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RECENT PROGRESS IN THE THEORY OF MINIMAL SURFACES [1]

by E. Bombieri

I. Introduction

In this talk I will report on some recent results in the theory of minimal surfaces. Many of them belong to the theory of higher dimensional minimal varieties and all of them are related to the point of view of Geometric Measure Theory and the Calculus of Variations. The important approach to the various aspects of the 2-dimensional Plateau problem provided by harmonic maps and the Hilbert space setting, will not be treated here. I should also stress the fact that this report is not and does not intend to be a survey of all important achievements of the last years, but rather its purpose is to present a few recent results connected with the central problems of the theory, namely existence, uniqueness and regularity of solutions to the Plateau problem from the point of view of the Calculus of Variations.

II. Currents and Varifolds

Let U be an open set of $\mathbf{R}^n$ and let T be a distribution on smooth differential m-forms φ with compact support in U. The boundary of T is the distribution defined by $(\partial T)(\psi) = T(d\psi)$ where d is the exterior differential; clearly ∂T is a distribution on $(m\text{-}1)$-forms. If T and ∂T are continuous with respect to the L^∞ topology on forms, one says that T is locally normal, and if in addition T has compact support in U one says that T is normal. Normal currents form a Banach space in the following way. Let $M(\varphi)$ be a norm on m-forms, and let $M(T)$ be the dual norm

$$M(T) = \sup \{T(\varphi);\ M(\varphi) \leqq 1\} ;$$

then $N(T) = M(T) + M(\partial T)$ is a norm in the space of normal currents. There is a very special norm on forms, called comass, such that the dual norm, called mass, coincides with m-dimensional area in case T is integra-

[1]) This article has already been published in *Contributions to Analysis*, papers communicated to an international Symposium in honour of A. Pfluger, ETH Zürich, April 1978. Monographie de l'*Ens. Math.* N° 27, Genève 1979.

tion on an m-dimensional simplex. It follows that if T is integration on an m-dimensional oriented compact manifold V, then its mass is $M(T) = m$-dimensional area of V.

Not all currents are obtained by integration on sets. An integral chain $\gamma = \Sigma n_\sigma \sigma$, where $n_\sigma \in \mathbf{Z}$ and σ are simplexes, determines in a natural fashion a current $\gamma(\varphi) = \Sigma n_\sigma \int_\sigma \varphi$; if f: spt $\gamma \to U$ is a Lipschitz map, then one can define $f_\# \gamma$ by means of $f_\# \gamma(\varphi) = \gamma(f^\# \varphi)$. Now a current T is rectifiable if it can be approximated in the mass norm by currents of type $f_\# \gamma, \gamma$ a finite polyhedral chain. If both T and ∂T are rectifiable, one says that T is an integral current. Integral currents are the appropriate generalization of the notion of oriented manifold with boundary, as far as integration of differential forms is concerned. Now the main point is: with respect to a certain very weak notion of convergence (the flat convergence) one has:

(a) a closure theorem, to the effect that if a limit of integral currents is normal, the limit still is an integral current;

(b) a compactness theorem, to the effect that bounded sets of integral (or normal) currents are precompact;

(c) an approximation theorem, to the effect that integral currents can be approximated in the strong norm by smooth C^1 deformations near the identity of suitable integral chains.

Only one warning: the results (a), (b) above hold only if we consider currents with compact support $\subset K$, where K is a compact Lipschitz neighborhood retract (for example K is convex or K has C^2 boundary). Also, the approximation theorem, although it says that integral currents are almost a countable union of C^1 manifolds, does not imply anything about the support of the current. For example, let $z_1, z_2, \ldots, z_m, \ldots$ be a countable dense subset of a compact set $K \subset \mathbf{R}^2$, let C_m be the circle $|z - z_m| = 2^{-m}$ with the usual orientation and let $T = \sum_m \int_{C_m}$. Then T is an integral current, but clearly spt $(T) \supset K$.

Since the mass $M(T)$ is lower semicontinuous on integral currents, one can use the closure and compactness theorems to obtain a solution to Plateau's problem in the following form. Let X be an integral current with support in a fixed nice compact set K; then there is an integral current T of least mass in the set $\{T;\ \partial T = \partial X,\ \text{spt}\ T \subset K\}$. In fact, one simply takes a minimizing sequence T_i such that $\partial T_i = \partial X$, $M(T_i) \to$ inf and takes a weak limit $T = \lim T_{i_k}$ on a suitable subsequence. Now $\partial T = \partial X$ and T is integral with spt $T \subset K$, by the compactness theorem, it is also obvious that $M(T) = \liminf M(T_i)$, by lower semicontinuity of mass.

