ACYCLIC MAPS

ACYCLIC MAPS

by Jean-Claude Hausmann and Dale Husemoller

In [K] M. Kervaire shows in the proof of Theorem 3 that by adding 2-cells and 3-cells to a homology sphere one obtains a homotopy sphere. This is a special case of a general procedure for killing part of the fundamental group of a space without changing its homology by adding 2-cells and 3-cells.

This technique was rediscovered and developed extensively by Quillen [Q] under the name "plus construction". For a space X and a perfect normal subgroup N of $\pi_1(X)$ there is a map $f: X \to X_N^+$ with $\pi_1(f): \pi_1(X) \to \pi_1(X_N^+)$ an epimorphism with kernel N and $H_*(f): H_*(X, f^*L) \to H_*(X_N^+, L)$ an isomorphism for any local coefficient system L on X_N^+ (or equivalently $\pi_1(X_N^+)$ — module L). The space X_N^+ can be obtained from X by adding 2-cells and 3-cells and is unique up to homotopy type. The homotopy fibre of $X \to X_N^+$ is acyclic, and following the terminology of algebraic geometers, the map is called acyclic. Twisted homology equivalence would also be suitable terminology for acyclic map.

The plus construction has already played an important role in many areas, for example, algebraic K-theory ([Q], [L]), stable homotopy theory [P], classification of manifolds [H], structures on manifolds ([H-V], [M-S]) and localization theory [V]. Further, Kan and Thurston ([K-T] see also [B-D-H]) have shown that for any CW-complex X there is a group G with a normal perfect subgroup N such that X is homotopy equivalent to $K(G, 1)_N^+$.

The aim of this paper is to give a general exposition of the basic properties of acyclic maps following the broad outlines of the subject given by Quillen. Some of these results in special cases are already in the literature, see for instance [A], [L] and [W]. The background needed for this paper consists only of standard material on homotopy theory: fibrations and cofibrations, Whitehead's criterion for a map to be a homotopy equivalence, homotopy sequence of a fibration, and the Serre spectral sequence. On the other hand, we do not use obstruction theory or semisimplicial techniques.

The paper is organized as follows:

§ 1 and 2. Various definitions of acyclic maps are given and the basic properties are worked out.

§ 3. Acyclic maps, up to homotopy equivalence, defined on a given space X are in bijective correspondence with the perfect normal subgroups of $\pi_1(X)$. Functorial aspects of acyclic maps are discussed.

§ 4. Dror's functor [D1] is shown to be the homotopy fibre of the plus construction and the plus construction is the homotopy cofiber of the Dror map. A strongly functorial plus construction can be deduced from this.

§ 5. We study acyclic maps $f: X \to Y$ with trivial action of $\ker \pi_1(f)$ on $\pi_*(X)$. In this situation there is a good relation between $\pi_*(X)$ and $\pi_*(Y)$ which is not the case for a general acyclic map.

§ 6. We classify acyclic maps $f: X \to Y$ into a fixed space Y for which $\ker \pi_1(f)$ acts trivially on $\pi_1(X)$ for $i > 2$.

For a general acyclic map there is a Dror-Postnikov decomposition of f generalizing the results of Dror [D1, D2]. It is an interesting problem to classify the n^{th}-stages of this decomposition in terms of invariants like those in Dror [D1, D2]. [1])

The authors thank M. Zisman for useful comments on the first version of this paper.

TABLE OF CONTENTS

[1]) Results in this direction have been recently obtained by W. Meier.

§ 1. Acyclic maps and homotopy equivalences

We will use the terminology CW-space for a space having the homotopy type of a CW-complex. The category of CW-spaces is the largest category of spaces for which the Whitehead characterization of homotopy equivalences holds.

(1.1) Definition. *A space X is acyclic provided the integral reduced homology* $\tilde{H}_*(X) = 0$.

In particular, an acyclic space X is path connected, its fundamental group $\pi_1(X)$ is perfect, i.e. $\pi_1(X)$ is equal to its commutator subgroup, and for any constant coefficient module L it follows that $\tilde{H}_*(X, L) = 0$. Recall that a local coefficient system L on X is a module over $\pi_1(X)$ and that

$$H_*(X, L) = H_*(C_*(\tilde{X}) \otimes_{\mathbf{Z}\pi_1(X)} L)$$

where $C_*(\tilde{X})$ is the chain complex over $\mathbf{Z}$ viewed as a $\mathbf{Z}\pi_1(Y)$-module. In general, $\tilde{H}(X, L) \neq 0$ for an acyclic space and a local coefficient system L.

(1.2) Definition/Proposition. *A map $f: X \to Y$ between path connected spaces is acyclic provided any of the following equivalent conditions hold:*

(a) *The homotopy fibre F of $f: X \to Y$ is an acyclic space.*

(b) *For any local coefficient system L on Y the induced morphism*

$$f_*: H_*(X, f^*L) \to H_*(Y, L)$$

is an isomorphism where f^*L is the induced local system on X.

(c) *The induced morphism*

$$f_*: H_*(X, f^*\mathbf{Z}\pi_1(Y)) \to H_*(Y, \mathbf{Z}\pi_1(Y))$$

is an isomorphism.

(d) *For the universal covering $\tilde{Y} \to Y$ of Y the map $X \times_Y \tilde{Y} \to \tilde{Y}$ defined by f induces an isomorphism*

$$H_*(X \times_Y \tilde{Y}) \to H_*(\tilde{Y}).$$

Proof. For (a) implies (b), we use the Serre spectral sequence for the fibration $F \xrightarrow{i} X \xrightarrow{f} Y$ where

$$E^2 = H_* \big(Y, H_* (F, i^* f^* L)\big) \Rightarrow H_* (X, f^* L).$$

Since $i^* f^* L$ is trivial on F, statement (a) gives $\tilde{H}_* (F, i^* f^* L) = 0$ and the edge morphism $H_* (X, f^* L) \to H_* (Y, L) = E^2_{*,0}$, which is induced by f, is an isomorphism.

Clearly (b) implies (c), which is a special case of (b), and for (c) implies (d) we use the following morphism of fibrations

$$\begin{array}{ccc} & \pi = \pi_1 (Y) & \\ & \swarrow \quad \searrow & \\ X \times_Y \tilde{Y} & \longrightarrow & \tilde{Y} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

This induces a morphism of the Serre spectral sequences which on the E^2-level is the given isomorphism from (c)

$$E^2 = H_* \big(X, f^* \mathbf{Z}\pi_1 (Y)\big) \to H_* \big(Y, \mathbf{Z}\pi_1 (Y)\big) = E^2.$$

Hence by the spectral mapping theorem $H_* (X \times_Y \tilde{Y}) \to H_* (\tilde{Y})$ is an isomorphism.

