§ 5. (p, q)-multipliers which are not measures

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However, even assuming that E is first countable and complete, one can in general no longer conclude that f^* is bounded (i.e., that $f^*(A)$ is bounded for every bounded subset A of E) whenever it is finite-valued. Counter-examples are easily given in the case of the familiar spaces $E = l^p(N)$ with $p \in (0, 1)$.

PART 2: APPLICATIONS TO MULTIPLIERS

§ 5. (p, q)-*multipliers which are not measures*

5.1 INTRODUCTION. In this section and the following one we will use the substance of § 3 to prove several apparently new properties of (p, q)-multipliers. Let G be a locally compact group [all topological groups will be assumed to be Hausdorff and, in this section, will be multiplicatively written with identity e]. Denote by $L^p(G)$, where $1 \leqq p \leqq \infty$, the usual Lebesgue space formed with a fixed left Haar measure λ_G on G; and by $C_c(G)$ the space of continuous complex-valued functions on G with compact supports.

For $a \in G$, define the left translation operator τ_a and the right translation operator ρ_a by

$$\tau_a g(x) = g(a^{-1} x) \quad \text{and} \quad \rho_a g(x) = g(xa^{-1});$$

respectively. A linear operator T from $C_c(G)$ into $L^q(G)$ is said to be a *(left)* (p, q)-*multiplier* if and only if

(i) T is continuous from $C_c(G)$, equipped with the norm induced by $L^p(G)$, into $L^q(G)$; and

(ii) T commutes with left translations, that is $T\tau_a = \tau_a T$ for all $a \in G$.

A *right* (p, q)-*multiplier* is defined in a similar manner with (ii) replaced by

(ii′) $T\rho_a = \rho_a T$ for all $a \in G$.

Let $L_p^q(G)$ denote the Banach space of (p, q)-multipliers equipped with the customary norm, denoted by $\|\cdot\|_{p,q}$, of continuous linear operators from a subspace of $L^p(G)$ into $L^q(G)$. That is, for each $T \in L_p^q(G)$, $\| T \|_{p,q}$ is the smallest real number K satisfying

$$\| Tg \|_q \leqq K \| g \|_p$$

for all $g \in C_c(G)$. [When $p \neq \infty$ it is more usual to define $L_p^q(G)$ as the space of unique continuous extensions to $L^p(G)$ of the (p, q)-multipliers.]

As an example, whenever $k \in C_c(G)$, the operator T_k, defined by

$$T_k : g \mapsto g * k$$

for all $g \in C_c(G)$, is (a) a (p, q)-multiplier for all (p, q) satisfying $1 \leqq p \leqq q \leqq \infty$; and (b) a (p, q)-multiplier for all $p, q \in [1, \infty]$ provided G is compact. [When G is noncompact it is known that $L_p^q = \{0\}$ whenever $p > q$—see [1], § 3.4.3. We also remark that, unless a more explicit reference is given, all the properties of the convolution operator between functions and functions and between functions and measures used in the sequel may be found in [2], § 4.19.] For convenience, we will sometimes write $\| k \|_{p,q}$ in place of $\| T_k \|_{p,q}$. Use will be made of the fact that

$$\left.\begin{aligned} \| k \|_{1,s} &= \| T_k \|_{1,s} = \| k \|_s, \\ \| k \|_{s,\infty} &= \| T_k \|_{s,\infty} = \| \Delta^{-1/s'} k \|_{s'}, \end{aligned}\right\} \tag{5.1}$$

where Δ denotes the modular function of G, as defined in [7], (15.11) and (15.15) and s' is defined by $1/s + 1/s' = 1$; cf. [1], Corollary 2.6.2 (i) and Theorem 1.4.

5.2 Definitions. If $T \in L_p^q(G)$, we say that:

(i) supp $T \subseteq W$, where W is a closed subset of G, if and only if supp $Tg \subseteq (\text{supp}\, g) \,.\, W$ for every $g \in C_c(G)$.

(ii) *T is a measure* μ if and only if $Tg = g * \mu$ for every $g \in C_c(G)$.

[When $k \in C_c(G)$, supp $T_k \subseteq W$ if and only if supp $k \subseteq W$; and in any case T_k is the measure $\mu = k\lambda_G$.]