The question arises to what extent this is a satisfactory solution to the problem: among all surfaces with a given boundary, find one with least area. This gives rise to the regularity problem, that is showing that the solutions thus found are indeed manifolds or manifolds outside a small singular set.

The theory of normal and integral currents, as developed by Federer and Fleming [F-F] in their fundamental paper of 1960, is essentially a theory of chains with real or integer coefficients, which has both all reasonable properties of algebraic topology and which at the same time yields reasonable spaces for the purpose of the calculus of variations. However, it is not entirely suitable to study the actual soap films which come out in physical experiments. One difficulty is because of orientation; for example, a Möbius band usually arises as a soap film off a wire which approaches a doubly covered circle. This difficulty can be overcome by working with currents with finite abelian coefficient group, for example with mod 2 coefficients. On the other hand, it became increasingly clear that in order to get a theory suitable for describing physical experiments one had to work in a more set theoretic fashion and give up the useful notion of boundary operator. A convenient theory is the theory of varifolds by Almgren. A varifold is simply a Radon measure on the Grassmann bundle of the space. An appropriate notion of rectifiable and integral varifold is developed, and analogs of the closure, compactness and approximation theorems can be obtained. There are some important differences however and it turns out that currents and varifolds complement each other in several respects.

We end this section by referring to Federer, [FH 1] Ch. IV and Almgren, [AF 1] for precise definitions and proofs of the basic properties of currents and varifolds. All these concepts can be extended to ambient spaces different from euclidean space and in particular to Riemannian manifolds.

III. Recent progress on existence problems

One of the great successes of the Calculus of Variations in the Large has been the proof of existence of closed geodesics on smooth compact Riemannian manifolds. The following striking result is a two dimensional extension.

THEOREM 1 (J. Pitts [P 1, 2]. *Every smooth, compact, three dimensional Riemannian manifold without boundary contains a non-empty, closed, imbedded, two dimensional, minimal submanifold without boundary.*

Suppose M is a smooth compact Riemannian manifold of dimension n and let $0 \leqq k \leqq n$. It is still an open question whether M always contains a regular closed minimal submanifold of dimension k. If $k = 1$, this is the problem of existence of simple closed geodesics, which can be treated using Morse theory. If $k \geqq 2$, existing results required suitable assumptions on M. For example, if $k = n - 1$ and $H_{n-1}(M, \mathbf{Z}) \neq 0$ one can find a closed current T with $\partial T = 0$ representing a given homology class and for which $M(T) = \min$; regularity theorems in dimension $n \leqq 7$ now imply that spt T is a smooth manifold. Lawson [L] proved that if $M = S^3$ then M contains closed minimal surfaces of arbitrarily high genus. For the general case, there has been a partially successful approach by Almgren [AF 2].

Let I^m be the unit m-cube, I_0^m its boundary, let (A, a) be a space with a base point a and let $\pi_m(A, \{a\})$ be the m dimensional homotopy group of equivalence classes of continuous mappings of (I^m, I_0^m) into (A, a). If $Z_k(M, G)$ is the group [1]) of k-cycles of M with coefficients in the abelian group G, then it is known that $\pi_m(Z_k(M, G), \{0\})$ is naturally isomorphic with the homology group $H_{m+k}(M; G)$ for $1 \leqq k \leqq \dim M$. Now let Π be a homotopy class of maps $\varphi: (I^m, I_0^m) \to (Z_k(M, G), \{0\})$ and consider the minimax problem

$$\inf_{\varphi \in \Pi} \sup_{t \in I^m} F(\varphi(t))$$

where F is a good function on $Z_k(M, G)$, in our case the mass. The point of Morse theory is: if $\Pi \neq 0$, then the solutions to the minimax problem are non-trivial critical points of the functional $F(T)$. Almgren succeeded in doing this on the space of k-currents and k-varifolds, obtaining non-trivial stationary varifolds in this way. Unfortunately, the regularity theory of stationary varifolds is still at a rudimentary stage, and Almgren's solution suffers of the same defects as for the earlier Federer and Fleming solution to Plateau's problem. However, Pitts has been able to restrict the class of competing maps φ in such a way so that the topological aspects are unchanged but the critical points share the most fundamental properties of locally minimizing currents and varifolds. For these special critical points he is then able to carry further the regularity theory and obtain eventually his theorem.