For (d) implies (a), note that $F \to X \times_Y \tilde{Y}$ is the fibre of $X \times_Y \tilde{Y} \to \tilde{Y}$. Since $H_* (X \times_Y \tilde{Y}) \to H_* (\tilde{Y})$ is an isomorphism on the horizontal edge of the spectral sequence, we see $\tilde{H}_0 (F) = 0$. Moreover, assuming inductively that $\tilde{H}_i (F) = 0$ for $i < n$, we deduce that $\tilde{H}_n (F) \doteq 0$ by looking at the spectral sequence terms $E^r_{0,n}$ which is $H_n (F)$ for $r = 2$ and zero for $r > n + 1$. This completes the proof the equivalence of (a), (b), (c), and (d).

(1.3) Proposition. *If $f : X \to Y$ is an acyclic map, then $f_* : \pi_1 (X) \to \pi_1 (Y)$ is an epimorphism with kernel a perfect normal subgroup.*

Proof. Since the fibre F of f is connected, the induced homomorphism f_* is an epimorphism, and since $\pi_1 (F)$ is perfect, $\ker (f_*) = \operatorname{im} \big(\pi_1 (F) \to \pi_1 (X)\big)$ is perfect.

(1.4) PROPOSITION. *Let* $f : X \to Y$ *be a map between path connected spaces. Then* $\pi_i(f) : \pi_i(X) \to \pi_i(Y)$ *is an isomorphism for all* $i \geqq 0$ *if and only if* f *is acyclic and* $\pi_1(f)$ *is an isomorphism.*

Proof. Let $F \to X$ be the homotopy fibre of f. The second conditions say that $\pi_1(F)$ is perfect and abelian respectively. Thus $\pi_1(F) = 0$ and on simply connected spaces F the homotopy $\pi_i(F) = 0$ if and only if the homology $\tilde{H}_i(F) = 0$. The proposition follows now from an application of the homotopy exact sequence.

(1.5) COROLLARY. *A map* $f : X \to Y$ *between path connected CW-spaces is a homotopy equivalence if and only if* f *is acyclic and* $\pi_1(f)$ *is an isomorphism.*

This is an immediate application of the Whitehead criterion for homotopy equivalence applied to (1.4).

In section 3 we will see that the subgroups $\ker(\pi_1(f))$ classify acyclic maps $f : X \to Y$ from X.

(1.6) *Remark.* Cohomology with local coefficients can be used to characterize acyclic maps. As with homology

$$H^*(X, L) = H^*(\mathrm{Hom}_{Z\pi_1(X)}(C^*(\tilde{X}), L))$$

defines cohomology with local coefficients. Then a map $f : X \to Y$ between path connected spaces is acyclic if and only if $f^* : H^*(Y, L) \to H^*(X, f^*L)$ is an isomorphism for each local coefficient system L on Y. The direct implication is checked exactly as (a) implies (b) using cohomology in (1.2). Conversely we show that $X \times_Y \tilde{Y} \to \tilde{Y}$ defined by f induces an isomorphism $H^*(\tilde{Y}) \to H^*(X \times_Y \tilde{Y})$. This is done as (c) implies (d) in (1.2) and as in (d) implies (a) in (1.2) we have $\tilde{H}^*(F) = 0$. Using the universal coefficient theorem, we deduce that $\tilde{H}_*(F) = 0$ and F is acyclic.

The cohomology characterization of acyclic maps is useful in obstruction theory.

(1.7) *Remark.* Let $f : X \to Y$ be an acyclic map and $\bar{Y}$ a connected covering of Y. Then the induced map $\bar{f} : X \times_Y \bar{Y} \to \bar{Y}$ is also acyclic. This follows directly from (1.2, (d)) or from the fact that f and $\bar{f}$ have the same fibre. When $\bar{Y}$ is the universal covering of Y, the space $X \times_Y \bar{Y} = \tilde{X}_N$ is the covering of X with fundamental group $N = \ker(\pi_1(f))$.

§ 2. Induced and coinduced acyclic maps

(2.1) Proposition. *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be two maps. If* f *and* g *are acyclic, then* gf *is acyclic. If* f *and* gf *are acyclic, then* g *is acyclic.*

Proof. Consider a local system L on Z, and using g^*L on Y $f^*g^*L = (gf)^*L$ on X, we apply (1.2) (b) to obtain the proposition.

(2.2) Proposition. *Consider the following cartesian square where either* f *or* g *is a fibration.*

$$\begin{array}{ccc} Y' \times_Y Y & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

If f *is acyclic, then* f' *is acyclic.*

Proof. Since either f or g is a fibration, we can change the other to be a fibration, if necessary, without changing the homotopy type of any of the four spaces. Now the homotopy fibre F of f is the actual fiber and F is also the homotopy fibre of f'. Now apply (1.2) (a).

(2.3) Proposition. *Consider the following cocartesian square where either* f *or* g *is a cofibration.*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ X' & \xrightarrow{f'} & X' \cup_X Y = Y' \end{array}$$

If f *is acyclic, then* f' *is acyclic.*

Proof. Since either f or g is a cofibration, we can change the other to be a cofibration, if necessary, without changing the homotopy type of any of the four spaces. Hence each map is an injection, and for a local coefficient system L on Y', we have two long exact sequences in homology

$$\begin{array}{ccccccc} \longrightarrow H_q(X, f^*g'^*L) & \xrightarrow[f_*]{} & H_q(Y, f'^*L) & \longrightarrow & H_q(Y, X; f'^*L) & \longrightarrow & \dots \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g'_*} & & \downarrow{\scriptstyle (g,g')_*} & & \\ \longrightarrow H_q(X', g'^*L) & \xrightarrow{f'_*} & H_q(Y', L) & \longrightarrow & H_q(Y', X'; L) & \longrightarrow & \dots . \end{array}$$

By hypothesis (1.2) (b) the morphism f_* is an isomorphism and thus $H_*(Y, X; f'^*L) = 0$. By excision $(g, g')_*$ is an isomorphism and thus $H_*(Y', X'; L) = 0$. Hence f'_* is an isomorphism and criterion (1.2) (b) is satisfied for f' to be an acyclic map which proves the proposition.

The previous proposition concerning acyclic maps in a cofibration will be the basic tool for most of the results which follow in sections 2 and 3. It was pointed out to us by Quillen.

(2.4) PROPOSITION. *Consider the following diagram of CW-spaces.*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, "g'"] \\ X' \arrow[r, "f'"] & Y' \end{tikzcd}$$

If g and g' are acyclic, and if $\pi_1(f)$ and $\pi_1(f')$ are isomorphisms then the diagram is cocartesian up to homotopy equivalence.

Proof. First replace f and g by equivalent cofibrations and form $h : X' \cup_X Y \to Y'$. The map $g'' : Y \to X' \cup_X Y$ is an acyclic map by (2.3) and $g' = hg''$. Thus h is acyclic by (2.1).