5.3 Adjoint multipliers. Let $T \in L_p^q(G)$ and define an adjoint T' of T by

$$g * T' h\,(e) = Tg * h\,(e) \tag{5.2}$$

for all $g, h \in C_c(G)$. Since $Tg * h(e) = \int_G Tg \,.\, \check{h}\, d\lambda_G$, where $\check{h}(x) = h(x^{-1})$, it is readily shown that T' commutes with right translations and that it may be extended to an operator from $(L^{q'})^\vee$ into $(L^{p'})^\vee$. We also infer from (5.2) that

$$g * T' h = Tg * h \tag{5.3}$$

everywhere on G, since $\tau_a(Tg * h) = \tau_a(Tg) * h = T(\tau_a g) * h$. It is plain from (5.3) that T is a measure μ if and only if T' is of the form $h \mapsto \mu * h$.

If we also assume that G is unimodular, so that the L^p norms of g and $\check{g}$ are identical, two applications of the converse to Hölder's inequality will show that

$$\| T' \|_{q',p'} = \| T \|_{p,q}, \tag{5.4}$$

where $1/p' + 1/p = 1$; thus T' is a right (q', p')-multiplier. Moreover (cf. [1], Corollary 2.6.2 (ii))

$$\| T_k' \|_{1,s} = \| k \|_{1,s} = \| k \|_s. \tag{5.5}$$

5.4 Rudin-Shapiro sequences. If U is a nonvoid open subset of G, by a *U-supported Rudin-Shapiro sequence* (briefly: a *U-RS-sequence*) on G we shall mean a sequence $(h_n)_{n \in N}$ of elements of $C_c(G)$ with the following properties:

$$\left.\begin{array}{c} \operatorname{supp} h_n \subseteq U, \\ \inf \| h_n \|_2 > 0, \quad \sup \| h_n \|_\infty < \infty, \\ \lim_{n \to \infty} \| h_n \|_{2,2} = 0. \end{array}\right\} \tag{5.6}$$

We do not know conditions on G which are necessary and sufficient for there to exist U-RS-sequences on G for a given U. When G is nondiscrete Abelian, U-RS-sequences may be constructed on G in a fairly explicit manner for every non-void open subset U of G (see Appendix A.2 below). Sufficient conditions applying in the non-Abelian case are given in Appendix A.3.

If (h_n) is a U-RS-sequence, we may construct positive integers $m_1 < m_2 < \ldots$ so that

$$\| h_{m_n} \|_{2,2} \leqq n^{-1} 2^{-n}.$$

Let $k_n = n h_{m_n}$. It then follows from (5.6) that

$$\| k_n \|_1 \geqq Bn, \tag{5.7}$$

$$\| k_n \|_s \leqq A^{1/s} n \quad (1 \leqq s \leqq \infty), \tag{5.8}$$

$$\| k_n \|_{2,2} \leqq 2^{-n}, \tag{5.9}$$

where A and B are positive and independent of n.

5.5 When G is infinite compact Abelian, Theorem 4.15 of [1] shows that there exists an operator belonging to $L_p^q(G)$ for every $p \in (1, \infty]$ and every $q \in [1, \infty)$ and which is not a measure. [Given an infinite Sidon subset of Γ, operators with this property are immediately constructible whether G is Abelian or not; cf. [7], (37.22).] When G is noncompact locally compact Abelian or infinite compact, it has recently been shown that there exists an operator belonging to $L_p^p(G)$ for every $p \in (1, \infty)$ which is not a bounded measure. [See [4] and [9]; the proof contained in [9] is constructive to some extent. See also [17].] We aim to show in 5.7 below that, if U is a relatively compact open subset of G, and if we are able to construct a U-RS-sequence on G, then we can construct an operator $T \in \bigcap \{L_p^q(G) : 1 < p \leqq q \leqq \infty\}$ such that supp $T \subseteq \bar{U}$ and T is not a measure. (If G is also unimodular, an analogous result holds for right (p, q)-multipliers.)

The inequality $p > 1$, along with the inequality $q < \infty$ if G is unimodular, is essential for the existence of such a T since every member of $L_1^q(G)$ is of the form $g \mapsto g * \mu$, where μ is a bounded measure if $q = 1$ or $\mu \in L^q(G)$ if $1 < q \leqq \infty$ (see [1], Corollary 2.6.2), and since $L_1^q(G) = L_{q'}^\infty(G)$ if G is unimodular (see (5.4) above). When G is non-compact, the inequality $p \leqq q$ is also essential since in this case $L_p^q(G) = \{0\}$ whenever $p > q$ (see [1], § 3.4.3). Concerning non-unimodular groups, see 5.8 below.

