

§3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)

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Note finally that, if \mathcal{L} is any line bundle on X and $D = \sum n_P P \in \text{Div } X$, then $\mathcal{L} \otimes \mathcal{O}(D)$ can be identified with the sheaf of germs of meromorphic sections σ of \mathcal{L} such that $\text{ord}_P \sigma \geq -n_P$.

We conclude this section with the following consequence of the Leray covering theorem ([3], p. 189 or [2], p. 44).

(2.16) PROPOSITION. *Let $f: X \rightarrow Y$ be a nonconstant holomorphic map of compact Riemann surfaces, and \mathcal{V} a vector bundle on X . Then the natural maps $H^i(Y, f_0(\mathcal{V})) \rightarrow H^i(X, \mathcal{V})$ are isomorphisms for all $i \geq 0$.*

Proof: If \mathfrak{U} is a sufficiently fine open covering of Y , then it is clear that, for each $U \in \mathfrak{U}$, $f_0(\mathcal{V})|_U$ is \mathcal{O}_Y -free, and that $f^{-1}(U)$ is a finite disjoint union of coordinate open sets in X , restricted to each of which \mathcal{V} is free. Since, for $i > 0$, $H^i(W, \mathcal{O}_W) = 0$ for any open $W \subset \mathbf{C}$, it follows that \mathfrak{U} and $\mathfrak{U}' = \{f^{-1}(U) : U \in \mathfrak{U}\}$ are Leray coverings for $f_0(\mathcal{V})$ and \mathcal{V} respectively. Now the natural maps $H^i(\mathfrak{U}, f_0(\mathcal{V})) \rightarrow H^i(\mathfrak{U}', \mathcal{V})$ are obviously bijective, q.e.d.

(2.17) *Remark.* Propositions (2.4) and (2.16) are valid (with the same proofs) even if X is not compact, provided we assume that f is *proper*.

(2.18) *Remark.* We know by (2.10) that any (compact) X admits a non-constant meromorphic function, i.e. a nonconstant holomorphic map $f: X \rightarrow \mathbf{P}^1$. Since \mathbf{P}^1 is covered by two coordinate neighbourhoods which (by (2.11) and (2.12)) constitute a Leray covering for any vector bundle on \mathbf{P}^1 , it follows by (2.16) that $H^i(X, \mathcal{V}) = 0$ for $i \geq 2$ for any compact Riemann surface X and any vector bundle \mathcal{V} on it. This proof is valid in the algebraic situation also. This is the reason for including the case $i \geq 2$ in (2.16) rather than appealing to (2.8). We also remark that the Leray theorem is almost trivial for H^1 ; the fact that for a Leray covering \mathfrak{U} , $H^2(\mathfrak{U}, \mathcal{F}) \rightarrow H^2(X, \mathcal{F})$ is surjective (which is what was needed above) is also trivial if we use resolutions.

§ 3. RIEMANN-ROCH THEOREM (PRELIMINARY FORM)

We fix a compact Riemann surface X .

(3.1) *Notation—Definition.* For any vector bundle \mathcal{V} on X , we set

$$h^i(\mathcal{V}) = \dim_{\mathbf{C}} H^i(X, \mathcal{V}), \quad i = 0, 1 \text{ and } \chi(\mathcal{V}) = h^0(\mathcal{V}) - h^1(\mathcal{V}).$$

The *genus* g of X is $h^1(\mathcal{O}_X)$.

(3.2) *Remark.* If $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$ is an exact sequence of vector bundles, then $\chi(\mathcal{V}) = \chi(\mathcal{V}') + \chi(\mathcal{V}'')$, as follows from the cohomology exact sequence (since $H^2 = 0$).

(3.3) *Definition.* The *degree* $\deg D$ of $D = \sum n(P)P \in \text{Div } X$ is $\sum n(P)$.

(3.4) **PROPOSITION.** *For any $D \in \text{Div}(X)$,*

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D = \deg D - g + 1.$$

Proof: (Serre [5], pp. 20-21). The assertion is a tautology for $D = 0$; hence we need only prove that it holds for $D \in \text{Div}(X)$ iff it holds for a divisor of the form $D' = D + P, P \in X$. Now $\mathcal{O}(D)$ is a subsheaf of $\mathcal{O}(D')$, and the quotient sheaf $\mathcal{Q} = \mathcal{O}(D')/\mathcal{O}(D)$ is concentrated at P with stalk isomorphic to $\mathcal{O}_P/\mathfrak{m}_P$. Hence $h^0(\mathcal{Q}) = 1$, and $h^1(\mathcal{Q}) = 0$. Now the exact sequence

