6. Proof of (4)

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6. Proof of (4)

For characters $\psi_1, \ldots, \psi_m$ on $GF(q)$, define the Jacobi sum

$$J(\psi_1, \ldots, \psi_m) = (-1)^{m+1} \sum_{\substack{x_1, \ldots, x_m \in GF(q) \\ x_1 + \ldots + x_m = 1}} \psi_1(x_1) \ldots \psi_m(x_m) .$$

We will use the well-known fact that if $\psi_1 \psi_2 \ldots \psi_m$ is nontrivial, then

$$(18) \qquad J(\psi_1, \ldots, \psi_m) = G(\psi_1 \ldots \psi_m)^{-1} \prod_{i=1}^{m} G(\psi_i) .$$

Let S denote the left side of (4). If χ_1, χ_2, or χ_3^2 is trivial, then it is easy to verify (4) directly, with use of (18) and (26) below. Thus assume that χ_1, χ_2, and χ_3^2 are nontrivial. With the change of variables

$$u = xy , \qquad v = x + y ,$$

we have

$$S = \sum_{u,\, v \in GF(q)} \chi_1(u)\, \chi_2(1+u-v)\, \chi_3(v^2-4u) \{1 + \phi(v^2-4u)\} .$$

It therefore remains to show that

$$(19) \qquad S_1 = \sum_{u,\, v} \chi_1(u)\, \chi_2(1+u-v)\, \chi_3(v^2-4u) = R(\chi_1, \chi_2, \chi_3\phi) .$$

Replace v by $u + 1 - v$ to get

$$(20) \qquad S_1 = \sum_{u,\, v} \chi_1(u)\, \chi_2(v)\, \chi_3(1+u^2+v^2-2u-2v-2uv) .$$

Replace u by u/t, and v by v/t, to get

$$(21) \qquad S_2 = -S_1 G(\chi_1\chi_2\chi_3^2)$$
$$= \sum_{t \neq 0} \sum_{u,\, v} \chi_1(u)\, \chi_2(v)\, \chi_3(t^2+u^2+v^2-2ut-2vt-2uv)\, \zeta^{T(t)} .$$

Since $\chi_1\chi_2\chi_3^2$ is nontrivial, the restriction $t \neq 0$ may be dropped. Then replace t by $t + u + v$ to get

$$S_2 = \sum_{t,\, u,\, v} \chi_1(u)\, \chi_2(v)\, \chi_3(t^2-4uv)\, \zeta^{T(t+u+v)} .$$

Replace u by ua and v by vb to get

$$(22) \qquad S_3 = S_2 \overline{G(\chi_1)}\, \overline{G(\chi_2)}$$
$$= \sum_{t} \sum_{a,\, b,\, u,\, v \neq 0} \chi_1(u)\, \chi_2(v)\, \chi_3(t^2-4uvab)\, \zeta^{T(t+a(u-1)+b(v-1))} .$$

Replace a by $a/(4uvb)$ to get

$$S_3 = \sum_t \sum_{a,\,b,\,u,\,v\neq 0} \chi_1(u)\,\chi_2(v)\,\chi_3(t^2-a)\,\zeta^{T\left(t+b(v-1)+\frac{a(u-1)}{4uvb}\right)}.$$

Since χ_1 is nontrivial, the restriction $a \neq 0$ may be dropped. Then replace a by $t^2 - a$ to get

$$S_3 = \sum_{a,\,t} \sum_{b,\,u,\,v\neq 0} \chi_1(u)\,\chi_2(v)\,\chi_3(a)\,\zeta^{T\left(t+b(v-1)+\frac{(1-u)(a-t^2)}{4uvb}\right)}$$

$$= -G(\chi_3) \sum_{u\neq 0,\,1} \sum_{b,\,v\neq 0} \chi_1(u)\,\chi_2(v)\,\chi_3\left(\frac{4uvb}{1-u}\right) \zeta^{T(b(v-1))} \sum_t \zeta^{T\left(t+\frac{t^2(u-1)}{4uvb}\right)}.$$

The inner sum on t equals $-\zeta^{T\left(\frac{uvb}{1-u}\right)}\,\phi\left(\frac{4uvb}{u-1}\right) G(\phi)$.

Hence

$$S_4 = S_3\,(G(\chi_3)\,G(\phi)\,\chi_3(-1))^{-1} \tag{23}$$

$$= \sum_{u\neq 0,\,1} \sum_{b,\,v\neq 0} \chi_1(u)\,\chi_2(v)\,\chi_3\phi\left(\frac{4uvb}{u-1}\right) \zeta^{T\left(b(v-1)+\frac{uvb}{1-u}\right)}.$$

Therefore,

$$S_4 = \sum_{u\neq 0,\,1} \sum_{v\neq 0} \chi_1\chi_3\phi(u)\,\chi_2\chi_3\phi(v)\,\overline{\chi}_3\phi\left(\frac{u-1}{4}\right) \sum_b \chi_3\phi(b)\,\zeta^{T\left(b(v-1)+\frac{buv}{1-u}\right)}.$$

