APPLICATION OF TOPOLOGY TO PROBLEMS ON SUMS OF SQUARES

by Z. D. DAI, T. Y. LAM [1]) and R. J. MILGRAM [1])

If $n = 1, 2, 4$ or 8, the classical n-square identities imply that the product of two sums of n squares in any commutative ring A is also a sum of n squares in A. On the other hand, by a classical theorem of Hurwitz [L, p. 137], one knows that the same statement cannot hold for other natural numbers n.

One can study the same problem over *fields* instead of over commutative rings. Here, the solution of the problem is also known, albeit somewhat different. According to a remarkable theorem of Pfister [P], if $n = 2^m$ is *any* 2-power, and if u, v are sums of n squares in a field F, then their product uv is also a sum of n squares in F. (This implies that the set of nonzero elements in F which are a sum of $n = 2^m$ squares in F is a group under multiplication.) On the other hand, Pfister has also shown that the above statement cannot hold for all fields if n is not of the form 2^m.

Back to sums of squares in commutative rings again, the above two paragraphs suggest that, in considering the multiplication problem, it is perhaps more reasonable to confine one's attention to *units* of a ring A which are sums of 2^m squares in A. Writing $n = 2^m$ and $U(A)$ for the group of units in A, one can ask:

(*) *If* $u, v \in U(A)$ *are sums of* n *squares in* A, *is* $uv \in U(A)$ *also a sum of* n *squares in* A?

This is equivalent to asking if the set of units in A which are a sum of $n = 2^m$ squares in A is a *group* under multiplication. This problem, first raised by R. Baeza, appeared as "Question 12" in Knebusch's collection [K_2] of open problems in the Proceedings of the Quadratic Form Conference in Kingston, Ontario in 1976. Generalizing the work of Pfister, Knebusch [K_1] has shown that (*) has an affirmative answer in case A is a (commutative) *semilocal* ring.

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In this note, we shall furnish the following solution to Baeza's problem:

THEOREM 1. *The answer to* $(*)$ *is affirmative (for all commutative rings* A*) iff* $n = 1, 2, 4$ *or* 8.

In view of the classical square identities mentioned before, we need only show the "only if" part of the theorem. The idea of the proof is to apply $(*)$ in a "generic" setting, and then use suitable topological machinery to derive the conclusion $n = 1, 2, 4$ or 8. The topological result needed here is Adams' famous theorem [A_1] on the nonexistence of Hopf invariant one. Surprisingly, this algebraic application of Adams' Theorem, though reasonably straightforward, seems to have escaped the notice of both algebraists and topologists.

Let n *be a natural number for which* $(*)$ *holds for any commutative ring* A. *We shall prove that* $n = 1, 2, 4$ *or* 8. (In the following, we do not need to assume n to be a power of 2 to begin with, though this would follow from Pfister's theorem.)

Let A be the ring obtained by localizing the polynomial ring

$$\mathbf{R}[x_1, \ldots, x_n, y_1, \ldots, y_n]$$

at the multiplicative set generated by

$$u = x_1^2 + \ldots + x_n^2 \quad \text{and} \quad v = y_1^2 + \ldots + y_n^2 .$$

Then, by $(*)$, the unit $uv \in U(A)$ is a sum of n squares in A, say

$$uv = \left(\frac{f_1}{u^r v^s}\right)^2 + \ldots + \left(\frac{f_n}{u^r v^s}\right)^2, \qquad f_i \in \mathbf{R}[x, y] .$$

Clearing denominators, we get a polynomial equation:

$$(1) \qquad \begin{aligned} &(x_1^2 + \ldots + x_n^2)^{2r+1}(y_1^2 + \ldots + y_n^2)^{2s+1} \\ &= f_1(x, y)^2 + \ldots + f_n(x, y)^2 . \end{aligned}$$

Now we make the following key observation:

LEMMA 1. *Each* $f_i(x, y)$ *above is a "biform" in* (x, y), *of bidegree* $(2r+1, 2s+1)$ *(i.e. viewing the* y's *as constants,* f_i *is a form of degree* $2r+1$ *in* x, *and, viewing the* x's *as constants,* f_i *is a form of degree* $2s+1$ *in* y*).*

Proof. View each f_i as a polynomial in x, and let f_{ij} denote its homogeneous component of degree j in x. We may write

$$f_i = f_{i,p} + f_{i,p+1} + \ldots + f_{i,q} \quad (1 \leqslant i \leqslant n)$$

where p, q are independent of i. If $q > 2r+1$, a comparison of terms of x-degree $2q$ on the two sides of (1) shows that

$$\sum_{i=1}^{n} f_{i,q}^2 = 0,$$

and hence $f_{i,q} = 0$ for all i. Similarly, if $p < 2r+1$, we must have $f_{i,p} = 0$ for all i. Hence, f_i is a form in x of degree $2r+1$. By symmetry, we infer that f_i is also a form in y of degree $2s+1$. Q.E.D.

