§5. k-SIMPLE ACYCLIC MAPS

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(4.4) *Remark.* The group $\tilde{N} = \pi_1 (A\tilde{X}_N)$ is a central extension of N (see the appendix) and, as $A\tilde{X}_N$ is acyclic, satisfies $H_1 (\tilde{N}) = H_2 (\tilde{N}) = 0$. Therefore $\tilde{N}$ is the universal central extension of N (see [K2]), namely one has the exact sequence $0 \to H_2 (N) \to \tilde{N} \to N \to 1$. Therefore, if $f : X \to X'$ is a map such that $\pi_1 (f)$ sends the perfect normal subgroup N of $\pi_1 (X)$ isomorphically onto a normal subgroup N' of $\pi_1 (X')$, then the induced map $Af : A\tilde{X}_N \to A\tilde{X}'_{N'}$ induces an isomorphism on the fundamental groups.

§ 5. k-SIMPLE ACYCLIC MAPS

In this section we study acyclic maps having simplicity properties. The first proposition generalizes some results of Dror [D1, Lemma 3.4].

(5.1) PROPOSITION. *Let $f : X \to Y$ be a map of path connected spaces with $\pi_1 (f)$ an isomorphism, and let N be a perfect normal subgroup of $\pi_1 (X) = \pi$. If f induces an isomorphism $H_* (X, \mathbf{Z} [\pi/N]) \xrightarrow{\sim} H_* (Y, \mathbf{Z} [\pi/N])$ and an isomorphism $\pi_i (X) \xrightarrow{\sim} \pi_i (Y)$ for $i \leqq k - 1$, then*

(1) $\pi_k (f) : \pi_k (X) \to \pi_k (Y)$ *is an epimorphism when N acts trivially on $\pi_k (Y)$, and*

(2) $\pi_k (f) : \pi_k (X) \to \pi_k (Y)$ *is an isomorphism when N acts trivially on $\pi_k (X)$ and $\pi_k (Y)$.*

Proof. Let $F \to \tilde{X}_N$ be the homotopy fibre of the covering map $\tilde{f} : \tilde{X}_N \to \tilde{Y}_N$. By hypothesis it follows easily that $\tilde{f}$ induces an isomorphism on integral homology and on $\pi_i (X) \to \pi_i (Y)$ for $i \leqq k - 1$. From the Serre spectral sequence we have $H_0 (\tilde{Y}_N, H_{k-1} (F)) = H_0 (N, H_{k-1} (F)) = 0$. Since $H_{k-1} (F) = \pi_{k-1} (F)$ is a quotient of $\pi_k (Y)$ on which the perfect group N acts trivially, it follows that $\pi_{k-1} (F) = 0$, which proves (1).

Under the hypothesis of (2) we have $\pi_i (F) = 0$ for $i < k$ and $H_0 (\tilde{Y}_N, H_k (F)) = H_0 (N, \pi_k (F)) = 0$. Since N acts trivially on $\pi_k (X)$ the induced morphism $\pi_k (F) \to \pi_k (X)$ must be trivial, which proves the proposition.

The following lemma, proved in [D2, Lemma 2.6], follows easily from the homology exact sequence.

$$H_1(G, M'') \to H_0(G, M') \to H_0(G, M) \to H_0(G, M'') \to 0$$

(5.2) LEMMA. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of* $\mathbf{Z}[G]$*-modules where* G *is a perfect group. Then* M' *and* M'' *are trivial G-modules if and only if* M *is a trivial G-module.*

(5.3) DEFINITION. *A space* X *is k-simple provided* $\pi_1(X)$ *acts trivially on* $\pi_k(X)$. *A map* $f: X \to Y$ *is k-simple provided* $\ker \pi_1(f) \subset \pi_1(X)$ *acts trivially on* $\pi_k(X)$.

(5.4) PROPOSITION. *Let* $f: X \to Y$ *be a map with homotopy fibre* A *where* $\pi_1(A)$ *is perfect. Then* f *is k-simple if and only if* A *is k-simple.*

Proof. In the homotopy exact sequence of any fibration

$$\pi_{k+1}(Y) \to \pi_k(A) \to \pi_k(X) \to \pi_k(Y),$$

see the appendix, $\pi_1(A)$ acts trivially on $\operatorname{im}(\pi_{k+1}(Y) \to \pi_k(A)) = M'$. If f is k-simple, then $\operatorname{im}(\pi_1(A)) = \ker(\pi_1(f))$ acts trivially on $\pi_k(X)$. Hence $\pi_1(A)$ acts trivially on $M' \subset \pi_k(A)$ and on the quotient $\pi_k(A)/M'$. By (5.2), it acts trivially on $\pi_k(A)$.

Conversely, $\ker(\pi_1(f))$ acts trivially on $\ker(\pi_k(f)) \subset \pi_k(X)$ and trivially on $\pi_k(Y) \supset \operatorname{im}(\pi_k(f))$. By (5.2), $\ker(\pi_1(f))$ acts trivially on $\pi_1(X)$. This proves the proposition.

(5.5) *Notations.* For a path connected space X and a perfect normal subgroup N of $\pi_1(X)$, we consider the following conditions:

(P_k). The group N acts trivially on $\pi_i(X)$ for $i \leqq k$.

(H_k). The group N acts trivially on $H_i(\tilde{X})$ for $i \leqq k$.

(5.6) PROPOSITION. *For all natural numbers* k *we have that* P_k *implies* H_k *and* H_k *implies* P_{k-1}. *In particular,* H_∞ *and* P_∞ *are equivalent.*

Proof. Consider the following commutative diagram where the rows and columns are fibrations.

