3. The angular defect in higher dimensions

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Theorem 1 exhibits a remarkable fact about the total angular defect of P. For, quite apart from the precise relationship between Δ and χ which it expresses, it shows that $\Delta(P)$ depends only on the topological type of P. It would be remarkable enough that $\Delta(P)$ is independent of the cellular subdivision of P; but, in fact, it does not vary if P is replaced by some other polyhedron homeomorphic to P. Thus $\Delta(P)$ may be said, paradoxically, to be defined by the geometry of P—and to be independent of that geometry! In fact the situation is even more remarkable, since the Euler characteristic is not only a topological invariant but even a *homotopy* invariant; this means that we may deform P continuously without changing $\chi(P)$—and thus without changing $\Delta(P)$.

3. The angular defect in higher dimensions

We look now at the possibility of obtaining a formula for the total angular defect for a polyhedron of arbitrary dimension. We will largely confine attention to *polytopes* (see [3]), that is, homeomorphs of [1] S^{n-1}, for some $n \geqslant 3$. As explained in the Introduction, we will no longer expect to find any significant relationship with the Euler characteristic, so we will concentrate on the question of whether, for such a polytope P, we may obtain a formula for $\Delta(P)$ in terms of V, E and F. Our first result is very general, but will prove to be applicable for certain standard polytopes.

Theorem 2. *Let P be an arbitrary polyhedron in which every edge is incident with precisely q faces, then*

$$\Delta(P) = \pi(2V - qE + 2F). \tag{3.1}$$

Proof. We have only to make a small modification of Pólya's argument. We proceed as in the proof of Theorem 1 as far as the relation (2.2). But now

$$\sum_{j=1}^{F} S_j = qE,$$

so that (2.2) implies that

$$q\pi E - 2\pi F = 2\pi V - \Delta,$$

from which (3.1) immediately follows.

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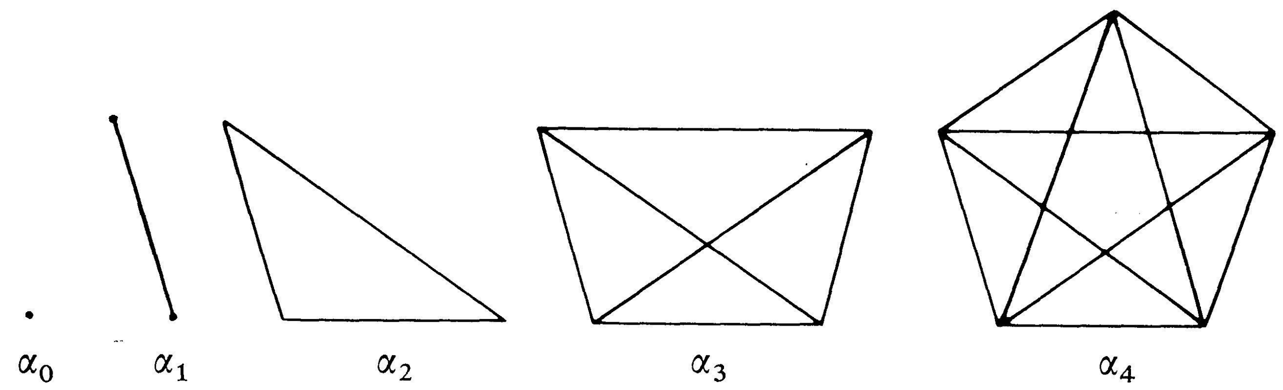
[1]) We explain later in the section why it is more convenient to talk of S^{n-1} than of S^n.

The restriction in the hypothesis of Theorem 2, that every edge be incident with precisely q faces, is very severe, except in the case that P is 2-dimensional. What is remarkable is that it is satisfied in the case of three standard polytopes. These we now describe. In doing so it will be convenient sometimes to adopt the notation of the Introduction, replacing V, E, F by N_0, N_1, N_2, and, generally, using N_i to designate the number of cells of dimension i in the polytope P.