Now let x, y be points on the unit sphere S^{n-1}. The equation (1) above implies that the n-tuple

$$(f_1(x, y), \ldots, f_n(x, y))$$

is also a point on S^{n-1}. Thus,

$$(x, y) \mapsto (f_1(x, y), \ldots, f_n(x, y))$$

induces a polynomial (and hence continuous) mapping:

$$\mu : S^{n-1} \times S^{n-1} \to S^{n-1}.$$

Fix a base point $b \in S^{n-1}$. Then the compositions

$$S^{n-1} \to S^{n-1} \times \{b\} \xrightarrow{\mu} S^{n-1}$$
$$S^{n-1} \to \{b\} \times S^{n-1} \xrightarrow{\mu} S^{n-1}$$

are *odd* mappings, since each f_i has bidegree $(2r+1, 2s+1)$. By the theorem of Borsuk [B], these odd mappings from S^{n-1} to itself must have odd (topological) degrees, say, $2r'+1$ and $2s'+1$. Thus, the mapping μ has "type" $(2r'+1, 2s'+1)$ in the sense of Hopf [H_1].

Now by the Hopf Construction, the map μ induces a continuous map $\sigma : S^{2n-1} \to S^n$. Let

$$H : \pi_{2n-1}(S^n) \to \mathbf{Z}$$

be the Hopf invariant on the homotopy group $\pi_{2n-1}(S^n)$. According to Hopf [H_1, § 6], the homotopy class $[\sigma] \in \pi_{2n-1}(S^n)$ has Hopf invariant

$$H[\sigma] = \pm (2r'+1)(2s'+1),$$

which is an odd number. By Adams' theorem [A_1] on the nonexistence of Hopf invariant one (or odd Hopf invariant), one knows that this is possible only if $n = 1, 2, 4$ or 8. This completes the proof of Theorem 1.

Adams' original solution of the Hopf invariant one problem took 85 pages, but there exists a proof using the powerful machinery of topological K-theory (cf. [A_6], [A_7, p. 137]) which, according to M. Atiyah, "can be written on a postcard". Thus, our Theorem 1 does admit a "short" proof. In fact, using K-theory, it is possible to obtain a more general version of Theorem 1. This will be deduced from the following topological statement:

THEOREM 2. *Let* $\mu : S^{k-1} \times S^{n-1} \to S^{n-1}$ *be a continuous mapping such that*

$$\mu(-x, y) = \mu(x, -y) = -\mu(x, y)$$

for all $x \in S^{k-1}$ *and* $y \in S^{n-1}$ (cf. [H_2]). *Then* $k \leqslant \rho(n)$, *where* ρ *is the Hurwitz-Radon function.*

(Recall that, if $n = 2^{4a+b} n_o$ where n_o is odd and $b = 0, 1, 2$ or 3, then, by definition, $\rho(n) = 8a + 2^b$.)

Before proving this theorem, let us first record several of its remarkable consequences in algebra. The first one is a result on real common zeros of biforms.

COROLLARY 1. *Let*

$$x = (x_1, \ldots, x_k),\ y = (y_1, \ldots, y_n).$$

Let $f_i(x, y)$ $(1 \leqslant i \leqslant n)$ *be biforms in* (x, y) *of odd bidegrees* $(2r_i + 1, 2s_i + 1)$. *If* $k > \rho(n)$, *then the real loci of* $f_i = 0$ *in the multiprojective space* $\mathbf{RP}^{k-1} \times \mathbf{RP}^{n-1}$ *have a common point.*

Proof. If otherwise, we would have a mapping μ as in Theorem 2 defined by

$$\mu(x, y) = \big(f_1(x, y)/g(x, y), \ldots, f_n(x, y)/g(x, y)\big)$$

where $g(x, y) = \left(\sum f_i(x, y)^2\right)^{1/2}$.

COROLLARY 2. *Let*

$$F(x, y) = F(x_1, \ldots, x_k; y_1, \ldots, y_n)$$

be a biform of bidegree (d, e) *where* d, e *are not multiples of* 4, *and* $k > \rho(n)$. *Suppose that*

$$F(x_o, y_o) = 0 \ (x_o \in \mathbf{R}^k, y_o \in \mathbf{R}^n) \Rightarrow x_o = 0 \text{ or } y_o = 0\,.$$

Then F *cannot be a sum of* n *squares in* $\mathbf{R}[x, y]$.

Proof. This is clear from the above Corollary and the argument given in Lemma 1.

The next corollary may be viewed as a nonlinear generalization of the classical Hurwitz-Radon Theorem [L, p. 137]:

COROLLARY 3. *For*

$$x = (x_1, ..., x_k), \; y = (y_1, ..., y_n)$$

and fixed integers $r, s \geqslant 0$, *the following statements are equivalent:*

(1) $(x_1^2 + ... + x_k^2)^{2r+1} (y_1^2 + ... + y_n^2)^{2s+1}$ *is a sum of* n *squares in* $\mathbf{R}[x, y]$;

(2) $(x_1^2 + ... + x_k^2)^{2r+1} (y_1^2 + ... + y_n^2)^{2s+1}$ *is a sum of* n *squares in* $\mathbf{Z}[x, y]$;

(3) $k \leqslant \rho(n)$.

