16. Use symbols correctly

forced not only to learn as he goes, but, at the same time, to decode as he goes. The double effort is needless. By spending another ten minutes writing a carefully worded paragraph, the author can save each of his readers half an hour and a lot of confusion. The paragraph should be a recipe for action, to replace the unhelpful code that merely reports the results of the act and leaves the reader to guess how they were obtained. The paragraph would say something like this: "For the proof, first substitute p for q, then collect terms, permute the factors, and, finally, insert and cancel a factor r."

A familiar trick of bad teaching is to begin a proof by saying: "Given ε, let δ be $\left(\frac{\varepsilon}{3M^2 + 2}\right)^{1/2}$". This is the traditional backward proof-writing of classical analysis. It has the advantage of being easily *verifiable* by a machine (as opposed to *understandable* by a human being), and it has the dubious advantage that something at the end comes out to be less than ε, instead of less than, say, $\left(\frac{(3M^2 + 7)\,\varepsilon}{24}\right)^{1/3}$. The way to make the human reader's task less demanding is obvious: write the proof forward. Start, as the author always starts, by putting something less than ε, and then do what needs to be done—multiply by $3M^2 + 7$ at the right time and divide by 24 later, etc., etc.—till you end up with what you end up with. Neither arrangement is elegant, but the forward one is graspable and rememberable.

16. Use symbols correctly

There is not much harm that can be done with non-alphabetical symbols, but there too consistency is good and so is the avoidance of individually unnoticed but collectively abrasive abuses. Thus, for instance, it is good to use a symbol so consistently that its verbal translation is always the same. It is good, but it is probably impossible; nonetheless it's a better aim than no aim at all. How are we to read "$\in$": as the verb phrase "is in" or as the preposition "in" ? Is it correct to say: "For $x \in A$, we have $x \in B$," or "If $x \in A$, then $x \in B$" ? I strongly prefer the latter (always read "$\in$" as "is in") and I doubly deplore the former (both usages occur in the same sentence). It's easy to write and it's easy to read "For x in A, we have $x \in B$"; all dissonance and all even momentary ambiguity is avoided. The same is

true for "$\subset$" even though the verbal translation is longer, and even more true for "$\leqq$". A sentence such as "Whenever a positive number is $\leqq 3$, its square is $\leqq 9$" is ugly.

Not only paragraphs, sentences, words, letters, and mathematical symbols, but even the innocent looking symbols of standard prose can be the source of blemishes and misunderstandings; I refer to punctuation marks. A couple of examples will suffice. First: an equation, or inequality, or inclusion, or any other mathematical clause is, in its informative content, equivalent to a clause in ordinary language, and, therefore, it demands just as much to be separated from its neighbors. In other words: punctuate symbolic sentences just as you would verbal ones. Second: don't overwork a small punctuation mark such as a period or a comma. They are easy for the reader to overlook, and the oversight causes backtracking, confusion, delay. Example: "Assume that $a \in X$. X belongs to the class C, ... ". The period between the two X's is overworked, and so is this one: "Assume that X vanishes. X belongs to the class C, ... ". A good general rule is: never start a sentence with a symbol. If you insist on starting the sentence with a mention of the thing the symbol denotes, put the appropriate word in apposition, thus: "The set X belongs to the class C, ... ".

The overworked period is no worse than the overworked comma. Not "For invertible X, X^* also is invertible", but "For invertible X, the adjoint X^* also is invertible". Similarly, not "Since $p \neq 0$, $p \in U$", but "Since $p \neq 0$, it follows that $p \in U$". Even the ordinary "If you don't like it, lump it" (or, rather, its mathematical relatives) is harder to digest than the stuffy-sounding "If you don't like it, then lump it"; I recommend "then" with "if" in all mathematical contexts. The presence of "then" can never confuse; its absence can.

A final technicality that can serve as an expository aid, and should be mentioned here, is in a sense smaller than even the punctuation marks, it is in a sense so small that it is invisible, and yet, in another sense, it's the most conspicuous aspect of the printed page. What I am talking about is the layout, the architecture, the appearance of the page itself, of all the pages. Experience with writing, or perhaps even with fully conscious and critical reading, should give you a feeling for how what you are now writing will look when it's printed. If it looks like solid prose, it will have a forbidding, sermony aspect; if it looks like computational hash, with a page full of symbols, it will have a frightening, complicated aspect. The golden mean is golden. Break it up, but not too small; use prose, but not too much. Intersperse enough displays to give the eye a chance to help the brain;

use symbols, but in the middle of enough prose to keep the mind from drowning in a morass of suffixes.

17. All communication is exposition

I said before, and I'd like for emphasis to say again, that the differences among books, articles, lectures, and letters (and whatever other means of communication you can think of) are smaller than the similarities.

When you are writing a research paper, the role of the "slips of paper" out of which a book outline can be constructed might be played by the theorems and the proofs that you have discovered; but the game of solitaire that you have to play with them is the same.

A lecture is a little different. In the beginning a lecture is an expository paper; you plan it and write it the same way. The difference is that you must keep the difficulties of oral presentation in mind. The reader of a book can let his attention wander, and later, when he decides to, he can pick up the thread, with nothing lost except his own time; a member of a lecture audience cannot do that. The reader can try to prove your theorems for himself, and use your exposition as a check on his work; the hearer cannot do that. The reader's attention span is short enough; the hearer's is much shorter. If computations are unavoidable, a reader can be subjected to them; a hearer must never be. Half the art of good writing is the art of omission; in speaking, the art of omission is nine-tenths of the trick. These differences are not large. To be sure, even a good expository paper, read out loud, would make an awful lecture—but not worse than some I have heard.

The appearance of the printed page is replaced, for a lecture, by the appearance of the blackboard, and the author's imagined audience is replaced for the lecturer by live people; these are big differences. As for the blackboard: it provides the opportunity to make something grow and come alive in a way that is not possible with the printed page. (Lecturers who prepare a blackboard, cramming it full before they start speaking, are unwise and unkind to audiences.) As for live people: they provide an immediate feedback that every author dreams about but can never have.

The basic problems of all expository communication are the same; they are the ones I have been describing in this essay. Content, aim and organization, plus the vitally important details of grammar, diction, and notation—they, not showmanship, are the essential ingredients of good lectures, as well as good books.