Since $\pi_1(f)$ is an isomorphism, it follows that $f'' : X' \to X' \cup_X Y$ has the property that $\pi_1(f'')$ is an isomorphism by the van Kampen theorem and $f' = hf''$. Thus $\pi_1(h)$ is an isomorphism. Now apply (1.5) to see that h is a homotopy equivalence. This proves the proposition.

(2.5) THEOREM. *Let $f : X \to Y$ be an acyclic map between CW-spaces with homotopy fibre $g : F \to X$. Then f is the homotopy cofibre of g.*

Proof. Let CF be the cone over F. The homotopy cofibre C of $g : F \to X$ is homotopy equivalent to $CF \cup_F X$ and we have the cocartesian square

$$\begin{tikzcd} F \arrow[r, "g"] \arrow[d] & X \arrow[r, "f"] \arrow[d, "v"'] & Y \\ CF \arrow[r] & C \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

Since $fg \simeq *$, it follows that we have a map $h : C \to Y$ such that $f \simeq hv$. Since f is acyclic, the map $F \to CF$ is acyclic and, by (2.3) v is acyclic. One deduces then, by (2.1) that h is acyclic. As $\pi_1(h)$ is onto (1.3), one has:

$$\ker(\pi_1(h)) = v(\ker \pi_1(f)) = v(\operatorname{Im} \pi_1(g)) = 1$$

So $\pi_1(h)$ is injective and, by (1.3) and (1.5), h is a homotopy equivalence.

(2.6) THEOREM. *Let $f: X \to Y$ be an acyclic map between CW-spaces and let $h_1, h_2 : Y \to Z$ be two maps. If $h_1 f \simeq h_2 f$, then it follows that $h_1 \simeq h_2$.*

Proof. By (2.5) we have cofibre sequence

$$F \xrightarrow{g} X \xrightarrow{f} Y \longrightarrow \Delta F$$

where ΔF is the reduced suspension of the acyclic space F. Since ΔF is simply connected and $\tilde{H}_*(\Delta F) = 0$, it is contractible, and the group $[\Delta F, Z]$ in the Puppe sequence is zero.

In general, the group $[\Delta F, Z]$ acts transitively on the fibres of the function $[Y, Z] \to [X, Z]$, so that in this case, $[Y, Z] \to [X, Z]$ is injective. This proves the theorem.

§ 3. CLASSIFICATION OF ACYCLIC MAP FROM A GIVEN SPACE

Let X be a path connected space. To each acyclic map $f: X \to Y$, we assign the kernel of $\pi_1(f) : \pi_1(X) \to \pi_1(Y)$ which is a perfect normal subgroup of $\pi_1(X)$ by (1.3). The object of this section is to show that this map from isomorphism classes of acyclic maps defined on X to perfect normal subgroups of $\pi_1(X)$ is a bijection.

(3.1) PROPOSITION. *Let $f: X \to Y$ and $f' : X \to Y'$ be two maps between CW-spaces such that f is acyclic. There exists a map $h: Y \to Y'$ with $hf \simeq f'$ if and only if* $\ker \pi_1(f) \subset \ker \pi_1(f')$, *and such an h is unique up to homotopy. In addition, if f' is acyclic, then h is acyclic, and h is a homotopy equivalence if and only if* $\ker \pi_1(f) = \ker \pi_1(f')$.

Proof. If h exists, then $\pi_1(f') = \pi_1(h) \circ \pi_1(f)$ and we have $\ker \pi_1(f) \subset \ker \pi_1(f')$. Conversely, we can suppose f is a cofibration and form the cocartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow{g} & Y' \cup_X Y \end{array}$$

where g is an acyclic map by (2.3). Now calculate

$$\pi_1(g) : \pi_1(Y') \to \pi_1(Y' \cup_X Y) = \pi_1(Y')_{*\pi_1(X)} \pi_1(Y)$$

by the vanKampen theorem. Since $\ker(\pi_1(f)) \subset \ker(\pi_1(f'))$, it follows that $\pi_1(g)$ is an isomorphism, and by (1.5) the map g is a homotopy equivalence. Let $g^* : Y' \cup_X Y \to Y'$ be a homotopy inverse of g. Then $h = g^* g' : Y \to Y'$ is the desired map with $hf = f'$. The map h is unique by (2.6).

The map h is acyclic by (2.1). Since $\pi_1(h)$ is an isomorphism if and only if $\ker \pi_1(f) = \ker \pi_1(f')$, the last statement follows from (1.5), and this proves the proposition.

(3.2) COROLLARY. *Let A be an acyclic CW-space. A map $f : A \to Z$ is null homotopic if and only if $\pi_1(f)$ is zero.*

Proof. We apply (3.1) to the acyclic map $A \to *$, and when $\pi_1(f)$ is zero, f factors $A \to * \to Z$ up to homotopy.

(3.3) PROPOSITION. *Let X be a path connected space, and let N be a perfect normal subgroup of $\pi_1(X)$. Then there exists an acyclic map $f : X \to Y$ with $\ker \pi_1(f) = N$. If X has the homotopy type of a CW-complex, then so does Y.*

Proof. First, we do the case where $N = \pi_1(X)$ is perfect. Let T_1 be a wedge of circles indexed by generators of N and $u : T_1 \to X$ a map such that $\pi_1(u)$ is surjective. We form the cofibre $v : X \to X^*$ of u, i.e. attach a 2-cell for each circle. By the van Kampen theorem it follows that $\pi_1(X^*) = 0$ and the homology exact sequence of the cofibration takes the form

$$0 \to H_q(X) \to H_q(X^*) \to 0 \quad \text{for} \quad q \geqq 3$$

and

$$0 \to H_2(X) \to H_2(X^*) \xrightarrow{\partial} H_1(T) \to H_1(X) = 0\,.$$

Since $H_1(T_1)$ is free abelian, it lifts back into $H_2(X^*)$, and since $\pi_2(X^*) \to H_2(X^*)$ is an isomorphism by the Hurewicz theorem, there is a wedge T_2 of two spheres and a map $w : T_2 \to X^*$ such that $\partial H_2(w) : H_2(T_2) \to H_1(T_1)$ is an isomorphism. Let $X^* \to Y$ denote the cofibre of $w : T_2 \to X^*$, and let $f : X \to Y$ denote the composite $X \to X^* \to Y$. The cofibration homology exact sequence takes the form

$$0 \to H_q(X^*) \to H_q(Y) \to 0 \quad \text{for} \quad q \geqq 4,\ q = 1$$

and

$$\begin{tikzcd}
 & & & H_1(T_1) & & \\
0 \arrow[r] & H_3(X^*) \arrow[r] & H_3(Y) \arrow[r] & H_2(T_2) \arrow[r] \arrow[u, "\wr"] & H_2(X^*) \arrow[r] \arrow[u] & H_2(Y) \arrow[r] & 0 \\
 & & & & H_2(X) \arrow[u] \arrow[ur, "H_2(f)"'] & &
\end{tikzcd}$$

From this, a quick examination of the homology sequence reveals that $H_*(f) : H_*(X) \to H_*(Y)$ is an isomorphism. Since Y is simply connected, every local system on Y is trivial, and $H_*(f)$ is an isomorphism for all coefficients. By (1.2) (c) the map f is an acyclic map with the desired properties.

