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Autor(en): **Schneider, Rolf**

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by Rolf SCHNEIDER

1. INTRODUCTION

In the geometry of convex bodies there is a certain type of questions which lead to some linear functional equations on the Euclidean sphere. Among these equations are, e.g., certain integral equations as well as relations between the values of a function at certain sets of finitely many arguments. What we mean by " linear functional equations on the sphere " is perhaps best described by giving some typical results:

Let E^d $(d \geqq 3)$ denote d-dimensional Euclidean space with scalar product $\langle\ ,\ \rangle$, and let

$$\Omega_d := \{ x \in E^d \mid \langle x, x \rangle = 1 \}$$

be its unit sphere.

THEOREM 1.1. *Let* f *be a real continuous function on* Ω_d *satisfying*

$$f(u_1) + \ldots + f(u_d) = 0 \tag{1.1}$$

for any d *pairwise orthogonal vectors* $u_1, \ldots, u_d \in \Omega_d$. *Then* f *is a spherical harmonic of degree* 2.

THEOREM 1.2. *Let* f *be a real continuous function on* Ω_d *satisfying*

$$f(u_1) + \ldots + f(u_{d+1}) = 0 \tag{1.2}$$

whenever $u_1, \ldots, u_{d+1}$ *are the vertices of a regular simplex inscribed in* Ω_d. *Then* f *is a sum of spherical harmonics of degrees* 0, 1, 2, 5 *if* $d = 3$, *and of degrees* 0, 1, 2 *if* $d \geqq 4$.

For $v \in \Omega_d$ let

$$s_v := \{ u \in \Omega_d \mid \langle u, v \rangle = 0 \}$$

be the great sphere with pole v; and let λ_v denote the $(d-2)$-dimensional Lebesgue measure on s_v.

THEOREM 1.3. *If* f *is a real, even, continuous function on* Ω_d *satisfying*

$$\int_{S_v} f \, d\lambda_v = 0 \tag{1.3}$$

for each $v \in \Omega_d$, *then* $f = 0$.

In the following, by a measure on Ω_d we understand a real valued, countably additive set function, defined on the σ-field of Borel subsets of Ω_d. A measure φ on Ω_d is called *even* (respectively *odd*) if $\varphi(B) = \varphi(B^*)$ ($\varphi(B) = -\varphi(B^*)$) for any two antipodal Borel subsets B, B^* of Ω_d.

THEOREM 1.4. *If* φ *is an even measure on* Ω_d *satisfying*

$$\int_{\Omega_d} |\langle u, v\rangle| \, d\varphi(u) = 0 \tag{1.4}$$

for each $v \in \Omega_d$, *then* $\varphi = 0$.

For $v \in \Omega_d$ let

$$S_v := \{ u \in \Omega_d \mid \langle u, v\rangle > 0 \}$$

be the open hemisphere with center v.

THEOREM 1.5. *If* φ *is an odd measure on* Ω_d *satisfying*

$$\varphi(S_v) = 0 \tag{1.5}$$

for each $v \in \Omega_d$, *then* $\varphi = 0$.

To functional relations of the types (1.1)–(1.5) one is lead by some uniqueness and characterization problems in the theory of convex bodies; and the theorems quoted above (all of which are essentially known—see section 3) have interesting geometric interpretations. The main purpose of this note is to treat the above functional equations from a unifying point of view and to exhibit them as special cases of one general equation.

In fact, each of these equations can be written in the form (see section 3)

$$\int_{\Omega_d} f(\delta u) \, d\varphi(u) = 0 \qquad \text{for each } \delta \in SO(d). \tag{1.6}$$

Here $SO(d)$ denotes the d-dimensional rotation group acting on Ω_d, f is a function and φ a measure on Ω_d. Equation (1.6) may be read in two ways: Either φ is given, then (1.6) is a functional equation for f, or f is given and φ is to be determined.

We shall now state a theorem which gives necessary and sufficient conditions for a pair f, φ in order that (1.6) be true. From these conditions the uniqueness theorems 1.1–1.5, and some others, can be immediately deduced via some elementary computations.

