V. Recent progress on regularity problems

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$u = v$. This corresponds to the well-known cone $x_1^2 + x_2^2 = x_3^2 + x_4^2$, which has indeed mean curvature 0. The other possibility for a geodesic is to end on the u-axis, those ending on the v-axis being obtained by a symmetrical reflection. Up to a homothetic transformation there is only one such geodesic. We introduce the new homothetically invariant parameters

$$\varphi = \operatorname{artg} \frac{v}{u}, \quad \theta = \operatorname{artg} \frac{v'}{u'},$$

$$\sigma = \theta - 3\varphi + \frac{\pi}{2}, \quad \psi = \theta + \varphi - \frac{\pi}{2}$$

and rewrite the equation for geodesics as

$$\begin{cases} \dot{\sigma} = -\frac{3}{2} \sin \sigma - \frac{7}{2} \sin \psi \\ \dot{\psi} = \frac{1}{2} \sin \sigma - \frac{3}{2} \sin \psi . \end{cases}$$

We are interested in the unique characteristic C which at time $t = -\infty$ starts at the saddle point $(\pi, 0)$ and at time $t = \infty$ ends at the origin $(0, 0)$. Since the diagonal $u = v$ goes in the line $\sigma = \psi$ in the (σ, ψ)-plane, if we follow C from $t = -\infty$ to a time t_0 for which $\sigma = \psi$, going back to the (u, v) plane we get a geodesic starting on the axis $v = 0$ and ending on $u = v$; clearly by applying a suitable homothety we may get a geodesic ending at $u = v = \frac{1}{\sqrt{2}}$ and a solution to our problem. It follows that our result will be proved if we show that the characteristic C crosses the line $\sigma = \psi$ infinitely many times. This in fact is obvious, because C ends at $(0, 0)$ and it is easily checked that $(0, 0)$ is a focal singular point, or vortex, of the differential system for σ, ψ.

It may be noted that the same construction gives other examples, like for the boundary $S^2\left(\frac{1}{\sqrt{2}}\right) \times S^2\left(\frac{1}{\sqrt{2}}\right)$, with almost exactly the same result.

V. Recent progress on regularity problems

The regularity theory of minimal currents and varifolds is fundamental if we want to obtain classical solutions to variational problems. Here the theory proceeds in two main directions: one is to prove stronger and better regularity theorems, the other is to produce more examples of singular minimal varieties to narrow the gap.

It is a classical result that a minimal surface is real analytic at every regular point. Let $V \subset \mathbf{C}^n \cong \mathbf{R}^{2n}$ be a complex analytic subvariety of $\mathbf{C}^n$; by Wirtinger's inequality, V is also an absolutely minimizing surface in $\mathbf{R}^{2n}$, hence V may carry singularities and the singular set can have co-dimension 2. If T is a minimal hypersurface in $\mathbf{R}^n$, singularities are harder to find: Simons' cone

$$x_1^2 + \ldots + x_4^2 = x_5^2 + \ldots + x_8^2$$

is the first and simplest example of a singular absolutely minimal hypersurface in $\mathbf{R}^8$. All these examples are real analytic sets and one could ask whether this is always the case. However there are topological obstructions for a singularity to be real analytic, as the following construction by Milani shows.

We can find an embedding of $\mathbf{P}^2(\mathbf{C})$, the complex projective plane, in $\mathbf{R}^9$ so that $\mathbf{P}^2(\mathbf{C})$ will be on the sphere S^8 given by $x_1^2 + \ldots + x_9^2 = 1$. By the general theory, there is a 5-dimensional current T with boundary $\mathbf{P}^2(\mathbf{C})$ which is absolutely minimizing and, by results of Allard on boundary regularity, one can show that spt (T) is a manifold in a neighborhood of its boundary. We conclude that the singular set of T is a compact subset of spt $(T) \setminus$ spt (∂T). Now assume that spt (T) is a real analytic set Σ. By Hironaka's theorem on resolution of singularities, together with a very important refinement obtained by Tognoli, there is a real analytic manifold Σ' and a proper $f: \Sigma' \to \Sigma$ which is an isomorphism outside f^{-1} (sing Σ). Thus Σ' is a real manifold with boundary $\mathbf{P}^2(\mathbf{C})$. This contradicts Thom's theorem that $\mathbf{P}^2(\mathbf{C})$ is a generator of infinite order of the cobordism ring, and the conclusion is that spt (T) is not a real analytic set.

Another beautiful example has been obtained by Lawson and Osserman [L-O] in their work on the Dirichlet problem on the minimal surface system in non-parametric form, in higher codimension. If $\eta: S^3 \to S^2$ is the Hopf map

$$\eta(z_1, z_2) = (|z_1|^2 - |z_2|^2, 2z_1 z_2)$$

where $(z_1, z_2) \in \mathbf{C} \times \mathbf{C} \cong \mathbf{R}^4$ and η is considered as $\eta(z_1, z_2) \in \mathbf{R} \times \mathbf{C} \cong \mathbf{R}^3$, they found that the Lipschitz function $f: \mathbf{R}^4 \to \mathbf{R}^3$ defined by

$$f(x) = \frac{\sqrt{5}}{2} |x| \, \eta\left(\frac{x}{|x|}\right) \quad \text{for } x \neq 0,$$

is a solution of the minimal surface system. This gives the first example of a non-parametric minimal Lipschitz cone, of dimension 4 in $\mathbf{R}^7$.

General regularity theorems for absolutely minimal currents have proved to be very difficult to obtain. The codimension 1 case has been treated with success; after previous work by Reifenberg, De Giorgi, Almgren, Miranda, Simons, finally Federer [FH 2] proved the sharp result that absolutely minimal hypersurfaces are non-singular in codimension less than 7. In particular, minimal hypersurfaces of dimension $\leqq 6$ are analytic manifolds. Also, in the codimension one non-parametric case Bombieri, De Giorgi and Miranda proved regularity in any dimension, a result to be contrasted with the Lipschitz singular cone of Lawson and Osserman.

In general codimensions, the only result was that the set of regular points is dense (Reifenberg, Morrey, Almgren) and only recently Almgren announced [AF 3] that minimal surfaces are regular almost everywhere. It seems likely that Almgren's new methods will in fact show that minimal surfaces are regular in codimension 2; in view of the examples provided by complex analytic varieties, this result would be sharp.

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