For a general perfect normal subgroup $N \subset \pi_1(X)$, let $g : \tilde{X}_N \to X$ be the covering corresponding to N, that is, $\operatorname{im}(\pi_1(g)) = N$, and let $f_0 : \tilde{X}_N \to Y_0$ be the acyclic map with $\ker(\pi_1(f_0)) = N = \pi_1(X_0)$ constructed in the previous paragraph. Change it up to homotopy into a cofibration, and form the following cocartesian diagram.

$$\begin{tikzcd}
X_N \arrow[r, "f_0"] \arrow[d, "g"'] & Y_0 \arrow[d] \\
X \arrow[r, "f"] & X \cup_{\tilde{X}_N} Y_0
\end{tikzcd}$$

By (2.3) the map f is acyclic. In order to calculate, we determine, using the van Kampen theorem, the group $\pi_1(X \cup_{\tilde{X}_N} Y_0) \cong \pi_1(X) *_{\pi_1(\tilde{X}_N)} \pi_1(Y_0)$. Since $\pi_1(Y_0) = 1$, it follows that $\pi_1(X \cup_{\tilde{X}_N} Y_0) \cong \pi_1(X)/\pi_1(\tilde{X}_N) = \pi_1(X)/N$. The morphism $\pi_1(f)$ is thus an epimorphism with kernel N. This proves the proposition.

(3.4) Definition. *Two acyclic maps $f : X \to Y$ and $f' : X \to Y'$ defined on X are equivalent provided there exists a homotopy equivalence $h : Y \to Y'$ with $hf \simeq f'$.*

Putting together propositions (3.1) and (3.3), we obtain the classification theorem.

(3.5) Theorem. *Let X be a path connected space with the homotopy type of a CW-complex. The function which assigns to an acyclic map*

$f: X \to Y$ the subgroup $\ker(\pi_1(f))$ *of* $\pi_1(X)$ *is a bijection from the set of equivalence classes of acyclic maps on* X *to the set of normal perfect subgroups of* $\pi_1(X)$.

Proof. The function is injective by (3.1) and surjective by (3.3).

In view of this theorem we see that the theory of acyclic maps is similar to the theory of covering spaces, in that, they are classified by certain subgroups of the fundamental group. By way of comparison, for covering maps $f: Y \to X$ over X, the group $\operatorname{im} \pi_1(f)$ is given, and $\pi_q(f)$ is an isomorphism for $q \geqq 2$. The homology of Y is related to that of X by a spectral sequence. For acyclic maps $f: X \to Y$ from X, the group $\ker \pi_1(f)$ classifies the objects. It is perfect and normal, and $f_*: H_*(X, f^{-1}L) \to H_*(Y, L)$ is an isomorphism for any local system L on Y. The higher homotopy groups of X and Y are not easily related in general (but see § 5).

(3.6) *Notations.* Let $\mathscr{P}$ be the category whose objects are pairs (X, N) where X is a pointed CW-space and N is a perfect normal subgroup of $\pi_1(X)$ and whose morphisms $f: (X, N) \to (X', N')$ are homotopy classes of maps $f: X \to X'$ with $\pi_1(f)(N) \subset N'$. Let (CW) be the category of pointed CW-spaces and homotopy classes of maps. We have two natural functors $\alpha: (CW) \to \mathscr{P}$ and $\beta: \mathscr{P} \to (CW)$ with $\beta\alpha$ the identity where $\beta(X, N) = X$ and $\alpha(X) = (X, N_0)$ for N_0 the maximal normal perfect subgroup of $\pi_1(X)$.

(3.7) THEOREM. *For* (X, N) *in* $\mathscr{P}$ *choose* $f: X \to X_N^+$ *an acyclic map with* $\ker(\pi_1(f)) = N$. *Then there is a functor* $\sigma: \mathscr{P} \to (CW)$ *and a morphism of functors* $f: \beta \to \sigma$ *such that* $\sigma(X, N) = X_N^+$ *and* $f(X, N) = f$.

Proof. This immediate from the universal property (3.1).

(3.8) *Remark.* The space X_N^+ is unique up to homotopy equivalence. The acyclic map $X \to X_N^+$ we had to choose is defined up to the composition with a homotopy equivalence of X_N^+. However, we shall give in Section 4 a stronger functorial way to construct acyclic maps without any choice, for instance the functorial plus construction $f: X \to X^+$ where $X^+ = \sigma\alpha(X)$

§ 4. The homotopy fibre of the plus construction

(4.1) Theorem. *Let* $u : AX \to X$ *be the fibre of* $X \to X^+$ *for a* CW*-space* X*. Then for any map* $f : W \to X$ *from an acyclic* CW*-space* W *into* X*, there is a map* $f' : W \to AX$ *with* $uf' \simeq f$ *and* f *is unique up to homotopy.*

Proof. We have the following diagram where the lower row is a fibre sequence.

$$\begin{tikzcd} & & W \arrow[dl, dashed, "f'"'] \arrow[d, "f"] & \\ \Omega(X^+) \arrow[r] & AX \arrow[r, "u"] & X \arrow[r, "\theta"] & X^+ \end{tikzcd}$$

Since $\pi_1(W)$ is perfect and $\pi_1(X^+)$ contains no nonzero perfect subgroups, $\pi_1(\theta f)$ is zero and by (3.2) the map θf is null homotopic. Then there is a map $f' : W \to AX$ with $uf' \simeq f$. Two factorizations f' of f differ by the action of a map $W \to \Omega(X^+)$. Since again $\pi_1(W)$ is perfect and $\pi_1(\Omega(X^+))$ abelian, π_1 of this map is zero so by (3.2) the map is null homotopic. Hence f' is unique, and this proves the theorem.

(4.2) *Remark.* Dror introduced the map $AX \to X$ having the universal property given in the previous theorem and proved for each CW-space X the map $AX \to X$ existed. He used a Posnikov tower construction starting with the covering of X corresponding to the maximal perfect normal subgroup of $\pi_1(X)$. By (2.5) we see that we can recover $X \to X^+$ as the cofibre of $AX \to X$.

All the properties of AX listed in [D1, Theorem 2.1] can be shown using the fact that AX is the fibre of $X \to X^+$. For instance we will in (5.4) give a sharper version of [D1, Theorem 2.1 (iv)].