2. The main theorem

Let us first give some definitions. In the following, all functions are complex valued and continuous. Let ω denote Lebesgue measure on Ω_d, normalized so that $\omega(\Omega_d) = 1$.

For functions f, g and measures φ on Ω_d we write $(f, g) := \int f \bar{g} d\omega$ and $(f, \varphi) := \int f d\varphi$, where the integrals are extended over Ω_d. For $n = 0, 1, 2, \ldots$ let $\mathfrak{H}_n$ denote the complex vector space of spherical harmonics of degree n on Ω_d; let $N_{d,n}$ be its dimension. If f is a function on Ω_d, we say that $\mathfrak{H}_n$ *occurs in* f if and only if the orthogonal projection of f onto $\mathfrak{H}_n$ does not vanish, i.e. if $(f, Y_n) \neq 0$ for some spherical harmonic Y_n of degree n (or, equivalently, if $\int f(u) C_n^\nu(\langle u, v\rangle) d\omega(u)$ does not vanish identically, where C_n^ν is the Gegenbauer polynomial of degree n and order $\nu = \frac{1}{2}(d-2)$). Analogously, we say that $\mathfrak{H}_n$ occurs in the measure φ if and only if $(Y_n, \varphi) \neq 0$ for some $Y_n \in \mathfrak{H}_n$. If f is a function on Ω_d and $\delta \in SO(d)$ is a rotation, the *left translate* δf of f by δ is defined by $(\delta f)(u) = f(\delta^{-1} u)$ for $u \in \Omega_d$.

Theorem 2.1. *Let* f *be a continuous function and* φ *a measure on* Ω_d. *In order that* $(\delta f, \varphi) = 0$ *for each* $\delta \in SO(d)$, *it is necessary and sufficient that none of the spaces* $\mathfrak{H}_n$, $n \in \{0, 1, 2, \ldots\}$, *occurs in both,* f *and* φ.

We remark that this theorem, together with its proof to be given below, carries over to the following more general situation: $SO(d)$ and Ω_d may be replaced, respectively, by a compact connected topological group G and by the homogeneous manifold G/K, where $K (= SO(d-1)$ in our case) is a closed subgroup of G. The rôle of the spherical harmonics is then played by their natural generalizations. We do not write down this generalization explicitly since we do not know any application of it.

Proof of Theorem 2.1. Let $\{ Y_{ni}; i=1, \ldots, N_{d,n} \}$ be an orthonormal basis of $\mathfrak{H}_n (n = 0, 1, 2, \ldots)$. Let us first suppose that f is a finite sum of spherical harmonics,

$$f = \sum_{n=0}^{k} \sum_{j=1}^{N_{n,d}} (f, Y_{nj}) Y_{nj} . \tag{2.1}$$

Since $\mathfrak{H}_n$ is invariant under the action of $SO(d)$ by left translation, we have a relation

$$(2.2) \qquad Y_{nj}(\delta^{-1} u) = \sum_{i=1}^{N_{d,n}} t^{n}_{ij}(\delta)\, Y_{ni}(u)$$

for each $\delta \in SO(d)$, by which continuous functions t^n_{ij} on $SO(d)$ are defined. It is well known that, for each $n \in \{0, 1, 2, \ldots\}$, the mapping $\delta \to (t^n_{ij}(\delta))$ is a unitary, irreducible matrix valued representation of the group $SO(d)$. From (2.1) and (2.2) we get

$$(2.3) \qquad (\delta f, \varphi) = \sum_{n=0}^{k} \sum_{i,j=0}^{N_{d,n}} t^{n}_{ij}(\delta)(f, Y_{nj})(Y_{ni}, \varphi) .$$

If f is an arbitrary continuous function on Ω_d, then f can be uniformly approximated by a sequence $f_1, f_2, \ldots$, where each f_k is a finite sum of spherical harmonics of those degrees n only, for which $\mathfrak{H}_n$ occurs in f (see, e.g., Weyl [22], p. 499).