Proof. (2) $\Rightarrow$ (1) is obvious.
(1) $\Rightarrow$ (3) follows from Corollary 2.
(3) $\Rightarrow$ (2): It is enough to prove (2) for $r = s = 0$. This follows from [G_1] or [G_2].

For a commutative ring A, let $S_m(A)$ denote the set of sums of m squares in A, and let $US_m(A) = U(A) \cap S_m(A)$.

COROLLARY 4. *For fixed integers* k *and* n, *the following statements are equivalent:*

(1) *For any commutative* $\mathbf{R}$*-algebra* A, $US_k(A) \cdot US_n(A) \subseteq US_n(A)$;

(2) *For any commutative ring* A, $S_k(A) \cdot S_n(A) \subseteq S_n(A)$;

(3) $k \leqslant \rho(n)$.

Proof. This is clear from Corollary 3 and the localization argument we have given before. (Note that Theorem 1 is a special case of this Corollary since it is well-known that $n \leqslant \rho(n)$ iff $n = 1, 2, 4$ or 8.)

We shall now begin the proof of Theorem 2, using tools from K-theory, especially Adams' work on the J-homomorphism. For any finite CW-complex X, let $KO(X)$ denote the K-group of virtual real vector bundles over X, and

$\widetilde{KO}(X)$ the reduced K-group (modulo trivial bundles). Let $J(X)$ denote the group of stable fiber homotopy equivalence classes of virtual sphere bundles over X, and $\tilde{J}(X)$ the reduced J-group. The canonical J-homomorphism $J: KO(X) \to J(X)$ induces a homomorphism $\tilde{J}: \widetilde{KO}(X) \to \tilde{J}(X)$. We shall use Adams' results in the following form (see [A_2, (7.4)], [A_4, (3.5)] and [A_5]):

ADAMS' THEOREM. *For* $X = \mathbf{RP}^m$, $\tilde{J}$ *is an isomorphism* $\widetilde{KO}(X) \cong \tilde{J}(X)$. *The group* $\widetilde{KO}(X)$ *is cyclic of order* $2^{\phi(m)}$ *where* $\phi(m)$ *is the number of positive integers* $\leqslant m$ *which are congruent to* 0, 1, 2 *or* 4 (mod 8). *A generator for* $\widetilde{KO}(X)$ *is given by the canonical line bundle* ξ_m *over* $\mathbf{RP}^m$.

On the product $S^{k-1} \times S^{n-1}$, we have an involution defined by

$$T(x, y) = (-x, -y);$$

let E be the quotient space $S^{k-1} \times S^{n-1}/T$. We have an $(n-1)$-sphere bundle $\eta: E \to \mathbf{RP}^{k-1}$: this is the associated sphere bundle of the Whitney sum $n \cdot \xi_{k-1}$. Note that E has an involution $\tau\,\overline{(x, y)} = \overline{(x, -y)}$ which on each fiber is the antipodal map.

Assume that we have a continuous map

$$\mu: S^{k-1} \times S^{n-1} \to S^{n-1},$$

as in Theorem 2. Then μ induces a map $\bar{\mu}: E \to S^{n-1}$ which is equivariant with respect to the involution τ on E and the antipodal map on S^{n-1}. We have a fiber map

$$(\eta, \bar{\mu}): E \to \mathbf{RP}^{k-1} \times S^{n-1} \quad (\text{trivial bundle over } \mathbf{RP}^{k-1})$$

which (by the theorem of Borsuk again) has *odd* degree d on each fiber. By the "mod d-Dold Theorem" [A_3, (1.1)], there exists an integer $e \geqslant 0$ such that $d^e \cdot \eta$ is fiber homotopy equivalent to a trivial bundle. Since $\tilde{J}(\mathbf{RP}^{k-1})$ is 2-primary, this implies that $\eta = 0$ in $\tilde{J}(\mathbf{RP}^{k-1})$. Pulling back to $\widetilde{KO}(\mathbf{RP}^{k-1})$, we have $n \cdot \xi_{k-1} = 0$ in $\widetilde{KO}(\mathbf{RP}^{k-1})$, so by Adams' Theorem, n is divisible by $2^{\phi(k-1)}$. Let $n = 2^{4a+b} n_o$ where n_o is odd and $b = 0, 1, 2$ or 3. If $k > \rho(n) = 8a + 2^b$, then

$$\phi(k-1) \geqslant \phi(8a+2^b) = 4a + b + 1,$$

contradicting $2^{\phi(k-1)} \mid n$. Therefore, we have $k \leqslant \rho(n)$ as desired.

In a recent communication to us, I. M. James has suggested a similar proof of Theorem 2. He points out that a more general discussion of similar structures from which Theorem 2 follows may be found in [W] and in [J, Sec. 7].

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Z. D. Dai

Institute of Mathematics
Peking, China

T. Y. Lam

University of California
Berkeley, California 94720

R. J. Milgram

Stanford University
Stanford, California 94305

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