(4.3) *Remark.* The Posnikov tower construction for $AX \to X$, when done in the category of simplicial sets, is functorial for maps of simplicial sets. For CW-spaces we obtain a functorial $AX \to X$ for maps using the geometric realization of simplicial sets. Since we can choose $X \to X_N^+$ to be the cofibre of $A(\tilde{X}_N) \to X$, we obtain a sharper version of the functoriality in (3.7) and (3.8), namely on the level of spaces and maps.

(4.4) *Remark.* The group $\tilde{N} = \pi_1(A\tilde{X}_N)$ is a central extension of N (see the appendix) and, as $A\tilde{X}_N$ is acyclic, satisfies $H_1(\tilde{N}) = H_2(\tilde{N}) = 0$. Therefore $\tilde{N}$ is the universal central extension of N (see [K2]), namely one has the exact sequence $0 \to H_2(N) \to \tilde{N} \to N \to 1$. Therefore, if $f : X \to X'$ is a map such that $\pi_1(f)$ sends the perfect normal subgroup N of $\pi_1(X)$ isomorphically onto a normal subgroup N' of $\pi_1(X')$, then the induced map $Af : A\tilde{X}_N \to A\tilde{X}'_N$ induces an isomorphism on the fundamental groups.

§ 5. k-SIMPLE ACYCLIC MAPS

In this section we study acyclic maps having simplicity properties. The first proposition generalizes some results of Dror [D1, Lemma 3.4].

(5.1) PROPOSITION. *Let $f : X \to Y$ be a map of path connected spaces with $\pi_1(f)$ an isomorphism, and let N be a perfect normal subgroup of $\pi_1(X) = \pi$. If f induces an isomorphism $H_*(X, \mathbf{Z}[\pi/N]) \xrightarrow{\sim} H_*(Y, \mathbf{Z}[\pi/N])$ and an isomorphism $\pi_i(X) \xrightarrow{\sim} \pi_i(Y)$ for $i \leqq k-1$, then*

(1) $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ *is an epimorphism when N acts trivially on $\pi_k(Y)$, and*

(2) $\pi_k(f) : \pi_k(X) \to \pi_k(Y)$ *is an isomorphism when N acts trivially on $\pi_k(X)$ and $\pi_k(Y)$.*

Proof. Let $F \to \tilde{X}_N$ be the homotopy fibre of the covering map $\tilde{f} : \tilde{X}_N \to \tilde{Y}_N$. By hypothesis it follows easily that $\tilde{f}$ induces an isomorphism on integral homology and on $\pi_i(X) \to \pi_i(Y)$ for $i \leqq k-1$. From the Serre spectral sequence we have $H_0(\tilde{Y}_N, H_{k-1}(F)) = H_0(N, H_{k-1}(F)) = 0$. Since $H_{k-1}(F) = \pi_{k-1}(F)$ is a quotient of $\pi_k(Y)$ on which the perfect group N acts trivially, it follows that $\pi_{k-1}(F) = 0$, which proves (1).

Under the hypothesis of (2) we have $\pi_i(F) = 0$ for $i < k$ and $H_0(\tilde{Y}_N, H_k(F)) = H_0(N, \pi_k(F)) = 0$. Since N acts trivially on $\pi_k(X)$ the induced morphism $\pi_k(F) \to \pi_k(X)$ must be trivial, which proves the proposition.

The following lemma, proved in [D2, Lemma 2.6], follows easily from the homology exact sequence.

$$H_1(G, M'') \to H_0(G, M') \to H_0(G, M) \to H_0(G, M'') \to 0$$

(5.2) LEMMA. *Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of* $\mathbf{Z}[G]$*-modules where* G *is a perfect group. Then* M' *and* M'' *are trivial* G*-modules if and only if* M *is a trivial* G*-module.*

(5.3) DEFINITION. *A space* X *is k-simple provided* $\pi_1(X)$ *acts trivially on* $\pi_k(X)$. *A map* $f: X \to Y$ *is k-simple provided* $\ker \pi_1(f) \subset \pi_1(X)$ *acts trivially on* $\pi_k(X)$.

(5.4) PROPOSITION. *Let* $f: X \to Y$ *be a map with homotopy fibre* A *where* $\pi_1(A)$ *is perfect. Then* f *is k-simple if and only if* A *is k-simple.*

Proof. In the homotopy exact sequence of any fibration

$$\pi_{k+1}(Y) \to \pi_k(A) \to \pi_k(X) \to \pi_k(Y),$$

see the appendix, $\pi_1(A)$ acts trivially on $\operatorname{im}(\pi_{k+1}(Y) \to \pi_k(A)) = M'$. If f is k-simple, then $\operatorname{im}(\pi_1(A)) = \ker(\pi_1(f))$ acts trivially on $\pi_k(X)$. Hence $\pi_1(A)$ acts trivially on $M' \subset \pi_k(A)$ and on the quotient $\pi_k(A)/M'$. By (5.2), it acts trivially on $\pi_k(A)$.

Conversely, $\ker(\pi_1(f))$ acts trivially on $\ker(\pi_k(f)) \subset \pi_k(X)$ and trivially on $\pi_k(Y) \supset \operatorname{im}(\pi_k(f))$. By (5.2), $\ker(\pi_1(f))$ acts trivially on $\pi_1(X)$. This proves the proposition.

(5.5) *Notations.* For a path connected space X and a perfect normal subgroup N of $\pi_1(X)$, we consider the following conditions:

(P_k). The group N acts trivially on $\pi_i(X)$ for $i \leqq k$.

(H_k). The group N acts trivially on $H_i(\tilde{X})$ for $i \leqq k$.

(5.6) PROPOSITION. *For all natural numbers* k *we have that* P_k *implies* H_k *and* H_k *implies* P_{k-1}. *In particular,* H_∞ *and* P_∞ *are equivalent.*

Proof. Consider the following commutative diagram where the rows and columns are fibrations.

$$\begin{array}{ccccc}
T & \longrightarrow & A\tilde{X}_N & \longrightarrow & A(BN) \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{X} & \longrightarrow & \tilde{X}_N & \longrightarrow & BN \\
\downarrow & & \downarrow & & \downarrow \\
F & \longrightarrow & \tilde{X}_N^+ & \longrightarrow & BN^+
\end{array}$$

By (5.4) condition P_k implies that $\pi_1(A\tilde{X}_N)$ acts trivially on $\pi_i(A\tilde{X}_N)$ for $i \leqq k$. Since $A\tilde{X}_N$ and $A(BN)$ are both acyclic and $\pi_1(A\tilde{X}_N) \xrightarrow{\sim} \pi_1(A(BN))$ is an isomorphism (by (4.4)), we deduce using (5.1) that $\pi_i(A\tilde{X}_N) \to \pi_i(A(BN))$ is an isomorphism for $i \leqq k$. Thus $\pi_i(T) = 0$ for $i \leqq k-1$ and $\pi_k(T)$ is a trivial module $\pi_1(A(BN))$ since it is a quotient of $\pi_{k+1}(A(BN))$. On the other hand, we have $H_0(\pi_1(A(BN)), \pi_k(T)) = 0$ since $\pi_k(T) = H_k(T)$ and thus $H_*(A\tilde{X}_N) \to H_*(A(BN))$ is an isomorphism. Therefore, $\pi_k(T) = 0$ and $H_i(\tilde{X}) \to H_i(F)$ is an isomorphism for $i \leqq k$. Hence P_k implies H_k since N acts trivially on $H_*(F)$.