Simplexes

FIGURE 5

SIMPLEXES are produced, as illustrated in Figure 5, by beginning with a single point α_0; we then take this existing structure, introduce another point and join it to the previous one, thus producing α_1 (an edge or line segment); again, we begin with this existing structure, introduce a single point, not in the linear space spanned by α_1, and join it to each of the existing points to produce α_2 (a triangle or 2-simplex); we continue by taking the structure of α_2, introducing a single point, not in the linear space spanned by α_2, and joining that point to each of the existing points to obtain α_3 (a tetrahedron or 3-simplex); etc. In the general case the $(n+1)$ points we have introduced are the vertices of an n-dimensional *simplex*, or n-simplex, α_n, whose cells are themselves simplexes formed by subsets of the $(n+1)$ points, so that there are $\binom{n+1}{1}$ vertices, $\binom{n+1}{2}$ edges, $\binom{n+1}{3}$ faces, $\binom{n+1}{4}$ tetrahedra, etc. Hence we see that, if P_1 is the boundary of α_n,

$$N(P_1)_k = \binom{n+1}{k+1}, \quad 0 \leqslant k \leqslant n-1 .$$

When all the edges are equal these structures are called *regular simplexes*, in [3] denoted α_i. The α_i of Figure 5 should be viewed as though they are in perspective since they were intentionally drawn to show a symmetric placement of the vertices in α_4.

If we remove the interior of α_n we obtain a cellular subdivision of S^{n-1}. It is for this reason that we prefer to speak in this section of S^{n-1} rather than S^n. Since every proper subset of the $(n+1)$ vertices of α_n span a cell of S^{n-1}, we see that, for this polytope, every edge is incident with precisely $(n-1)$ faces, so that we may apply Theorem 2 with $q = n - 1$. Since for this polytope, with $n \geqslant 3$,

$$V = \binom{n+1}{1}, \quad E = \binom{n+1}{2}, \quad F = \binom{n+1}{3}, \tag{3.2}$$

we have, from Theorem 2,

COROLLARY 1. *Let* P_1 *be the polytope obtained by subdividing* S^{n-1} *as the boundary of an* n*-simplex,* $n \geqslant 3$. *Then*

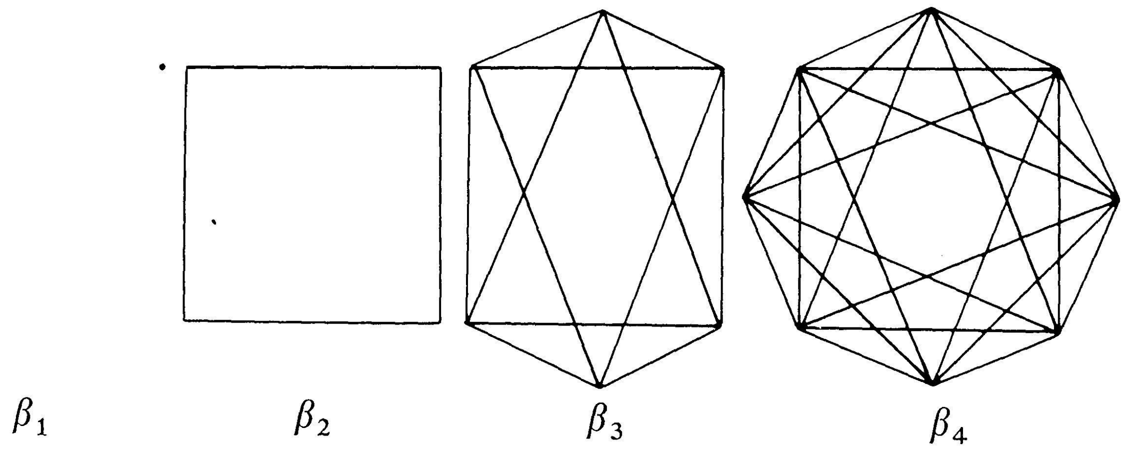
$$\Delta(P_1) = -\frac{\pi}{6}(n-4)(n+1)(n+3).$$

Proof. We have, from (3.1) and (3.2)

$$\begin{aligned}\Delta(P_1) &= \pi\left(2(n+1) - (n-1)\frac{n(n+1)}{2} + 2\frac{(n+1)n(n-1)}{6}\right)\\ &= \frac{\pi}{6}(n+1)\big(12 - 3n(n-1) + 2n(n-1)\big)\\ &= -\frac{\pi}{6}(n+1)(n^2-n-12).\end{aligned}$$

It is interesting to note that, while a simplex is convex, $\Delta(P_1)$ is negative for $n \geqslant 5$ (and zero for $n = 4$).