Let us now suppose that $\mathfrak{H}_n$ does not occur in both, f and φ ($n = 0, 1, 2, \ldots$). Approximate f as explained above. Then if $(Y_{ni}, \varphi) \neq 0$ for some n and some $i \in \{1, \ldots, N_{d,n}\}$, the space $\mathfrak{H}_n$ does not occur in f. Therefore we have $(f_k, Y_{nj}) = 0$ ($k = 1, 2, \ldots$) for each $j \in \{1, \ldots, N_{d,n}\}$, since f_k is a finite sum of spherical harmonics of degrees other than n. This shows that $(f_k, Y_{nj})(Y_{ni}, \varphi) = 0$ for each possible choice of k, n, i, j, and hence $(\delta f_k, \varphi) = 0$ by (2.3). For $k \to \infty$ we get $(\delta f, \varphi) = 0$, which proves one half of the theorem.

In order to prove the other direction of Theorem 2.1, we multiply equation (2.3) by $\overline{t^n_{km}(\delta)}$ and integrate over $SO(d)$ with respect to the normalized Haar measure μ. Using one of the well known orthogonality relations for the matrix elements of unitary, irreducible representations of a compact group, namely

$$N_{d,n} \int_{SO(d)} t^{n}_{ij}(\delta)\, \overline{t^{n}_{km}(\delta)}\, d\mu(\delta) = \delta_{ik}\, \delta_{jm} ,$$

we arrive at

$$N_{d,n} \int_{SO(d)} (\delta f, \varphi)\, \overline{t^{n}_{km}(\delta)}\, d\mu(\delta) = (f, Y_{nm})(Y_{nk}, \varphi) ,$$

provided f is a finite sum of spherical harmonics. By approximation, this holds for arbitrary continuous f. If now (1.6) is assumed, we get $(f, Y_{nm})(Y_{nk}, \varphi) = 0$ for $n = 0, 1, 2, \ldots$ and $k, m \in \{1, \ldots, N_{d,n}\}$, which shows that $\mathfrak{H}_n$ does not occur in both, f and φ. Theorem 2.1 is proved.

3. Applications

In this section we want to indicate how Theorems 1.1 – 1.5 come out as corollaries of Theorem 2.1. Some other consequences of this theorem will also be mentioned. Furthermore, we give references concerning equations (1.1) – (1.5) and we review some geometric applications of the relevant uniqueness theorems.

In order to get the equations (1.1) – (1.3) from (1.6) one has to choose the measure φ appropriately. The condition, e.g., that (1.1) be true for any d pairwise orthogonal unit vectors $u_1, \ldots, u_d \in \Omega_d$ is equivalent to the equation

$$(3.1) \qquad f(\delta u_1) + \ldots + f(\delta u_d) = 0 \qquad \text{for each } \delta \in SO(d),$$

where now $u_1, \ldots, u_d$ is a fixed d-tuple of pairwise orthogonal unit vectors. Equation (3.1) results from (1.6) if φ is the discrete measure concentrated in $u_1, \ldots, u_d$ and assigning the same weight to each of these points. If now (3.1) holds, then Theorem 2.1 shows that $(f, Y_{ni}) = 0$ $(i = 1, \ldots, N_{d,n})$ must hold for each $n \in \{0, 1, 2, \ldots\}$ for which

$$Y_n(u_1) + \ldots + Y_n(u_d) \neq 0$$

for some spherical harmonic Y_n of degree n. If we choose for Y_n the zonal harmonic defined by $Y_n(u) = C_n^\nu(\langle u_1, u \rangle)$, where C_n^ν is the Gegenbauer polynomial of degree n and order $\nu = \frac{1}{2}(d-2)$, then we get

$$\sum_{r=1}^{d} Y_n(u_r) = C_n^\nu(1) + (d-1)\, C_n^\nu(0) \neq 0 \qquad \text{for } n \neq 2\,.$$

From the completeness of the system of spherical harmonics we conclude $f \in \mathfrak{H}_2$. That each element of $\mathfrak{H}_2$ is in fact a solution of (3.1) may be shown directly.