Next, assume H_k holds. Then $H_i(\tilde{X}) \to H_i(F)$ is an isomorphism for $i \leqq k$ by the comparision theorem for spectral sequences of fibrations with trivial actions. Since $\pi_1(F)$ is abelian, $\pi_1(F) = 0$ and $\pi_i(T) = 0$ for $i \leqq k-1$. Hence $\pi_1(A\tilde{X}_N)$ acts trivially on $\pi_i(A\tilde{X}_N)$ for $i \leqq k-1$. Using (5.4), we deduce P_{k-1} and the proposition.

(5.7) THEOREM. *Let $f : X \to Y$ be an acyclic map between CW-spaces which is k-simple for all $k \geqq 2$ with $N = \ker \pi_1(f)$. Then the following is a fiber sequence*

$$\tilde{X} \to Y \xrightarrow{\alpha'} [B\pi_1(X)]_N^+$$

where α' is induced by $\alpha : X \to B\pi_1(X)$ as in (3.1) *and $\pi_1(\alpha)$ is the identity.*

Proof. As in the previous proposition, we have a diagram of fibrations

$$\begin{tikzcd}
T \arrow[r] \arrow[d] & AX_N \arrow[r] \arrow[d] & A(BN) \arrow[d] \\
\tilde{X} \arrow[r] \arrow[d] & \tilde{X}_N \arrow[r] \arrow[d] & BN \arrow[d] \\
F \arrow[r] \arrow[dr] & \tilde{Y} \arrow[r] \arrow[d] & BN^+ \arrow[d] \\
 & Y \arrow[r] & {[B\pi_1(X)]_N^+}
\end{tikzcd}$$

We prove $\tilde{X} \to F$ is a homotopy equivalence with the same argument used in (5.6) to show P_k implies H_k. Since F is also the fibre of $X_N^+ \to [B\pi_1(X)]_N^+$ we have proved the theorem.

(5.8) *Remark.* Using (5.1), we see that for an acyclic map $f: X \to Y$ which is k-simple for all $k \geqq 2$, the homotopy groups $\pi_*(Y)$ can be computed in terms of $\pi_*(X)$ and $\pi_*(B\pi_1(X)_N^+) \cong \pi_*(BN)^+$ for $i \geqq 2$. Some computations of $\pi_*(BN^+)$ for a certain perfect group N can be found for instance in [H, Chapter 7].

§ 6. Acyclic maps into a given space

In this section we study acyclic maps $f: X \to Y$ into a fixed space Y. Two such map $f: X \to Y$ and $f': X' \to Y$ are called equivalent provided there is a homotopy equivalence $h: X \to X'$ with $f \simeq f'h$. Let $AC(Y)$ denote the class of equivalence classes of acyclic $f: X \to Y$ over Y where X and Y are CW-spaces.

(6.1) Definition. *An extension data over a space Y is a triple (Φ, i, Φ) where*

(a) *Φ is an extension $1 \to N \to G \to \pi_1(Y) \to 1$ with N perfect,*

(b) *$i: BG \to BG_N^+$ is an acyclic map with* $\ker(\pi_1(i)) = N$ *(whose equivalence class is well defined by* (3.5)), *and*

(c) *$\phi: Y \to BG_N^+$ is a 2-connected map.*

Two triples of extension data (Φ, i, ϕ) and (Φ', i', ϕ') are called equivalent provided there exists an isomorphism $g: G \to G'$ making the following diagrams commutative (up to homotopy for the second one).

$$\begin{tikzcd} G \arrow[rr, "g"] \arrow[dr] & & G' \arrow[dl] \\ & \pi_1(Y) & \end{tikzcd} \qquad \begin{tikzcd} BG \arrow[rr, "Bg"] \arrow[d, "i"'] & & BG' \arrow[d, "i'"] \\ BG_N^+ \arrow[rr, "Bg+"] & & B(G')_N^+ \\ & Y \arrow[ul, "\phi"] \arrow[ur, "\phi'"'] & \end{tikzcd}$$

where $N' = g(N)$ and Bg^+ is the unique homotopy equivalence determined by g with (3.1).

We denote by $ED(Y)$ the class of equivalence classes of extension data.

(6.2) DEFINITION. *The data map ρ is the function $\rho : AC(Y) \to ED(Y)$ which assigns to an acyclic map $f : X \to Y$ the class $\rho(f) = (\Phi, i, \phi)$ of extension data defined as follows:*

(a) Φ *is the extension* $1 \to \ker \pi_1(f) \to \pi_1(X) \to \pi_1(Y) \to 1$.

(b) (c) *With the well defined $j : X \to BG$ for $G = \pi_1(X)$ we form the cocartesian diagram*

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & BG \arrow[d, "i"] \\ Y \arrow[r, "\phi"] & Y \cup_X BG \end{tikzcd}$$

Since f is acyclic, i is acyclic, and since $\pi_1(j)$ is an isomorphism, $\ker(\pi_1(i)) = N$. Thus $Y \cup_X BG$ is BG_N^+ up to equivalence.

Now we have to check that the map $\phi : Y \to Y \cup_X BG = BG_N^+$ is 2-connected. Since $\pi_1(j)$ is an isomorphism, $\pi_1(\phi)$ is also an isomorphism. The fact that $\pi_2(\phi)$ is surjective comes from the diagram.

$$\begin{tikzcd} \pi_2(Y) \arrow[d, "\pi_2(\phi)"'] & \pi_2(\tilde{Y}) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "\pi_2(\tilde{\phi})"] & H_2(\tilde{Y}) \arrow[d] & H_2(\tilde{X}_N) \arrow[l, "\sim"'] \arrow[dl] \\ \pi_2(BG_N^+) & \pi_2(BN^+) \arrow[l, "\sim"'] \arrow[r, "\sim"] & H_2(N) & \end{tikzcd}$$

The surjectivity on the right is a classical result of Hopf which follows easily from the Serre spectral sequence of the fibration $\tilde{X} \to \tilde{X}_N \to BN$.

Now using (2.5) a simple argument, left to the reader, shows that $\rho : AC(Y) \to ED(Y)$ is well defined.