We now turn to our second example of a polytope.



Cross Polytopes

FIGURE 6

CROSS POLYTOPES may be introduced by recognizing that an important aspect of n-dimensional space is the possibility of having n mutually perpendicular lines through any point 0. For example, each regular simplex α_{n-1} (of Figure 5) involves n points equidistant from 0. Now if we choose to take points equidistant from 0 in both directions we obtain the cellular subdivision of the $(n-1)$-sphere called a *cross polytope*. These have $2n$ vertices and their $(n-1)$-cells consist of 2^n of the α_{n-1}'s. Figure 6 illustrates the cases where n is equal to 1, 2, 3, and 4 respectively. Thus β_1 is a pair of points (vertices) and we can think of progressing from β_i to β_{i+1} by beginning with β_i, introducing a pair of diametrically opposed points (vertices), not in the linear space spanned by β_i, and then joining each of these points to the existing points of β_i (but not to each other). The polytope β_n, which we will call P_2, is, in fact, homeomorphic to S^{n-1}. It can easily be shown by induction that

$$N(P_2)_k = 2^{k+1}\binom{n}{k+1}, \quad 0 \leqslant k \leqslant n-1 .$$

Thus, in particular, for this polyhedron P_2,

$$V = 2n, \quad E = 2n(n-1), \quad F = \frac{4}{3}n(n-1)(n-2) \tag{3.3}$$

We now prove

PROPOSITION 1. *In the polytope* P_2 *every edge is incident with precisely* $(2n-4)$ *faces,* $n \geqslant 2$.

Proof. We first assert that it is plain that in β_n every vertex is incident with precisely $(2n-2)$ edges. This follows by an easy induction on n. For β_{n-1} has $(2n-2)$ vertices and every vertex is, by induction, incident with $(2n-4)$ edges. Thus a vertex of β_{n-1} is incident with $((2n-4)+2)$ edges of β_n, while a new vertex of β_n is incident with $(2n-2)$ edges of β_n.

Now suppose that, in β_{n-1}, every edge is incident with $(2n-6)$ faces—this is certainly true if $n = 3$. Then an edge of β_{n-1} is incident with $((2n-6)+2)$ faces of β_n, while a new edge of β_n is incident with $(2n-4)$ faces of β_n (since a vertex of β_{n-1} is incident with $(2n-4)$ edges of β_{n-1}).

This proof illustrates how we pass from β_{n-1} to β_n by "stepping up dimensions by 1". This is the point of view of topologists, who introduced such an idea into combinatorial topology without, perhaps, realizing that it had already been introduced by the geometers. Topologists call the passage from β_{n-1} to β_n

suspension, and apply this idea to arbitrary topological spaces. Thus the suspension of X is obtained by joining X to two independent points or, equivalently, by taking two cones with base X and joining them together along their bases.

Returning to P_2, we are now ready to prove

COROLLARY 2. *Let P_2 be the polytope obtained by subdividing S^{n-1} as a cross polytope. Then*

