{2n+1}$), this shows for instance the existence of a free subgroup Γ, no element of which except 1 has the eigenvalue 1 (resp. the eigenvalue 1 with multiplicity >1).

2. Any semi-simple element x of G is contained in a maximal torus [1: 11.10]; x is called regular if it is contained in exactly one maximal torus. We shall say that x is *strongly regular* if it is not contained in any non-maximal torus, i.e., if the cyclic group generated by x is Zariski-dense in a maximal torus.

The following result contains Theorem C of the introduction.

COROLLARY 1. *Assume G to be defined over K. Then $G(K)$ contains a non-commutative free subgroup Γ all of whose elements $\neq 1$ are strongly regular. Given $m \geqq 2$, the set of m-tuples $(g_i) \in G(K)^m$ which generate freely a subgroup with that property is Zariski dense in G^m.*

The field K contains a field of definition L of G which is finitely generated over its prime field. Let $\bar{L}$ be an algebraic closure of L in our universal field Ω. Then the subfield generated by $\bar{L}$ and K has infinite transcendence degree over $\bar{L}$. Let S be the set of singular elements of G (i.e., of elements $g \in G$ such that $\operatorname{Ad} g$ has the eigenvalue one with multiplicity $>$ rk G). It is algebraic, defined over $\bar{L}$. Fix a maximal L-torus T of G [1: 18.2]. Every proper closed subgroup of T is contained in the kernel of a rational character [1: 8.2]. The characters are all defined over a finite separable extension L' of L [1: 8.11] and form a countable set. For $\lambda \in X^*(T)$, $\lambda \neq 1$, let $T_\lambda = \ker \lambda$, and V_λ the Zariski-closure of ${}^G T_\lambda$. The V_λ and S form a countable set $\mathscr{V}$ of proper algebraic subsets of G which are all defined over $\bar{L}$.

Our assertion is now a special case of the Theorem.

3. We can now prove the Corollary in the introduction. Let Ω be an algebraically closed extension of K. Since $G(K)/H(K)$ may be identified to an orbit of $G(K)$ in $G(\Omega)/H(\Omega)$ it suffices to show:

COROLLARY 2. *Assume K to be algebraically closed. Then every $\gamma \in \Gamma - \{1\}$, operating by left translations on $G(K)/H(K)$, has exactly $\chi(G, H)$ fixed points.*

For $\gamma \in \Gamma - \{1\}$, let F_γ be the fixed point set of γ in $G(K)/H(K)$, and let T_γ be the maximal torus in which the cyclic group generated by γ is dense. Clearly, F_γ is also the set of fixed points of $T_\gamma(K)$. Thus, if F_γ is non-empty, then T_γ is conjugate to a subgroup of H, and H has maximal rank. Assume this is the case and let T_0 be a maximal K-torus of H. Since the maximal tori of H (or G) are conjugate, it is elementary that F_γ may be identified with $\operatorname{Tr}(T_0, T_\gamma)/N_H(T_0)$. But, if $x \in \operatorname{Tr}(T_0, T_\gamma)$, then $\operatorname{Tr}(T_0, T_\gamma) = x \cdot N_G(T_0)$, whence the Corollary.

4. We now generalize slightly the Corollary in case H contains a maximal torus of G, dropping again the assumption that K is algebraically closed. Assume instead

(*) *The maximal K-tori of H are conjugate under $H(K)$.*

If T_0 is a maximal K-torus of H, we then set

$$\chi(G(K), H(K)) = [N_{G(K)}(T_0) : N_{H(K)}(T_0)] .$$

If K is algebraically closed, then (*) is fulfilled and $\chi(G(K), H(K))$ is our previous $\chi(G, H)$. We again set $\chi(G(K), H(K)) = 0$ if H does not contain any maximal torus of G.

COROLLARY 3. *Let* Γ *be as in Theorem 2. Let* H *be a closed* K-*subgroup of maximal rank and assume* (*) *to be satisfied. Then* $\gamma \in \Gamma - \{1\}$ *acts freely if* T_γ *is not conjugate under* $G(K)$ *to* T_0 *and has* $\chi(G(K), H(K))$ *fixed points otherwise.*

The argument is the same as before: F_γ is also the set of fixed points of T_γ. The latter is defined over K. If $F_\gamma \neq \emptyset$, then there exists $x \in G(K)$ such that ${}^xT_\gamma \in H$, hence by (*),

$$\mathrm{Tr}_{G(K)}(T_0, T_\gamma) \neq \emptyset\,,$$

and we have, as above, bijections

$$F_\gamma = \mathrm{Tr}_{G(K)}(T_0, T_\gamma)/N_{H(K)}(T_0) = N_{G(K)}(T_0)/N_{H(K)}(T_0)\,.$$

5. (i) If $K = \mathbf{R}, \mathbf{C}$ or also is a non-archimedean local field with finite residue field, then $G(K)$, endowed with the topology stemming from K, is a Lie group over K, and in particular is a locally compact topological group. In that case, we can use in Theorem 2 a category argument instead of Lemma 2: the $f_w^{-1}(Z)$, being proper algebraic subsets, have no interior point, the intersection of their complement is then dense by Baire's theorem, whence the last assertion of Theorem 2 with "Zariski-dense" replaced by "dense in the K-topology".

(ii) In [4] it is asked whether the hyperbolic n-space admits a non-commutative free group of isometries which acts freely. More generally, one has the

PROPOSITION. *Let* S *be a connected semi-simple non-compact Lie group with finite center,* U *a maximal compact subgroup of* L *and* $X = L/U$ *the symmetric space of non-compact type of* S. *Then* S *contains a non-commutative free subgroup which acts freely on* X.

If $\mathrm{rk}\, S \neq \mathrm{rk}\, U$, this could be deduced from Corollary 2. However, the existence of one such subgroup can be proved much more directly in all cases: if $S = \mathbf{SL}_2(\mathbf{R})$ or $\mathbf{PSL}_2(\mathbf{R})$, then we may take for Γ a free subgroup of finite index in $\mathbf{SL}_2(\mathbf{Z})$ or $\mathbf{SL}_2(\mathbf{Z})/\{\pm 1\}$. If S is of dimension >3, then it contains a copy of $\mathbf{SL}_2(\mathbf{R})$ or of $\mathbf{PSL}_2(\mathbf{R})$, and therefore a *discrete* non-commutative free subgroup Γ. No element $\gamma \in \Gamma - \{1\}$ is contained in a compact subgroup of S, hence Γ acts freely on X.

A similar argument would be valid over a non-archimedean local field K for the Bruhat-Tits buildings attached to semi-simple K-groups.