(6.3) THEOREM. *Let Y be a CW-space. The map $\rho : AC(Y) \to ED(Y)$ surjective and its restriction to the subclass $AC_S(Y)$ of $AC(Y)$ of $f : X \to Y$ which are k-simple for all $k \geqq 2$ is a bijection.*

Proof. To show ρ is surjective, consider extension data (Φ, i, ϕ) and form the cartesian square

$$\begin{tikzcd}
X = Y \times_T BG \arrow[r, "\alpha"] \arrow[d, "f"'] & BG \arrow[d, "i"] \\
Y \arrow[r, "\phi"] & BG_N^+ = T
\end{tikzcd}$$

Now f is acyclic by (2.2), and since its fiber is the same as i, we deduce by (5.2) that f is k-simple for all $k \geqq 2$.

Next, let $\rho(f) = (\Phi_0, i_0, \phi_0)$ and we show this extension data is equivalent to (Φ, i, ϕ). Using the homotopy exact sequences for $X \to Y$ and $BG \to BG_N^+$ and the fact that ϕ is 2-connected, we deduce from the five lemma that $\pi_1(\alpha) : \pi_1(X) \to G$ is an isomorphism. The following diagram shows that (Φ_0, i_0, ϕ_0) is equivalent to (Φ, i, ϕ) and ρ is surjective.

$$\begin{tikzcd}
 & & B\pi_1(X) \arrow[dl, "B\pi_1(\alpha)"'] \arrow[ddd, "i_0"] \\
X \arrow[urr, "j"] \arrow[r, "\alpha"'] \arrow[d, "f"'] & BG \arrow[d, "i"] & \\
Y \arrow[r, "\phi"] \arrow[drr, "\phi_0"'] & BG_N^+ & \\
 & & Y \cup_X B\pi_1(X) \arrow[ul, "B\pi_1(\alpha)^+"']
\end{tikzcd}$$

Now, if $f : X \to Y$ is an acyclic map which is k-simple for all $k \geqq 2$ and with $\rho(f) = (\Phi, i, \phi)$, then we form the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "d"] \arrow[dr, "f"'] & Y \times_T BG \arrow[r] \arrow[d, "f_0"] & BG \arrow[d, "i"] \\
 & Y \arrow[r, "\phi"] & BG_N^+
\end{tikzcd} \qquad (G = \pi_1(X))$$

As we have seen in the proof the surjectivity of ρ, the map f_0 is acyclic and k-simple for $k \geqq 2$. The map d induces an isomorphism on the fundamental groups and on homology with $\mathbf{Z}\,\pi_1(Y)$ twisted coefficients. By (5.3), the map d is a homotopy equivalence. This proves that the acyclic map f is equivalent to the induced map f_0. Thus ρ restricted to $AC_S(U) \to ED(Y)$ is a bijection.

(6.4) *Remark.* This theorem leaves open the question of the fibres of the function.

$$\rho : AC(Y) \to ED(Y).$$

In the next theorem we factor an acyclic map by ones having simplicity properties.

(6.5) *Remark.* In theorem (6.3), if one fixes an extension $\Phi : 1 \to N \to G \to \pi_1(Y) \to 1$, then the same proof permits us to classify acyclic maps $f : X \to Y$ which are k-simple for $k > 2$ together with an identification $d : \pi_1(X) \to G$ such that $\Phi d = \pi_1(f)$. The objects of $ED(Y)$ have to be replaced by couples (i, ϕ) where $i : BG \to BG_N^+$ is as above and $\phi : Y \to BG_N^+$ is 2-connected with the following diagram commuting up to homotopy.

$$\begin{tikzcd} & B\pi_1(Y) \arrow[rr, "B\Phi"] \arrow[dr, "B\Phi^+"] & & BG \arrow[dl, "i"] \\ Y \arrow[ur] \arrow[rr, "\phi"] & & BG_N^+ & \end{tikzcd}$$

This is what is done implicitely in [H, Sections 2 and 4]. Observe that we are dealing here with classes which are sets.

(6.6) Lemma. *Let X be a CW-space and N a perfect normal subgroup of $\pi_1(X)$. Let $X \to P_nX$ denote the nth stage of the Postnikov decomposition of X. Then for all $n \geqq 1$ we have that*

(1) $\pi_j(X_N^+) \to \pi_j((P_nX)_N^+)$ *is an isomorphism for $j \leqq n$ and an epimorphism for $j = n + 1$, and*

(2) $\pi_j(A\tilde{X}_N) \to \pi_j(A(P_n\tilde{X}_N))$ *is an isomorphism for $j \leqq n$ and an epimorphism for $j = n + 1$.*

Proof. Consider the following homotopy commutative diagram of fibre sequences

$$\begin{tikzcd} T \arrow[r] \arrow[d] & A\tilde{X}_N \arrow[r] \arrow[d] & A(P_n\tilde{X}) \arrow[d] \\ F \arrow[r] \arrow[d] & \tilde{X}_N \arrow[r] \arrow[d] & P_n\tilde{X}_N \arrow[d] \\ G \arrow[r] & (\tilde{X}_N)^+ \arrow[r] & (P_n\tilde{X}_N)^+ . \end{tikzcd}$$

Clearly $\pi_i(F) = 0$ for $i \leqq n+1$. The spaces $\tilde{X}_N$ and $P_n\tilde{X}_N$ have the same $(n+1)$-skeleton and the same can be assumed for $\tilde{X}_N^+$ and $(P_n\tilde{X}_N)^+$. Hence $\pi_i(G) = 0$ for $i \leqq n+1$. Now (1) follows because G is the fibre of $X_N^+ \to (P_nX)^+$.

By comparing Serre spectral sequences, we obtain the surjectivity of

$$H_0\big(N, H_{n+1}(F)\big) \to H_0\big(N, H_{n+1}(G)\big) = H_{n+1}(G) = \pi_{n+1}(G)\,.$$

Thus $\pi_j(T) = 0$ for $j \leqq n$ and (2) follows.

(6.7) Theorem. *Let $f: X \to Y$ be a map between CW-spaces. Then there is a factorization*

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & X_2 \arrow[d, "j_1"] \arrow[ddr, "\alpha_2"] & \\
 & X_1 \arrow[d, "j_0"] \arrow[dr, "\alpha_1"] & \\
X \arrow[uur, "\beta_2"] \arrow[ur, "\beta_1"] \arrow[r, "\beta_0"] \arrow[rr, bend right, "f"'] & X_0 \arrow[r, "\mathrm{id}"] & Y
\end{tikzcd}$$

such that β_i is i-connected and α_i is an acyclic map which is k-simple for $k > i$.

Such a decomposition is unique up to a homotopy equivalence.

Proof. The ith stage X_i is defined by the cartesian diagram

$$\begin{tikzcd}
Y \times_T P_i(X) \arrow[r] \arrow[d] & P_iX \arrow[d] \\
Y \arrow[r] & (P_iX)_N^+ = T
\end{tikzcd}$$

where $N = \ker(\pi_1(X) \to \pi_1(Y))$. By (6.6) the map β_i is i-connected since the fiber of the two vertical arrows is $A(P_n\tilde{X})_N$. Now by (5.4) we see that α_i is simple for $k > i$.