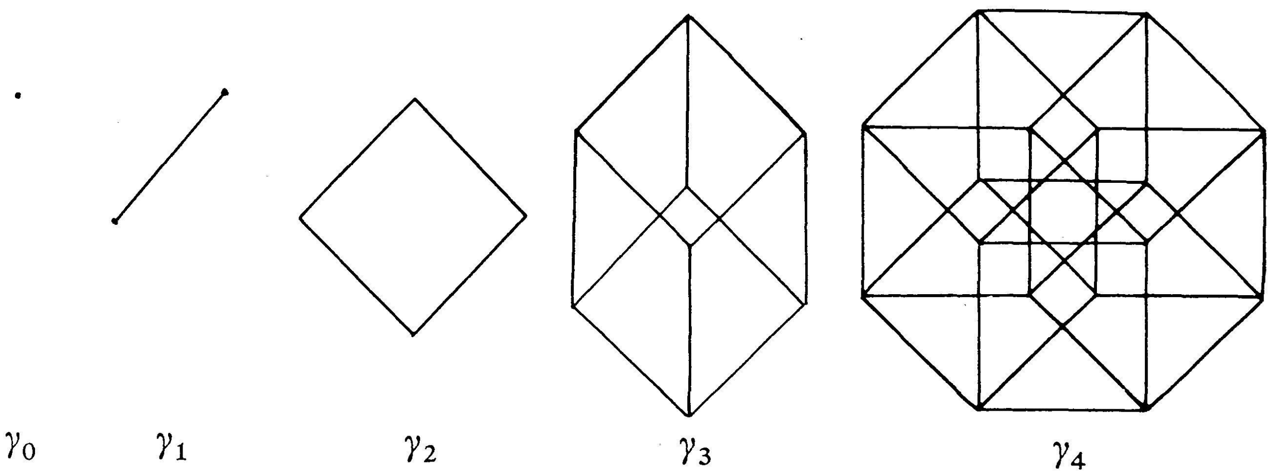
$$\Delta(P_2) = -\frac{4\pi}{3} n(n^2-3n-1).$$

Proof. We assemble the facts from (3.1), (3.3) and Proposition 1 to infer that

$$\begin{aligned}\Delta(P_2) &= \pi\left(4n - 4n(n-1)(n-2) + \frac{8}{3} n(n-1)(n-2)\right) \\ &= \frac{4\pi n}{3}\left(3 - (n-1)(n-2)\right) \\ &= -\frac{4\pi}{3} n(n^2-3n-1).\end{aligned}$$

Here we note that $\Delta(P_2)$ is negative for $n \geqslant 4$.

Finally we turn to our third example of a polytope.



Parallelotopes

FIGURE 7

PARALLELOTOPES are illustrated in Figure 7. The passage from γ_i to γ_{i+1} is achieved by translating γ_i (not along any of its own lines) from its initial to a final position and then joining in pairs each of the original points with the corresponding point of the translated figure. If all edges have the same length the polytope is called a *measure polytope*. The quantities N_k can be computed by considering the passage from γ_i to γ_{i+1}. Thus we readily obtain the inductive relation

$$N(\gamma_{i+1})_k = 2N(\gamma_i)_k + N(\gamma_i)_{k-1}, \quad k \leqslant i. \tag{3.4}$$

Now γ_n is, combinatorially, a hypercube—we specialize the construction by taking γ_1 to be the unit interval and always translating orthogonally by unit distance. Thus the boundary of γ_n is topologically S^{n-1}. We call the boundary P_3 and infer from (3.4) that

$$N(P_3)_k = 2^{n-k}\binom{n}{k}, \quad 0 \leqslant k \leqslant n-1.$$

Note that, for $n = 3$, we get, combinatorially, the unit cube, with 8 vertices, 12 edges, and 6 faces. In general the polytope P_3, with $n \geqslant 3$, yields the values

$$V = 2^n, \quad E = 2^{n-1}n, \quad F = 2^{n-3}n(n-1). \tag{3.5}$$

By an argument very similar to, but simpler than, that of Proposition 1, we may show

PROPOSITION 3. *In the polytope* P_3, *with* $n \geqslant 3$, *every edge is incident with* $(n-1)$ *faces.*

We are now ready to prove

COROLLARY 4. *Let* P_3 *be the polytope obtained by subdividing* S^{n-1} *as the boundary of an* n*-dimensional parallelotope,* $n \geqslant 3$. *Then*

$$\Delta(P_3) = -2^{n-2}\pi(n^2-n-8).$$

Proof. From (3.1), (3.5) and Proposition 3 we have

$$\begin{aligned}\Delta(P_3) &= \pi\left(2^{n+1} - 2^{n-1}n(n-1) + 2^{n-2}n(n-1)\right)\\ &= 2^{n-2}\pi\left(8 - n(n-1)\right)\\ &= -2^{n-2}\pi(n^2-n-8).\end{aligned}$$

Here we note that $\Delta(P_3)$ is negative for $n \geqslant 4$.

