3. Grassmann manifolds and classifying vectorbundles

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$\times\ \kappa_i$ matrix with 1's just above the diagonal and zeros everywhere else. Then the Gerstenhaber-Hesselink theorem says that $\overline{0(\kappa)} \supset 0(\kappa')$ iff $\kappa < \kappa'$. (Note the reversion of the order with respect to the result on orbits described in 2.6. above.)

3. GRASSMANN MANIFOLDS AND CLASSIFYING VECTORBUNDLES

In order to describe how the various manifestations of the specialization order are connected to each other we need to define Grassmann manifolds, the classifying vectorbundles over them and their Schubert cell decompositions (in section 4 below).

3.1 *Grassmann Manifolds.* Fix two numbers $m, n \in \overline{\mathbf{N}}$. Then the Grassmann manifold $\mathbf{G}_n(\mathbf{C}^{n+m})$ consists of all n-dimensional subspaces of $\mathbf{C}^{n+m}$. Thus for example $\mathbf{G}_1(\mathbf{C}^{m+1})$ is the m-dimensional complex projective space $\mathbf{P}^m(\mathbf{C})$. Let $\mathbf{C}_{reg}^{n\times(n+m)}$ be the space of all complex $n \times (n+m)$ matrices of rank n. Let $\mathbf{GL}_n(\mathbf{C})$ act on this space by multiplication on the left. Then the quotient space $\mathbf{C}_{reg}^{n\times(n+m)}/\mathbf{GL}_n(\mathbf{C})$ is $\mathbf{G}_n(\mathbf{C}^{n+m})$. The identification is done by associating to $M \in \mathbf{C}_{reg}^{n\times(n+m)}$ the subspace of $\mathbf{C}^{n+m}$ generated by the rows of M.

$\mathbf{G}_n(\mathbf{C}^{n+m})$ inherits a natural holomorphic manifold structure from $\mathbf{C}^{n\times(n+m)}$. For a detailed description of $\mathbf{G}_n(\mathbf{C}^{n+m})$ see e.g. [16] or [23].

3.2. *The Classifying bundle.* We define a holomorphic vectorbundle ξ_m over $\mathbf{G}_n(\mathbf{C}^{n+m})$ as follows. For each x let the fibre over x, $\xi_m(x)$, be the quotient space $\mathbf{C}^{n+m}/x$. More precisely define the bundle η_n over $\mathbf{G}_n(\mathbf{C}^{n+m})$ by

$$\eta_n = \{(x, v) \in \mathbf{G}_n(\mathbf{C}^{n+m}) \times \mathbf{C}^{n+m} \mid v \in x\} \tag{3.3}$$

with the obvious projection $(x, v) \mapsto x$. Then ξ_m is the quotient bundle of the trivial vectorbundle $\mathbf{G}_n(\mathbf{C}^{n+m}) \times \mathbf{C}^{n+m}$ over $\mathbf{G}_n(\mathbf{C}^{n+m})$ by η_n. Both ξ_m and η_n can be used as universal or classifying bundles (cf. [16] for η_n as a universal bundle). Let E be an m-dimensional vectorbundle over a complex analytic manifold M. Let $\Gamma(E) = \Gamma(E, M)$ be the space of all holomorphic sections of E, i.e. the space of all holomorphic maps $s : M \to E$ such that $ps = id$, where $p : E \to M$ is the bundle projection. The universality, or classifying, property of ξ_m in the setting of complex analytic manifolds now takes the following form. Suppose $V \subset \Gamma(E)$ is an $(n+m)$-dimensional subspace such that for each $x \in M$ the vectors $s(x)$, $s \in V$ span $E(x)$, the fibre of E over x. Now identify $V \simeq \mathbf{C}^{n+m}$ and associate to $x \in M$

the point of $\mathbf{G}_n(\mathbf{C}^{n+m})$ represented by $\operatorname{Ker}(V \to E(x))$. This gives a holomorphic map $\Psi_E : M \to \mathbf{G}_n(\mathbf{C}^{n+m})$ such that the pullback of ξ_m by means of Ψ_E is isomorphic to E, $\Psi_E^! \xi_m \simeq E$. It is universality properties such as this one which account for the importance of the bundles ξ_m and η_n in differential and algebraic topology [16], algebraic geometry and also system and control theory (cf. [22, 23] and the references therein for the last mentioned).

The bundle ξ_m has a number of obvious holomorphic sections, viz. the sections defined by $\varepsilon_i(x) = e_i \bmod x$ where e_i is the i-th standard basis vector of $\mathbf{C}^{n+m}$, $i = 1, \ldots, n + m$. And, as a matter of fact, it is not difficult to show that $\Gamma(\xi_m, \mathbf{G}_n(\mathbf{C}^{n+m}))$ is $(n+m)$-dimensional and that the $\varepsilon_1, \ldots, \varepsilon_{n+m}$ form a basis for this space of holomorphic sections; cf. subsection 8.1 below.

4. Schubert Cells

4.1. *Schubert Cells.* Consider again the Grassmann manifold $\mathbf{G}_n(\mathbf{C}^{m+n})$. Let $\underline{A} = (A_1, \ldots, A_n)$ be a sequence of n-subspaces of $\mathbf{C}^{n+m}$ such that $0 \neq A_1 \subset A_2 \subset \ldots \subset A_n$ with each containment strict. To each such sequence $\underline{A}$ we associate the closed subset

$$SC(\underline{A}) = \{x \in \mathbf{G}_n(\mathbf{C}^{m+n}) \mid \dim(x \cap A_i) \geqslant i\} \tag{4.2}$$

and call it the closed Schubert-cell of the sequence $\underline{A}$. In particular if

$$0 < \gamma_1 < \gamma_2 < \ldots < \gamma_n \leqslant n + m$$

is a strictly increasing sequence of natural numbers less than or equal to $n + m$ then we define (setting $\gamma = (\gamma_1, \ldots, \gamma_n)$)

$$SC(\gamma) = SC(\mathbf{C}^{\gamma_1}, \ldots, \mathbf{C}^{\gamma_n}) \tag{4.3}$$

where $\mathbf{C}^r$ is viewed as the subspace of all vectors in $\mathbf{C}^{n+m}$ whose last $n + m - r$ coordinates are zero.

4.4 *Flag Manifolds and the Bruhat Decomposition.* A *flag* in $\mathbf{C}^{n+m}$ is a sequence of subspaces $\underline{F} = F_1 \subset \ldots \subset F_{n+m} \subset \mathbf{C}^{n+m}$ such that $\dim F_i = i$. Let $Fl(\mathbf{C}^{n+m})$ denote the analytic manifold of all flags in $\mathbf{C}^{n+m}$. There is a natural holomorphic mapping $Fl(\mathbf{C}^{n+m}) \to \mathbf{G}_n(\mathbf{C}^{n+m})$ given by associating to a flag $\underline{F}$ its n-th element F_n. The flag manifold can be seen as the space of all cosets Bg, $g \in \mathbf{GL}_{n+m}(\mathbf{C})$ where B is the Borel subgroup of all lower triangular matrices