For two decompositions (X'_i) and (X''_i) of $f: X \to Y$ satisfying the above conditions, we have $P_iX'_i = P_iX''_i$ and both X'_i and X''_i map into X_i, constructed above, such that the resulting diagrams are homotopy commutative. The connectivity of the β_i and (5.1) shows that these maps are all homotopy equivalences. This proves the theorem.

(6.8) *Remarks.* This theorem (6.7) coincides with the Dror results for Y a point [D1, Theorem 1.3] and $Y = S^n$ [D2]. An interesting problem is to describe the ith stage X_i in terms of invariants of X_{i-1} as in [D1] and [D2]. (See the footnote in the introduction.)

Appendix — Simplicity properties of fibers

In the proof of (5.4) we used the fact that for a fibration $F \to E \xrightarrow{f} B$ the action of $\pi_1(F)$ on $\mathrm{Im}(\partial: \pi_{k+1}(B) \to \pi_k(F))$ is trivial. This assertion does not seem to be in the literature so we include a proof here.

We extend the mapping sequence of the fibration f to $\Omega B \to F \to E \xrightarrow{f} B$ and study F as the total space of a principal fibration with fibre the H-space ΩB. If G is an H-space, then $\pi_1(G)$ acts trivally on $\pi_*(G)$ because the covering transformations $\tilde{G} \to G$ on the universal covering $\tilde{G}$ of G are homotopic to the identity. This is proved by lifting a loop to a path in $\tilde{G}$ and using the H-space structure on $\tilde{G}$ to deform the identity along this path to the covering transformation defined by the homotopy class of the loop. Recall that a principal fibration is induced from $G \to E_G \to B_G$ up to fibre homotopy equivalence.

(A.1) Proposition. *Let* $G \to X \xrightarrow{\pi} Y$ *be a principal fibration with fibre* G *acting on* X. *Then we have:*

(a) $\mathrm{im}(\pi_1(G) \to \pi_1(X))$ *acts trivially on* $\pi_*(X)$, *and*

(b) $\pi_1(X)$ *acts trivially on* $\mathrm{im}(\pi_*(G) \to \pi_*(X))$.

Proof. For (a) we have the following commutative diagram induced by a covering transformation $T: \tilde{G} \to \tilde{G}$.

$$\begin{tikzcd}
 & \tilde{G} \arrow[rr] \arrow[dd] & & \tilde{G} \times \tilde{X} \arrow[rr] \arrow[dd] & & \tilde{X} \arrow[dd] \\
\tilde{G} \arrow[ur, "T"] \arrow[rr] \arrow[dr] & & \tilde{G} \times \tilde{X} \arrow[ur, "T \times 1"] \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[ur, "T'"] \arrow[dr] & \\
 & G \arrow[rr] & & G \times X \arrow[rr] & & X
\end{tikzcd}$$

The covering transformation T defines T', and since T is homotopic to the identity so is T'. This proves (a).

For (b) we use the following commutative diagram where T' is any covering transformation of $\tilde{X}$.

$$\begin{tikzcd}
\tilde{G} \arrow[rr] \arrow[dr] & & \tilde{G} \times \tilde{X} \arrow[rr] \arrow[dd] & & \tilde{X} \arrow[dd] \\
 & \tilde{G} \times \tilde{X} \arrow[ur, "1 \times T'"] \arrow[rr] \arrow[dr] & & \tilde{X} \arrow[ur, "T'"] \arrow[dr] & \\
 & & G \times X \arrow[rr] & & X
\end{tikzcd}$$

Now the inclusion $i : \tilde{G} \to \tilde{X}$ is the composite of the first horizontal row, and $T'i$ and i are homotopic by $i_t(g) = g \cdot \tilde{\alpha}(t)$ where $g \in G$ and α is a lifting of the loop α corresponding to the covering transformation T'. This proves the proposition.

For a general fibration $f : E \to B$ with fibre F the mapping sequence $\Omega B \to F \to E \to B$ allows us to deduce the next proposition from the previous one.

(A.2) Proposition. *Let* $f : E \to B$ *be a fibration with fibre* $F \to E$. *Then we have:*

(a) $\operatorname{im}(\pi_2(B) \to \pi_1(F))$ *acts trivially on* $\pi_*(F)$, *and*

(b) $\pi_1(F)$ *acts trivially on* $\operatorname{im}(\partial : \pi_{i+1}(B) \to \pi_i(F))$.

REFERENCES

[B-D-H] BAUMSLAG, G., E. DYER and A. HELLER. The topology of discrete groups. *(To appear).*

[D 1] DROR, E. Acyclic Spaces. *Topology 11* (1972), pp. 339-348.

[D 2] —— Homology Spheres. *Israel Journal of Math. 15* (1973).

[G] GESTERN, S. M. Higher *K*-theory of rings. Algebraic *K*-theory I. *Springer Lecture Notes 341* (1972), pp. 3-42.

[H] HAUSMANN, J.-C. Manifolds with a given homology and fundamental group. *Comm. Math. Helv. 53* (1978), pp. 113-134.

[H-V] HAUSMANN, J.-C. and P. VOGEL. Reduction of structures on manifolds by semi-s-cobordism. *Topology and algebra. Proceedings of a Colloquium in honor of B. Eckmann. L'Enseignement mathématique*, Université de Genève, 1978, pp. 117-124.

[K] KERVAIRE, M. Smooth homology spheres and their fundamental groups. *Trans AMS* (1969), pp. 67-72.

[K2] —— Multiplicateurs de Schur et *K*-théorie. *Essays on topology and related Topics (dedicated to G. de Rham)*, Springer-Verlag 1970.

[K-T] KAN, D. and W. THURSTON. Every connected space has the Homology of a $K(\pi, 1)$. *Topology 15* (1976), pp. 253-258.

[L] LODAY, J. L. *K*-theoric algébrique et représentation de groupes. *Ann. Ec. Norm. Sup. Paris* (1976), pp. 309-377.

[M-S] MACDUFF, D. and G. SEGAL. Classifying spaces for foliations. *(To appear).*

[Q] QUILLEN, D. Cohomology of Groups. *Actes Congrès Int. Math. Nice, T. 2* (1970), pp. 47-51.

[P] PRIDDY, S. Transfer Symmetric Groups and Stable Homotopy Theory. *Springer Lecture Notes 341* (1972), pp. 244-259.

[V] VOGEL, P. Localization with respect to a class of maps. *(To appear).*

[W] WAGONER, J. Delooping Classifying Spaces in Algebraic *K*-theory. *Topology 11* (1972), pp. 349-370.

(Reçu le 22 novembre 1977)

Jean-Claude Hausmann

Section de Mathématiques
Université de Genève
Case 124
1211 Genève 24 (Suisse)

Dale Husemoller

Haverford College
Haverford, Pennsylvania, 19041
(USA)

vide-leer-empty