Theorem 1.1 is, for $d = 3$, due to Blaschke [3], who used it to derive the following geometric result: *If the vertices of the boxes circumscribed to a given convex body $K \subset E^3$ lie on some fixed sphere, then* K *is a solid ellipsoid.* Chakerian [7] has generalized this theorem by induction with respect to the dimension and has drawn another geometric consequence.

Theorem 1.2 is proved similarly, though in this case the decision whether for given n the inequality

$$Y_n(u_1) + \ldots + Y_n(u_{d+1}) \neq 0$$

is valid for some $Y_n \in \mathfrak{H}_n$ or not, is a bit more difficult. For $d = 3$, Theorem 1.2 is due to Meissner [13]; for general d it is contained in [19] (Hilfssatz 4.4). The geometric problem leading to equation (1.2) and more generally to equations of type (1.6) with discrete φ, can be described as follows: Let $P \subset E^d$ be a d-dimensional convex polytope; a convex body $K \subset E^d$ is called a *rotor* of P if to each $\delta \in SO(d)$ there exists a translation τ of E^d such that $\tau\delta K$ is contained in the polytope P and touches each of its $(d-1)$-dimensional faces. Roughly speaking, a rotor can be completely turned inside P, always gliding along its facets. As an example, we mention the bodies of constant width one, which are the rotors of the unit cube. Meissner [13] has determined all the nontrivial (i.e. non-spherical) rotors of the threedimensional regular polyhedra. The general problem of determining all pairs (P, K) where $P \subset E^d$ is a d-dimensional convex polytope (not necessarily bounded) and K is a nontrivial rotor of P, has been completely solved in [19].

In order to obtain Theorem 1.3 we choose φ as the measure concentrated on a fixed great sphere s_a and proportional to the $(d-2)$-dimensional Lebesgue measure on s_a. Then (1.6) holds if and only if (1.3) holds for each $v \in \Omega_d$. For a spherical harmonic $Y_n \in \mathfrak{H}_n$ an easy computation gives (see [17], formula (5); put $\alpha = 0$)

$$(Y_n, \varphi) = \int_{s_a} Y_n \, d\lambda_a = \omega_{d-1} \, C_n^{\nu}(1)^{-1} \, C_n^{\nu}(0) \, Y_n(a) \, ,$$

where ω_{d-1} is the surface area of the unit sphere in E^{d-1} (this equation can also be derived by a limit process from the Funk-Hecke-formula; compare (3.4) below). If Y_n is properly chosen, this is $\neq 0$ for even n. Thus by Theorem 2.1, a function satisfying

$$\int_{s_a} f(\delta u) \, d\lambda_a = 0 \qquad \text{for each } \delta \in SO(d)$$

must be orthogonal to each spherical harmonic of even degree, hence if f itself is even, it must vanish identically.

For $d = 3$, Theorem 1.3 is due to Minkowski [14], who used it to prove that *a convex body of constant girth is also a body of constant width.* Other proofs of Theorem 1.3 for $d = 3$ may be found in Funk [8], Bonnesen-Fenchel [6] (p. 136-138); proofs for $d \geqq 3$ have been given by Petty [16] and Schneider [17]. Funk [8] (p. 287) remarked the following geometric consequence of Theorem 1.3: *The spherical ball is the only centrally symmetric convex body with the property that all intersections of the body with* *planes through its center have the same surface area.* Other geometric applications of Theorem 1.3 may be found in [2], [4], [12], [20].

In order to get the equations (1.4) and (1.5) from (1.6) one has to choose the function f appropriately. Both equations can be written in the form

$$\int_{\Omega_d} g(\langle u, v\rangle)\, d\varphi(u) = 0 \qquad \text{for each } v \in \Omega_d\,, \tag{3.2}$$

where $g(t) = |t|$ in the case of (1.4), and $g(t) = 1$ for $t > 0$ and $= 0$ for $t \leqq 0$ in the case of (1.5). If we put $f(u) = g(\langle u, a\rangle)$ for some fixed $a \in \Omega_d$, then (3.2) is equivalent to

$$\int_{\Omega_d} f(\delta u)\, d\varphi(u) = 0 \qquad \text{for each } \delta \in SO(d)\,.$$