The fact that $\Delta(P_1)$, $\Delta(P_2)$, and $\Delta(P_3)$ are all different (except for $n = 3$) shows that the total angular deficiency has no chance of being a topological invariant for polyhedra of dimension $\geqslant 3$. On the other hand it is still striking that Δ depends only on the cellular structure and is independent of the underlying geometric structure.

Remarks. (a) The polytopes P_1, P_2, P_3 not only enjoy the property that each edge of P_i is incident with the same number of faces of P_i, $i = 1, 2, 3$—the property we used to calculate $\Delta(P_i)$ from Theorem 1—they also enjoy the property that each face has the same number of sides. This latter property could also have been used to calculate $\Delta(P)$. Thus if P is a polyhedron subdivided so that each face has the same number s of sides, then one may show that

$$\Delta(P) = 2\pi V - \pi F(s-2). \tag{3.6}$$

It is easy to deduce either of the formulae (3.1), (3.6) from the other if the polyhedron P enjoys *both* the relevant properties. For if every edge of P is incident with q faces and every face of P has s sides, then

$$qE = sF. \tag{3.7}$$

Of course there is an equality corresponding to (3.7) in higher dimensions.

(b) The polytopes P_2 and P_3 may be regarded as *dual*, in the sense that there is a one-one correspondence between the cells of P_2 of dimension k and the cells

FIGURE 8

Data for P_1, P_2, P_3 when $n = 4$

Name of polytope (P_i)	N_0	N_1	N_2	N_3	*Number of sides on each* N_2	*Number of faces incident with each edge*	Δ
Simplex (P_1)	5	10	10	5	3	3	0
Cross polytope (P_2)	8	24	32	16	3	4	-16π
Parallelotope (P_3)	16	32	24	8	4	3	-16π

of P_3 of dimension [1] $(n-1) - k$. Moreover, the incidence relations are carried over by this duality; thus if, in P_2, every $(i-1)$-face is incident with [2] s_i i-faces, then, in P_3, every $(n-i)$-face is incident with s_i $(n-i-1)$-faces (and there is a symmetrical statement interchanging P_2 and P_3). In this sense P_1 is selfdual. Figure 8 displays these dualities for $n = 4$, as well as the value of Δ.

3. Historical comment and summary

René Descartes (1596–1650) and Leonhard Euler (1707–1783) worked on these subjects independently—yet, as we have seen, Pólya (1887–) has shown that their seemingly different formulae for convex polyhedra homeomorphic to S^2 are entirely equivalent to each other. One might believe from the evidence that Euler may have known about Descartes' work on this subject. That would be an erroneous assumption since Descartes' work on this matter [5] was not printed until a century after Euler's death (see [1], p. 56).

Euler [6] offered a variety of verifications but no formal proof of his formula. We have observed that each of the formulae is somewhat surprising by itself and that their connection rather defies intuition since at first glance they seem to be dealing with different qualitative aspects of polyhedra. As a matter of fact neither Euler's nor Descartes' formula is easy to *prove* independently; yet, as we have seen, it is not at all difficult to follow Pólya's proof that the two formulae are equivalent.

The formulae diverge in higher dimensions so that their relationship is a special phenomenon of dimension 2. Euler's formula was generalized by Ludwig Schläfli [9], a Swiss mathematician of the 19th century (1814–1895), who described, in effect, the Euler-Poincaré characteristic of an n-dimensional sphere S^n, subdivided as a *polytope*, a combinatorial structure attributed by Coxeter to Reinhold Hoppe [11]. Poincaré (1854-1912) gave a definition of the Euler-Poincaré characteristic for arbitrary polyhedra, and one proves now, by invoking the topological invariance of the homology groups (see [12]) that the Euler-Poincaré characteristic is a topological invariant.

[1]) The precise form of this duality shows how "correct" it is to regard S^{n-1} as $(n-1)$-dimensional, rather than n-dimensional.

[2]) In fact, $s_i = 2(n-i-1)$.