3. Preliminaries

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **22 (1976)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **10.05.2024**

3. Preliminaries

Since in this paper we are concerned for the most part with square-integrable kernels $K(x, y)$, and the Lebesgue integral is employed throughout, equalities and inequalities between functions, therefore, are generally to be understood as holding "almost everywhere" (a.e.).

For convenience we take $a = 0$, $b = \pi$, which we may do without any loss of generality, and consider the class of L^2 kernels $K(x, y)$ with $0 \leqslant x, y \leqslant \pi$. In our later work we will need to direct our attention primarily to one of the two variables x, y. Let us choose this to be the first and extend K to be periodic in this variable. There are many ways, of course, to accomplish this task, but a not unreasonable procedure is to first define

$$K^{(r)}(x, y) \equiv \frac{\partial^r K(x, y)}{\partial x^r} \quad (r = 0, 1, \ldots, s) \tag{3.1}$$

for some preselected nonnegative integer s, and then assume that $K(x, y)$ is extended, as an even function of x if s is even, and as an odd function of x if s is odd, into the domain $-\pi \leqslant x \leqslant 0$, and thence as a periodic function of x with period 2π. This approach ensures that, under suitable restrictions, the classical Fourier series for $K^{(s)}(x, y)$, viewed as a function of its first variable, consists solely of sine terms.

In order to enhance the character of the analogies in which we are interested, we shall say that $K^{(s)}(x, y)$ *is in* Lip α if

$$| K^{(s)}(x+h, y) - K^{(s)}(x-h, y) | < | h |^{\alpha} A(y) \quad (0 < \alpha \leqslant 1)$$

where $A(y)$ is nonnegative and square-integrable. More generally, for $p \geqslant 1$, $K^{(s)}(x, y)$ *will be said to be in* Lip (α, p) if

$$\int_0^{\pi} | K^{(s)}(x+h, y) - K^{(s)}(x-h, y) |^p \, dx < | h |^{\alpha p} A^p(y) \quad (0 < \alpha \leqslant 1)$$

with $L^2 \, A \geqslant 0$. In similar fashion, $K^{(s)}(x, y)$ *will be said to be relatively uniformly of bounded variation* if for all $N \geqslant 1$ and arbitrary choice of partition $0 \leqslant x_0 \leqslant x_1 \leqslant \ldots \leqslant x_N \leqslant \pi$,

$$\sum_{n=1}^{N} | K^{(s)}(x_n, y) - K^{(s)}(x_{n-1}, y) | < B(y)$$

where $L^2 \, B \geqslant 0$. The comparable definitions appropriate whenever the roles of x and y are reversed should be obvious.

Two-variable kernels behave very much like their one-variable analogues as regards integrated Lipschitz conditions. Indeed, the following can be easily established:

PROPERTY 4. Kernels in Lip (α, p) also belong to Lip (α, q) for all $1 \leqslant q < p$. Kernels in Lip α are automatically in Lip (α, p) for all $p \geqslant 1$.

PROPERTY 5. Kernels which are relatively uniformly of bounded variation belong to Lip (1, 1).

PROPERTY 6. If $K(x, y)$ is absolutely continuous in x, for almost all y, and

$$\int_0^\pi \left[\int_0^\pi | K^{(1)}(x, y) |^p \, dx\right]^{2/p} dy < \infty,$$

$p > 1$, then $K(x, y)$ is in Lip $(1, p)$.

PROPERTY 7. If a kernel belongs both to Lip (α, p) and to Lip (β, q) with $1 \leqslant p < q$, then it belongs to Lip (γ, r) for all $p \leqslant r \leqslant q$, where

$$\gamma = \alpha \frac{p(q-r)}{r(q-p)} + \beta \frac{q(r-p)}{r(q-p)} .$$

A somewhat deeper result is

PROPERTY 8. Whenever $1 \leqslant p \leqslant q$, $pq(\alpha-\beta) \geqslant q - p$, kernels in Lip (α, p) are automatically also in Lip (β, q).

4. GROWTH ESTIMATES FOR SINGULAR VALUES

We come now to the main thrust of our narrative. The characteristic values associated with a given L^2 kernel $K(x, y)$, $0 \leqslant x, y \leqslant \pi$, are those special values of λ for which there exist nontrivial solutions of the homogeneous Fredholm integral equation

$$\phi(x) = \lambda \int_0^\pi K(x, y) \phi(y) \, dy .$$

The singular values are those positive values μ for which there exist nontrivial $\phi(x)$, $\Psi(x)$ satisfying the coupled equations