1) Here $Z_k(M, G)$ is made into a topological group by means of the flat norm.

IV. Recent progress on uniqueness problems

It is a well-known fact that even for absolutely minimizing surfaces the minimum need not be unique. It is expected that in case there are two or more absolute minima, a small deformation of the boundary will separate them, restoring uniqueness of absolute minimum. This has been recently done by F. Morgan [M], who proves that almost every C^3 closed curve in $\mathbf{R}^3$ bounds a unique minimal surface of least area. It would be however too technical to describe this result in more detail and, going to the opposite point of view, I will give an explicit example of a 2 dimensional compact manifold in $\mathbf{R}^4$ bounding infinitely many oriented stationary manifolds of dimension 3. In fact, our example will be the Clifford flat torus

$$x_1^2 + x_2^2 = x_3^2 + x_4^2 = 1/2$$

which is also minimal in S^3.

THEOREM 2. *The Clifford flat torus in* $\mathbf{R}^4$ *bounds infinitely many 3-dimensional manifolds with mean curvature* 0 *at every point.*

We sketch the proof of this result, which is implicit in the paper [B-DG-G] on minimal cones and the Bernstein problem.

Let $p + q = n - 2$ and consider the action of $SO(p) \times SO(q)$ on $\mathbf{R}^n = \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$. Let $u = (x_1^2 + \ldots + x_{p+1}^2)^{1/2}$, $v = (x_{p+2}^2 + \ldots + x_{p+q+2}^2)^{1/2}$. If we consider minimal hypersurfaces in $\mathbf{R}^n$ invariant by $SO(p) \times SO(q)$, we may describe them in the form $f(u, v) = 0$ with u, v as above, and thus as a curve Γ in the quadrant $u, v \geqq 0$. If we now represent Γ parametrically as $(u(t), v(t))$ the condition of mean curvature 0 on the hypersurface means that

$$u'' v' - u' v'' + [p(u')^2 + q(v')^2]\left(\frac{u'}{v} - \frac{v'}{u}\right) = 0$$

or in other words that Γ is a geodesic for the metric

$$ds^2 = u^p v^q [(du)^2 + (dv)^2].$$

In our case

$$ds^2 = (uv)[(du)^2 + (dv)^2]$$

and the requirement that our hypersurface has the Clifford torus as boundary means that we have to find all geodesics which start at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and end at $uv = 0$. There is exactly one such geodesic ending at (0, 0), namely

$u = v$. This corresponds to the well-known cone $x_1^2 + x_2^2 = x_3^2 + x_4^2$, which has indeed mean curvature 0. The other possibility for a geodesic is to end on the u-axis, those ending on the v-axis being obtained by a symmetrical reflection. Up to a homothetic transformation there is only one such geodesic. We introduce the new homothetically invariant parameters

$$\varphi = \operatorname{artg} \frac{v}{u}, \quad \theta = \operatorname{artg} \frac{v'}{u'},$$

$$\sigma = \theta - 3\varphi + \frac{\pi}{2}, \quad \psi = \theta + \varphi - \frac{\pi}{2}$$

and rewrite the equation for geodesics as

$$\begin{cases} \dot{\sigma} = -\frac{3}{2} \sin \sigma - \frac{7}{2} \sin \psi \\ \dot{\psi} = \frac{1}{2} \sin \sigma - \frac{3}{2} \sin \psi . \end{cases}$$

We are interested in the unique characteristic C which at time $t = -\infty$ starts at the saddle point $(\pi, 0)$ and at time $t = \infty$ ends at the origin $(0, 0)$. Since the diagonal $u = v$ goes in the line $\sigma = \psi$ in the (σ, ψ)-plane, if we follow C from $t = -\infty$ to a time t_0 for which $\sigma = \psi$, going back to the (u, v) plane we get a geodesic starting on the axis $v = 0$ and ending on $u = v$; clearly by applying a suitable homothety we may get a geodesic ending at $u = v = \frac{1}{\sqrt{2}}$ and a solution to our problem. It follows that our result will be proved if we show that the characteristic C crosses the line $\sigma = \psi$ infinitely many times. This in fact is obvious, because C ends at $(0, 0)$ and it is easily checked that $(0, 0)$ is a focal singular point, or vortex, of the differential system for σ, ψ.