$$0 \rightarrow \mathcal{O}(D) \rightarrow \mathcal{O}(D') \rightarrow \mathcal{Q} \rightarrow 0$$

yields the exact sequence

$$\begin{aligned} 0 \rightarrow H^0(X, \mathcal{O}(D)) &\rightarrow \dots \rightarrow H^0(X, \mathcal{Q}) \rightarrow H^1(X, \mathcal{O}(D)) \\ &\rightarrow H^1(X, \mathcal{O}(D')) \rightarrow 0, \end{aligned}$$

so that $\chi(\mathcal{O}(D')) - \chi(\mathcal{O}(D)) = 1$. Since $\deg D' - \deg D = 1$, the desired assertion follows, q.e.d.

(3.5) **COROLLARY.** $h^0(D) \geq \deg D - g + 1$.

(3.6) **COROLLARY.** *For any $P \in X$, there exists a nonconstant meromorphic function on X , holomorphic in $X - P$, with a pole of order $\leq g + 1$ at P .*

Proof: For $D = (g+1)P$, $h^0(D) \geq 2$ by (3.4), i.e. $H^0(X, \mathcal{O}(D))$ contains a nonconstant element.

(3.7) **COROLLARY.** *For any vector bundle \mathcal{V} on X , and any $P \in X$, $H^1(X - \{P\}, \mathcal{V}) = 0$.*

Proof: By (3.6), there exists a holomorphic map $f: X \rightarrow \mathbf{P}^1$ with $P = f^{-1}(\infty)$. Now use (2.11), (2.12), (2.16) and (2.17).

(3.8) **COROLLARY.** $g = 0$ iff $X \approx \mathbf{P}^1$.

Proof: $g = 0$ for $X = \mathbf{P}^1$ by Laurent's theorem. Conversely, if $g = 0$, then there exists by (3.6) a meromorphic function f on X with just one

simple pole and no other singularities. It is easy to see that $f: X \rightarrow \mathbf{P}^1$ is then an isomorphism.

(3.9) COROLLARY. *If $D \sim D'$, then $\deg D = \deg D'$.*

Proof: $D \sim D'$ implies $\mathcal{O}(D) \approx \mathcal{O}(D')$, hence $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D'))$. Hence $\deg D = \deg D'$ by (3.4).

(3.10) *Definition.* The *degree* of a line bundle \mathcal{L} is the degree of any $D \in \text{Div } X$ such that $\mathcal{L} \approx \mathcal{O}(D)$, i.e. the degree of the divisor of any meromorphic section of \mathcal{L} .

(3.11) *Remark.* The above definition is justified by (2.11) and (3.9). It is clear that the map $\deg : \text{Pic } X \rightarrow \mathbf{Z}$ is a group homomorphism.

(3.13) *Definition.* The *degree* of a vector bundle \mathcal{V} is that of the line bundle $\det \mathcal{V} = \bigwedge^r \mathcal{O}_x \mathcal{V}, r = \text{rank } \mathcal{V}$.

(3.14) *Remark.* The stalk of $(\det \mathcal{V})^{-1} = \text{Hom}(\det \mathcal{V}, \mathcal{O}_x)$ at any $P \in X$ consists \mathcal{O}_P -multilinear alternate maps $\mathcal{V}_P \times \dots \times \mathcal{V}_P$ (r times) $\rightarrow \mathcal{O}_P$.

(3.15) PROPOSITION. *If $0 \rightarrow \mathcal{V}' \rightarrow \mathcal{V} \rightarrow \mathcal{V}'' \rightarrow 0$ is an exact sequence of vector bundles, then $\deg \mathcal{V} = \deg \mathcal{V}' + \deg \mathcal{V}''$.*

Proof: $\det \mathcal{V} \approx \det \mathcal{V}' \otimes \det \mathcal{V}''$.

(3.16) PROPOSITION. (Riemann-Roch theorem, preliminary form). *For any vector bundle \mathcal{V} on X ,*

$$\chi(\mathcal{V}) = \deg \mathcal{V} + \text{rank } \mathcal{V} \cdot \chi(\mathcal{O})$$

Proof: In view of (3.15), (3.2) and (2.11), the proposition follows from (3.4) by induction on rank \mathcal{V} .

§ 4. THE DEGREE OF THE CANONICAL LINE BUNDLE

Recall that the canonical line bundle K on X is the sheaf of holomorphic 1-forms.

(4.1) THEOREM. $\deg K = 2g - 2 = -2\chi(\mathcal{O})$.