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & A\tilde{X}_N \arrow[r] \arrow[d] & A(BN) \arrow[d] \\
\tilde{X} \arrow[r] \arrow[d] & \tilde{X}_N \arrow[r] \arrow[d] & BN \arrow[d] \\
F \arrow[r] & \tilde{X}_N^+ \arrow[r] & BN^+
\end{tikzcd}$$

By (5.4) condition P_k implies that $\pi_1(A\tilde{X}_N)$ acts trivially on $\pi_i(A\tilde{X}_N)$ for $i \leqq k$. Since $A\tilde{X}_N$ and $A(BN)$ are both acyclic and $\pi_1(A\tilde{X}_N) \xrightarrow{\sim} \pi_1(A(BN))$ is an isomorphism (by (4.4)), we deduce using (5.1) that $\pi_i(A\tilde{X}_N) \to \pi_i(A(BN))$ is an isomorphism for $i \leqq k$. Thus $\pi_i(T) = 0$ for $i \leqq k-1$ and $\pi_k(T)$ is a trivial module $\pi_1(A(BN))$ since it is a quotient of $\pi_{k+1}(A(BN))$. On the other hand, we have $H_0(\pi_1(A(BN)), \pi_k(T)) = 0$ since $\pi_k(T) = H_k(T)$ and thus $H_*(A\tilde{X}_N) \to H_*(A(BN))$ is an isomorphism. Therefore, $\pi_k(T) = 0$ and $H_i(\tilde{X}) \to H_i(F)$ is an isomorphism for $i \leqq k$. Hence P_k implies H_k since N acts trivially on $H_*(F)$.

Next, assume H_k holds. Then $H_i(\tilde{X}) \to H_i(F)$ is an isomorphism for $i \leqq k$ by the comparision theorem for spectral sequences of fibrations with trivial actions. Since $\pi_1(F)$ is abelian, $\pi_1(F) = 0$ and $\pi_i(T) = 0$ for $i \leqq k-1$. Hence $\pi_1(A\tilde{X}_N)$ acts trivially on $\pi_i(A\tilde{X}_N)$ for $i \leqq k-1$. Using (5.4), we deduce P_{k-1} and the proposition.

(5.7) THEOREM. *Let $f: X \to Y$ be an acyclic map between CW-spaces which is k-simple for all $k \geqq 2$ with $N = \ker \pi_1(f)$. Then the following is a fiber sequence*

$$\tilde{X} \to Y \xrightarrow{\alpha'} [B\pi_1(X)]_N^+$$

where α' is induced by $\alpha: X \to B\pi_1(X)$ as in (3.1) and $\pi_1(\alpha)$ is the identity.

Proof. As in the previous proposition, we have a diagram of fibrations

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & AX_N \arrow[r] \arrow[d] & A(BN) \arrow[d] \\
\tilde{X} \arrow[r] \arrow[d] & \tilde{X}_N \arrow[r] \arrow[d] & BN \arrow[d] \\
F \arrow[r] \arrow[dr] & \tilde{Y} \arrow[r] \arrow[d] & BN^+ \arrow[d] \\
 & Y \arrow[r] & {[B\pi_1(X)]_N^+}
\end{tikzcd}$$

We prove $\tilde{X} \to F$ is a homotopy equivalence with the same argument used in (5.6) to show P_k implies H_k. Since F is also the fibre of $X_N^+ \to [B\pi_1(X)]_N^+$ we have proved the theorem.

(5.8) *Remark.* Using (5.1), we see that for an acyclic map $f: X \to Y$ which is k-simple for all $k \geqq 2$, the homotopy groups $\pi_*(Y)$ can be computed in terms of $\pi_*(X)$ and $\pi_*(B\pi_1(X)_N^+) \cong \pi_*(BN)^+$ for $i \geqq 2$. Some computations of $\pi_*(BN^+)$ for a certain perfect group N can be found for instance in [H, Chapter 7].

§ 6. Acyclic maps into a given space

In this section we study acyclic maps $f: X \to Y$ into a fixed space Y. Two such map $f: X \to Y$ and $f': X' \to Y$ are called equivalent provided there is a homotopy equivalence $h: X \to X'$ with $f \simeq f'h$. Let $AC(Y)$ denote the class of equivalence classes of acyclic $f: X \to Y$ over Y where X and Y are CW-spaces.

(6.1) Definition. *An extension data over a space* Y *is a triple* (Φ, i, Φ) *where*

(a) Φ *is an extension* $1 \to N \to G \to \pi_1(Y) \to 1$ *with* N *perfect,*

(b) $i: BG \to BG_N^+$ *is an acyclic map with* $\ker(\pi_1(i)) = N$ *(whose equivalence class is well defined by* (3.5)), *and*

(c) $\phi: Y \to BG_N^+$ *is a 2-connected map.*

Two triples of extension data (Φ, i, ϕ) and (Φ', i', ϕ') are called equivalent provided there exists an isomorphism $g: G \to G'$ making the following diagrams commutative (up to homotopy for the second one).

$$\begin{tikzcd} G \arrow[rr, "g"] \arrow[dr] & & G' \arrow[dl] \\ & \pi_1(Y) & \end{tikzcd} \qquad \begin{tikzcd} BG \arrow[rr, "Bg"] \arrow[d, "i"'] & & BG' \arrow[d, "i'"] \\ BG_N^+ \arrow[rr, "Bg+"] & & B(G')_N^+ \\ & Y \arrow[ul, "\phi"] \arrow[ur, "\phi'"'] & \end{tikzcd}$$