It may be noted that the same construction gives other examples, like for the boundary $S^2\left(\frac{1}{\sqrt{2}}\right) \times S^2\left(\frac{1}{\sqrt{2}}\right)$, with almost exactly the same result.

V. Recent progress on regularity problems

The regularity theory of minimal currents and varifolds is fundamental if we want to obtain classical solutions to variational problems. Here the theory proceeds in two main directions: one is to prove stronger and better regularity theorems, the other is to produce more examples of singular minimal varieties to narrow the gap.

It is a classical result that a minimal surface is real analytic at every regular point. Let $V \subset \mathbf{C}^n \cong \mathbf{R}^{2n}$ be a complex analytic subvariety of $\mathbf{C}^n$; by Wirtinger's inequality, V is also an absolutely minimizing surface in $\mathbf{R}^{2n}$, hence V may carry singularities and the singular set can have codimension 2. If T is a minimal hypersurface in $\mathbf{R}^n$, singularities are harder to find: Simons' cone

$$x_1^2 + \dots + x_4^2 = x_5^2 + \dots + x_8^2$$

is the first and simplest example of a singular absolutely minimal hypersurface in $\mathbf{R}^8$. All these examples are real analytic sets and one could ask whether this is always the case. However there are topological obstructions for a singularity to be real analytic, as the following construction by Milani shows.

We can find an embedding of $\mathbf{P}^2(\mathbf{C})$, the complex projective plane, in $\mathbf{R}^9$ so that $\mathbf{P}^2(\mathbf{C})$ will be on the sphere S^8 given by $x_1^2 + \dots + x_9^2 = 1$. By the general theory, there is a 5-dimensional current T with boundary $\mathbf{P}^2(\mathbf{C})$ which is absolutely minimizing and, by results of Allard on boundary regularity, one can show that spt (T) is a manifold in a neighborhood of its boundary. We conclude that the singular set of T is a compact subset of spt $(T) \setminus$ spt (∂T). Now assume that spt (T) is a real analytic set Σ. By Hironaka's theorem on resolution of singularities, together with a very important refinement obtained by Tognoli, there is a real analytic manifold Σ' and a proper $f: \Sigma' \to \Sigma$ which is an isomorphism outside f^{-1} (sing Σ). Thus Σ' is a real manifold with boundary $\mathbf{P}^2(\mathbf{C})$. This contradicts Thom's theorem that $\mathbf{P}^2(\mathbf{C})$ is a generator of infinite order of the cobordism ring, and the conclusion is that spt (T) is not a real analytic set.

Another beautiful example has been obtained by Lawson and Osserman [L-O] in their work on the Dirichlet problem on the minimal surface system in non-parametric form, in higher codimension. If $\eta: S^3 \to S^2$ is the Hopf map

$$\eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1z_2)$$

where $(z_1, z_2) \in \mathbf{C} \times \mathbf{C} \cong \mathbf{R}^4$ and η is considered as $\eta(z_1, z_2) \in \mathbf{R} \times \mathbf{C} \cong \mathbf{R}^3$, they found that the Lipschitz function $f: \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$f(x) = \frac{\sqrt{5}}{2} |x| \, \eta\left(\frac{x}{|x|}\right) \quad \text{for } x \neq 0,$$

is a solution of the minimal surface system. This gives the first example of a non-parametric minimal Lipschitz cone, of dimension 4 in $\mathbf{R}^7$.

General regularity theorems for absolutely minimal currents have proved to be very difficult to obtain. The codimension 1 case has been treated with success; after previous work by Reifenberg, De Giorgi, Almgren, Miranda, Simons, finally Federer [FH 2] proved the sharp result that absolutely minimal hypersurfaces are non-singular in codimension less than 7. In particular, minimal hypersurfaces of dimension $\leqq 6$ are analytic manifolds. Also, in the codimension one non-parametric case Bombieri, De Giorgi and Miranda proved regularity in any dimension, a result to be contrasted with the Lipschitz singular cone of Lawson and Osserman.

In general codimensions, the only result was that the set of regular points is dense (Reifenberg, Morrey, Almgren) and only recently Almgren announced [AF 3] that minimal surfaces are regular almost everywhere. It seems likely that Almgren's new methods will in fact show that minimal surfaces are regular in codimension 2; in view of the examples provided by complex analytic varieties, this result would be sharp.

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