§1. Genus and self maps

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **27 (1981)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.05.2024**

THEOREM. *Let* $M = M(n_1, \ldots, n_k)$ *be a generalized flag manifold for which the Conjecture C holds. Then*

$$G(M) = \{[M]\}.$$

In particular the Grassmann manifolds $G_p(\mathbf{C}^{p+q})$ *for* $p \neq q$ *and the flag manifolds* $U(n)/T^n$ *are all generically rigid.*

§1. GENUS AND SELF MAPS

Let P denote a fixed set of primes. Two P-sequences

$$S_1, S_2 : P \to E(X_0)$$

are called *equivalent*, if there exist maps $h(0) \in E(X_0)$ and

$$h(p) \in \operatorname{im}\left(E(X_p) \xrightarrow{\text{can}} E(X_0)\right)$$

such that for all $p \in P$ one has

$$h(0)\, S_1(p) = S_2(p)\, h(p).$$

Definition 1.1. We denote by P-Seq $\left(E(X_0)\right)$ the set of equivalence classes of P-sequences in $E(X_0)$.

If P is a finite set of primes and X a nilpotent space of finite type, then there is a canonical map

$$\theta : G(X) \to P\text{-Seq}\left(E(X_0)\right).$$

It is defined as follows. Let $Y \in G(X)$ and $P = \{p_1, \ldots, p_n\}$. Then the localization Y_P is a pull-back of maps $X_{p_i} \xrightarrow{\lambda_i} X_0$, i.e. $Y_P \simeq \operatorname{hoinvlim}\{X_{p_i} \xrightarrow{\lambda_i} X_0\}$. The maps λ_i induce equivalences $\bar{\lambda}_i \in E(X_0)$ and we put

$$\theta(Y) = \{[\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n]\}.$$

If Y_P may also be represented by $\operatorname{hoinvlim}\{X_{p_i} \xrightarrow{\mu_i} X_0\}$, then there exist maps $h(0) \in E(X_0)$ and $\tilde{h}(p_i) \in E(X_{p_i})$, $i \in \{1, \ldots, n\}$ rendering the diagrams

$$\begin{tikzcd}
X_{p_i} \arrow[r, "\bar{h}(p_i)"] \arrow[d, "\lambda_i"'] & X_{p_i} \arrow[d, "\mu_i"] \\
X_0 \arrow[r, "h(0)"'] & X_0
\end{tikzcd}$$

homotopy commutative and thus inducing hoinvlim $\{\lambda_i\} \simeq$ hoinvlim $\{\mu_i\}$. Hence

$$\{[\bar{\lambda}_1, \ldots, \bar{\lambda}_n]\} = \{[\bar{\mu}_1, \ldots, \bar{\mu}_n]\} \in P\text{-Seq}\,(E\,(X_0))$$

and therefore θ is well defined.

LEMMA 1.2. *Let X be a nilpotent space of finite type and let P denote a finite set of primes. Then*

$$\theta : G\,(X) \to P\text{-Seq}\,(E\,(X_0))$$

is surjective with fibers of the form

$$\theta^{-1}\,(\theta\,(Y)) = \{Z \in G\,(X) \mid Z_P \simeq Y_P\}\,.$$

Proof. Let $P = \{p_1, \ldots, p_n\}$ and

$$\{[\bar{f}_1, \ldots, \bar{f}_n]\} \in P\text{-Seq}\,(E\,(X_0))\,.$$

Let $e_i : X_{p_i} \to X_0$ denote the canonical maps and put

$$f_i = \bar{f}_i \circ e_i : X_{p_i} \to X_0\,.$$

Define $W =$ hoinvlim $\{f_i\}$; W comes equipped with a canonical map $f : W \to X_0$. Let Z be the homotopy pull back of $W \xrightarrow{f} X_0 \xleftarrow{\text{can}} X_{\bar{P}}$, where $\bar{P}$ denotes the set of primes complementary to P. Then $Z \in G\,(X)$ and

$$\theta\,(Z) = \{[\bar{f}_1, \ldots, \bar{f}_n]\}\,;$$

thus θ is surjective. It is clear from the definition of θ that for $Y, Z \in G\,(X)$ one has $\theta\,(Y) = \theta\,(Z)$ if and only if $Y_P \simeq Z_P$.