If now (3.2) holds, then Theorem 2.1 shows that $(Y_{ni}, \varphi) = 0$ $(i = 1, \ldots, N_{d,n})$ must hold for each $n \in \{0, 1, 2, \ldots\}$ for which

$$\int_{\Omega_d} g(\langle u, a\rangle)\, Y_n(u)\, d\omega(u) \neq 0 \tag{3.3}$$

for some spherical harmonic Y_n of degree n. Here we should observe that g is only piecewise continuous in the case of equation (1.5), so that we cannot apply Theorem 2.1 verbally. It is, however, not difficult to generalize Theorem 2.1 appropriately. In order to decide whether (3.3) holds we apply the Funk-Hecke-formula (see, e.g., Müller [15], p. 20)

$$\int_{\Omega_d} g(\langle u, a\rangle)\, Y_n(u)\, d\omega(u) \tag{3.4}$$

$$= \omega_{d-1}\, C_n^{\nu}(1)^{-1} \int_{-1}^{1} g(t)\, C_n^{\nu}(t)\,(1-t^2)^{\frac{1}{2}(d-3)}\, dt\; Y_n(a)\,.$$

If we now make the additional assumption that $(Y_{ni}, \varphi) = 0$ $(i = 1, \ldots, N_{d,n})$ for each n for which

$$\int_{-1}^{1} g(t)\, C_n^{\nu}(t)\,(1-t^2)^{\frac{1}{2}(d-3)}\, dt = 0\,, \tag{3.5}$$

then we get $(Y_{ni}, \varphi) = 0$ $(i = 1, \ldots, N_{d,n})$ for $n = 0, 1, 2, \ldots$ and hence $(h, \varphi) = 0$ for each continuous function h on Ω_d, which shows that $\varphi = 0$. In special applications this additional assumption turns out to be less formal than it might seem. For instance, if $g(t) = |t|$, then (3.5) holds exactly for odd n, hence it suffices to assume that φ is even. Thus Theorem 1.4 comes out. Theorem 1.5 is obtained similarly.

Theorem 1.4, for $d = 3$ and special measures, may be found in the book of Blaschke [5] (p. 152, 154-155). The general case is due to A. D. Aleksandrov [1] (§8) (though in the form of a seemingly more special geometric theorem); compare also Petty [16] (p. 1545-1546). The case $d = 3$ (and φ specialized) of Theorem 1.5 is due to Funk [9]; another proof (of special cases in geometric formulation) has been given by Kubota [11]. The common generalization of both uniqueness theorems, which is given above, may be found in [18]. To this paper we refer also for references to the known geometric consequences of Theorems 1.4 and 1.5, as well as for some new applications thereof.

The question leading to Theorem 1.5 can be generalized in the following way: Let $D \subset \Omega_d$ be any domain, and let us say for the moment that D is *non-special* if and only if every measure φ on Ω_d for which $\varphi(D') = 0$ for each domain D' (properly) congruent to D, must vanish identically; otherwise D is called *special.* If D is a spherical cap of radius $\alpha \in (O, \pi)$, it has been shown that D is special if and only if α is contained in a certain set of values which is denumerable and dense in (O, π) (Ungar [21], more general in [18]). Ungar [21] has given an example of a non-circular special domain on Ω_3. Now Theorem 2.1 (if generalized to piecewise continuous functions) allows, at least theoretically, to decide whether a given domain $D \subset \Omega_d$ is non-special: For this to be the case it is necessary and sufficient that

$$\int_D Y_{ni}\, d\omega \neq 0$$

for each $n \in \{0, 1, 2, \dots\}$ and some $i \in \{1, \dots, N_{d,n}\}$. Thus the answer depends on the computation of denumerably many definite integrals.

Finally we mention that a special case of the 2-dimensional analogue of Theorem 2.1 was used by Görtler [10] in characterizing those pairs of plane convex domains whose mixed area is invariant under arbitrary (non-simultaneous) motions of the domains.

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Technische Universität Berlin
Mathematisches Institut
D-1 Berlin

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