APPENDIX — SIMPLICITY PROPERTIES OF FIBERS

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where $N = \ker(\pi_1(X) \to \pi_1(Y))$. By (6.6) the map β_i is i-connected since the fiber of the two vertical arrows is $A(P_n\tilde{X})_N$. Now by (5.4) we see that α_i is simple for $k > i$.

For two decompositions (X'_i) and (X''_i) of $f : X \to Y$ satisfying the above conditions, we have $P_iX'_i = P_iX''_i$ and both X'_i and X''_i map into X_i, constructed above, such that the resulting diagrams are homotopy commutative. The connectivity of the β_i and (5.1) shows that these maps are all homotopy equivalences. This proves the theorem.

(6.8) *Remarks.* This theorem (6.7) coincides with the Dror results for Y a point [D1, Theorem 1.3] and $Y = S^n$ [D2]. An interesting problem is to describe the ith stage X_i in terms of invariants of X_{i-1} as in [D1] and [D2]. (See the footnote in the introduction.)

Appendix — Simplicity properties of fibers

In the proof of (5.4) we used the fact that for a fibration $F \to E \xrightarrow{f} B$ the action of $\pi_1(F)$ on $\mathrm{Im}(\partial : \pi_{k+1}(B) \to \pi_k(F))$ is trivial. This assertion does not seem to be in the literature so we include a proof here.

We extend the mapping sequence of the fibration f to $\Omega B \to F \to E \xrightarrow{f} B$ and study F as the total space of a principal fibration with fibre the H-space ΩB. If G is an H-space, then $\pi_1(G)$ acts trivally on $\pi_*(G)$ because the covering transformations $\tilde{G} \to G$ on the universal covering $\tilde{G}$ of G are homotopic to the identity. This is proved by lifting a loop to a path in $\tilde{G}$ and using the H-space structure on $\tilde{G}$ to deform the identity along this path to the covering transformation defined by the homotopy class of the loop. Recall that a principal fibration is induced from $G \to E_G \to B_G$ up to fibre homotopy equivalence.

(A.1) Proposition. *Let* $G \to X \xrightarrow{\pi} Y$ *be a principal fibration with fibre* G *acting on* X. *Then we have:*

(a) $\mathrm{im}(\pi_1(G) \to \pi_1(X))$ *acts trivially on* $\pi_*(X)$, *and*

(b) $\pi_1(X)$ *acts trivially on* $\mathrm{im}(\pi_*(G) \to \pi_*(X))$.

Proof. For (a) we have the following commutative diagram induced by a covering transformation $T : \tilde{G} \to \tilde{G}$.

$$\begin{tikzcd}
& \tilde{G} \arrow[rr] \arrow[dd] & & \tilde{G} \times \tilde{X} \arrow[rr] \arrow[dd] & & \tilde{X} \arrow[dd] \\
\tilde{G} \arrow[ur, "T"] \arrow[rr] \arrow[dr] & & \tilde{G} \times \tilde{X} \arrow[ur, "T \times 1"] \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[ur, "T'"] \arrow[dr] & \\
& G \arrow[rr] & & G \times X \arrow[rr] & & X
\end{tikzcd}$$

The covering transformation T defines T', and since T is homotopic to the identity so is T'. This proves (a).

For (b) we use the following commutative diagram where T' is any covering transformation of $\tilde{X}$.

$$\begin{tikzcd}
\tilde{G} \arrow[rr] \arrow[dr] & & \tilde{G} \times \tilde{X} \arrow[rr] \arrow[dd] & & \tilde{X} \arrow[dd] \\
& \tilde{G} \times \tilde{X} \arrow[ur, "1 \times T'"] \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[ur, "T'"] \arrow[dr] & \\
& & G \times X \arrow[rr] & & X
\end{tikzcd}$$

Now the inclusion $i : \tilde{G} \to \tilde{X}$ is the composite of the first horizontal row, and $T'i$ and i are homotopic by $i_t(g) = g \cdot \tilde{\alpha}(t)$ where $g \in G$ and α is a lifting of the loop α corresponding to the covering transformation T'. This proves the proposition.

For a general fibration $f : E \to B$ with fibre F the mapping sequence $\Omega B \to F \to E \to B$ allows us to deduce the next proposition from the previous one.

(A.2) Proposition. *Let* $f : E \to B$ *be a fibration with fibre* $F \to E$. *Then we have:*

(a) $\operatorname{im}(\pi_2(B) \to \pi_1(F))$ *acts trivially on* $\pi_*(F)$, *and*

(b) $\pi_1(F)$ *acts trivially on* $\operatorname{im}(\partial : \pi_{i+1}(B) \to \pi_i(F))$.