The next lemma provides a sufficient condition for θ to be monic "at the basepoint".

LEMMA 1.3. Let X be a nilpotent space of finite type. Suppose that there exists a finite set of primes P with complement $\bar{P}$ such that

a) $Y \in G(X)$ implies $Y_{\bar{P}} \simeq X_{\bar{P}}$

b) every $f \in E(X_0)$ can be written as $f_1 \circ f_2$ with $f_1 \in \operatorname{im}\left(E(X_P) \overset{\text{can}}{\rightarrow} E(X_0)\right)$ and $f_2 \in \operatorname{im}\left(E(X_{\bar{P}}) \rightarrow E(X_0)\right)$.

Then for $\theta: G(X) \rightarrow P\text{-Seq}\left(E(X_0)\right)$ as above, one has $\theta^{-1}(\theta(X)) = \{X\}$.

Proof. Let $Y \in G(X)$ with $\theta(Y) = \theta(X)$. Then $Y_P \simeq X_P$ by the definition of θ, and $Y_{\bar{P}} \simeq X_{\bar{P}}$ by assumption. Hence Y may be obtained as a homotopy pull back of the form

$$\begin{tikzcd} & X_P \arrow[dr, "\alpha"] & \\ Y \arrow[ur] \arrow[dr] & & X_0 \\ & X_{\bar{P}} \arrow[ur, "\beta"'] & \end{tikzcd}$$

If α induces $\bar{\alpha} \in E(X_0)$ and if $\gamma = \bar{\alpha}^{-1} \circ \beta$, then Y is also a pull back of the form

$$\begin{tikzcd} & X_P \arrow[dr, "\text{can}"] & \\ Y \arrow[ur] \arrow[dr] & & X_0 \\ & X_{\bar{P}} \arrow[ur, "\gamma"'] & \end{tikzcd}$$

Let $\bar{\gamma} \in E(X_0)$ be the map induced by γ and write $\bar{\gamma} = f_1 f_2$ with

$$f_1 \in \operatorname{im}\left(E(X_P) \rightarrow E(X_0)\right), \quad f_2 \in \operatorname{im}\left(E(X_{\bar{P}}) \rightarrow E(X_0)\right).$$

Choose a lift $\tilde{f}_1^{-1} \in E(X_P)$ of f_1^{-1} and a lift $\tilde{f}_2 \in E(X_{\bar{P}})$ of f_2. Then $f_1^{-1}\gamma = \text{can} \circ \tilde{f}_2$ and one can form a commutative diagram,

$$\begin{tikzcd}
X_P \arrow[rrr, "\tilde{f}_1^{-1}"] \arrow[dr, "\text{can}"'] & & & X_P \arrow[dl, "\text{can}"] \\
& X_0 \arrow[r, "f_1^{-1}"] & X_0 & \\
X_{\bar{P}} \arrow[ur, "\gamma"] \arrow[rrr, "\tilde{f}_2"'] & & & X_{\bar{P}} \arrow[ul, "\text{can}"']
\end{tikzcd}$$

which shows that $Y \simeq X$.

§2. The case of generalized flag manifolds

The following result is an easy consequence of [F].

Lemma 2.1. Let M be a generalized flag manifold. Then the following holds.

a) If $g(\lambda) \in Gr(M_0)$ is a grading map with $\lambda \in \mathbf{Z}_Q^*$ for some (not necessarily finite) set of primes Q, then $g(\lambda)$ lifts to a homotopy equivalence $\tilde{g}(\lambda): M_Q \to M_Q$.

b) Let P be an arbitrary set of primes with complement $\bar{P}$. Then every

$$f \in \langle Gr(M_0), N(H)/H \rangle$$

may be written in the form $f = f_1 \circ f_2$ with

$$f_1 \in \operatorname{im}\big(E(M_P) \to E(M_0)\big)$$

and

$$f_2 \in \operatorname{im}\big(E(M_{\bar{P}}) \to E(M_0)\big).$$

Proof. Let $\lambda = k/l$ with k and l relatively prime integers. Then $g(k)$ and $g(l)$ lift to equivalences

$$\tilde{g}(k), \tilde{g}(l): M_Q \to M_Q$$