An example of beginning with concrete problems

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the actual (more recursive) limit. The reference is: *Soviet Math. Dokl.*, vol. 15 (1974), No. 1, pp. 299-303.

The three papers on recursive analysis are remarkable in staying away from more controversial issues concerning constructivity. In particular, the first paper is probably the earliest use of recursive functions to elucidate constructive analysis.

Of the three papers on topology, I can out of ignorance have very little to say. The paper 1950*b* was done when Specker visited the Institute for Advanced Study. In it he is able to make interesting contributions to group theory as a topologist. It contains elegant ideas which received further development, for instance, almost two decades later in G. Nöbeling, "Verallgemeinerung eines Satzes von Herrn E. Specker", *Inventiones math.*, *6* (1968), 41-55. Let F be the Abelian group of sequences of integers with $\{a_n\} + \{b_n\} = \{a_n + b_n\}$. Specker shows that every countable subgroup of F is free and that F contains a nonfree subgroup of cardinality aleph-one (hence, F itself is not free). Furthermore, let F_b be the subgroup of F consisting of all bounded sequences of integers. Then it is shown that every subgroup of F_b with cardinality aleph-one is a free group. This last theorem is generalized by Nöbeling to show that for an arbitrary set X (rather than just the set of integers), the group of all bounded sequences from X is free and possesses a characteristic basis.

Specker's interest in the group F and its subgroups is suggested by the problem of determining the algebraic structure of the first cohomogy group of an infinite complex, a problem studied in his dissertation 1949*a*. The paper 1950*a* considers end lattices and introduces what is known as Specker compactization which is investigated extensively, for instance, in Herbert Abels, Specker-Kompaktifizierung von lokal kompakten topologischen gruppen, *Math. Z.*, *135* (1974), 325-361.

An example of beginning with concrete problems

In March 1966, Specker lectured in England on his result that Ramsey's theorem does not hold in recursive set theory. Afterwards, he was persuaded to present it at the Logic Colloquium of 1969 (and publish it as 24 in the above list). More exactly, Specker proves that there is a recursive (Σ_0) partition of the 2-elements sets of natural numbers which possesses no recursively enumerable (Σ_1) infinite sets of indiscernibles and that for every recursive partition there is always a Δ_3 set of indiscernibles. This suggests both a question of more exact answers for the case 2 and a more general question of extending the study from 2 to greater n. In fact C. G. Jockush generalizes and settles both questions shortly afterwards *(Journal of symbolic logic*, vol. 37, 1972, pp. 268-280): for every $n \geqslant 2$ and every recursive partition of n-elements sets of natural numbers, there is some Π_n set of indiscernibles; for every $n \geqslant 2$, there is some recursive partition of n-elements sets (into two classes) such that there is no Σ_n set of indiscernibles. In 1977, results by Kirby and Paris aroused widespread interest because their work yields some more mathematical examples of the incompleteness of Peano arithmetic (see, e.g., the last chapter of *Handbook of mathematical logic*, December 1977). At this juncture, several people observed that Jockusch's generalization of Specker's result actually yields quite directly rather similar results, one form of which says simply that Ramsey's theorem is undecidable in the weak (or predicative) second order extension of Peano arithmetic. (A version of the derivation is reported in my *Popular lectures on mathematical logic* being published in Beijing.)

This is in my opinion an illustration of how a good choice of an apparently isolated concrete problem can relate to more substantial developments in a surprising way. Pursuing this line of thought, I would also like to consider another seemingly small beginning by Specker which has been followed by larger results and new vistas.

Logic and quantum mechanics

In 1960, Specker began his consideration of propositions which are not simultaneously decidable with an ancient story about applying to marry a certain princess. Three boxes A, B, C are each empty or contain a ball. The problem is to guess which box is empty and which is not by selecting to open any two boxes which both are conjectured to be empty or nonempty. The boxes are connected in such a way that one can open any two boxes but then the third can no longer be opened. Moreover, the construction is such that whenever any two boxes are open, exactly one of them is empty. Hence nobody was able to win. Finally, somebody insisted on opening two boxes exactly one of which he conjectured to be empty. As he happened to guess right and the third box could no longer be opened, he won the hand of the princess. In this example, the three propositions "x is empty" ($x = A, B, C$) are not simultaneously decidable.