Since χ_2 and $\chi_3\phi$ are nontrivial,

$$S_4 = -G(\chi_3\phi) \sum_{u,\,v} \chi_1\chi_3\phi(u)\,\chi_2\chi_3\phi(v)\,\overline{\chi}_3\phi\left(\frac{1-u-v}{4}\right),$$

so

$$S_4 = -\chi_3(4)\,G(\chi_3\phi)\,J(\chi_1\chi_3\phi,\,\chi_2\chi_3\phi,\,\overline{\chi}_3\phi). \tag{24}$$

Combining (21)-(24), we get

$$S_1 = \frac{\chi_3(-4)\,G(\chi_3)\,G(\phi)\,G(\chi_3\phi)\,J(\chi_1\chi_3\phi,\,\chi_2\chi_3\phi,\,\overline{\chi}_3\phi)}{G(\chi_1\chi_2\chi_3^2)\,\overline{G(\chi_1)}\,\overline{G(\chi_2)}}. \tag{25}$$

Applying (7) with $l = 2$, we find that for any character χ_3,

$$\chi_3(-4)\,G(\chi_3)\,G(\phi)\,G(\chi_3\phi) = \chi_3\phi(-1)\,qG(\chi_3^2). \tag{26}$$

Since χ_1 and χ_2 are nontrivial, it follows from (25) and (26) that

$$S_1 = \frac{\chi_3\phi(-1)\,G(\chi_3^2)\,G(\chi_1)\,G(\chi_2)\,J(\chi_1\chi_3\phi,\,\chi_2\chi_3\phi,\,\bar{\chi}_3\phi)}{q\,G(\chi_1\chi_2\chi_3^2)}. \tag{27}$$

Since $\chi_1\chi_2\chi_3\phi$ and $\chi_3\phi$ are nontrivial, (19) now follows from (18) and (27).

Remark. We evaluated S (the left side of (4)) only under the assumption that $\chi_1\chi_2\chi_3^2$ and $(\chi_1\chi_2\chi_3)^2$ were nontrivial. We now indicate how S can be simply evaluated in terms of Gauss sums when this assumption is dropped. If χ_1, χ_2, or χ_3^2 is trivial, one can easily evaluate S directly from its definition. If $\chi_1\chi_2\chi_3^2$ is trivial, then one can evaluate S_1 (and hence S) from (20), by first replacing u by u^{-1}, then replacing v by vu^{-1}, to obtain

$$S_1 = \sum_{u,\, v} \bar{\chi}_1\bar{\chi}_2\bar{\chi}_3^2(u)\,\chi_3(1+u^2+v^2-2u-2v-2uv)\,\chi_2(v)\,.$$

Finally, suppose that χ_1, χ_2, χ_3^2, and $\chi_1\chi_2\chi_3^2$ are nontrivial. Then S_1 can be evaluated from (27).

7. Proof of (5)

Let E denote the left side of (5). Since $\chi_1\chi_2$ is nontrivial,

$$E + 1 + \chi_1\chi_2(-1) = \sum_{\substack{x,\, y \neq 0 \\ x+y\neq -1}} \chi_1\chi_3\left(\frac{1+x}{y}\right)\chi_2\chi_3\left(\frac{1+y}{x}\right)\;\chi_1\chi_2(y-x)\,.$$

Set $t = \dfrac{1+x}{y}$, $\quad u = \dfrac{1+y}{x}$, $\quad$ so

$$x = \frac{t+1}{ut-1}, \qquad y = \frac{u+1}{ut-1}\,.$$

Then

$$\begin{aligned} E + 1 + \chi_1\chi_2(-1) &= \sum_{\substack{u,\, t\neq -1 \\ ut\neq 1}} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2\left(\frac{t-u}{1-ut}\right) \\ &= \sum_{u,\, t\neq -1} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2(t-u)\,\bar{\chi}_1\bar{\chi}_2(1-ut)\,. \end{aligned}$$

Since $\chi_1\chi_3$ and $\chi_2\chi_3$ are nontrivial,

$$E = \sum_{u,\, t\neq 0} \chi_1\chi_3(t)\,\chi_2\chi_3(u)\,\chi_1\chi_2(t-u)\,\bar{\chi}_1\bar{\chi}_2(1-ut)\,.$$

Replace t by t/u to obtain

$$\begin{aligned} E &= \sum_{u,\, t\neq 0} \chi_1\chi_3(t)\,\bar{\chi}_1^2(u)\,\chi_1\chi_2(t-u^2)\,\bar{\chi}_1\bar{\chi}_2(1-t) \\ &= \sum_{u,\, t\neq 0} \chi_1\chi_3(t)\,\bar{\chi}_1\bar{\chi}_2(1-t)\,\bar{\chi}_1(u)\,\chi_1\chi_2(t-u)\,\{1 + \phi(u)\}\,. \end{aligned}$$