5.6 Lemma. Let k be a continuous function supported by a relatively compact open subset U of G, and let $c = c(U) > 0$ denote $\inf \{\Delta(x)^{-1} : x \in U\}$, where Δ is the modular function for G. Then functions $u, v \in C_c(G)$ with $\| u * v \|_\infty \leqq 1$ may be constructed so that

$$| u * T_k v(e) | \geqq (c/2) \| k \|_1.$$

Proof. Let $\{\delta_\alpha\}$ be an approximate identity on G comprised of non-negative functions with compact supports and which each satisfy $\int_G \delta_\alpha d\lambda_G = 1$. Since $\check{k} * \delta_\alpha$ tends to $\check{k}$ in $L^1(G)$, we may select $v = \check{\delta}_\alpha$ so that

$$\| (v*k)^\vee \|_1 = \| \check{k} * \check{v} \|_1 \geqq \frac{3}{4} \| \check{k} \|_1. \tag{5.10}$$

Define a compactly supported function g on G by $g(x) = \overline{v * k(x)} / | v * k(x) |$ if $v * k(x) \neq 0$, and $g(x) = 0$ otherwise. Let $u_\alpha = \delta_\alpha * \check{g}$. Then $u_\alpha \in C_c(G)$ and, since $u_\alpha (v * k)^\vee$ tends to $\check{g} (v * k)^\vee$ in $L^1(G)$, we may select α so that

$$\left| \int_G u_\alpha (v * k)^\vee \, d\lambda_G \right| \geqq \frac{3}{4} \left| \int \check{g} (v * k)^\vee \, d\lambda_G \right|. \tag{5.11}$$

Putting $u = u_\alpha$, we then have from (5.10) and (5.11)

$$\begin{aligned} \left| u * T_k v(e) \right| &= \left| \int_G u (v * k)^\vee \, d\lambda_G \right| \\ &\geqq \frac{3}{4} \left| \int_G \check{g} (v * k)^\vee \, d\lambda_G \right| \\ &= \frac{3}{4} \| (v * k)^\vee \|_1 \geqq \frac{1}{2} \| \check{k} \|_1 \\ &\geqq (c/2) \| k \|_1. \end{aligned}$$

Moreover, $\| u * v \|_\infty = \| \check{v} * \check{u} \|_\infty \leqq \| \check{v} \|_1 \| \check{u} \|_\infty \leqq 1$, as required.

5.7 THEOREM. (1) Let (h_n) be a U-RS-sequence on a locally compact group G, where U is a relatively compact open subset of G, and let $(k_n)_{n \in N}$ be defined as in 5.4. A continuum of sequences $(\omega_n) \in l^1_+(N)$ may be constructed for which the series

$$\sum_{n \in N} \omega_n T_{k_n} \tag{5.12}$$

converges normally in $L^q_p(G)$ for every pair (p, q) satisfying $1 < p \leqq q < \infty$ to a unique operator, T say, such that

(i) supp $T \subseteq \overline{U}$, and

(ii) T is not a measure.

(2) With the further condition that G is unimodular, the theorem remains valid if we replace throughout left multipliers and their related concepts by right multipliers and their correspondingly related concepts.

PROOF. (1) For each $n \in N$, Lemma 5.6 shows that we may select and fix $u_n, v_n \in C_c(G)$ such that

$$\| u_n * v_n \|_\infty \leqq 1, \left| u_n * T_{k_n} v_n (e) \right| \geqq (c/2) \| k_n \|_1, \tag{5.13}$$

where $c = \inf \{ \Delta(x)^{-1} : x \in U \} > 0$ does not depend on n.

We aim to apply 3.2, taking:

H = the space of linear maps from $C_c(G)$ into $L^1_{loc}(G)$, the topology on H being that of pointwise convergence;

$I = \{(p, q) : 1 < p \leqq q < \infty\}$;

$E_{(p,q)} = L_p^q(G)$ with its standard norm;

$E = \mathscr{E}$;

$f_n : T \mapsto | u_n * Tv_n(e) |$;

$x_n = T_{k_n}$.

It is clear that 3.2 (i) holds and that f_n is continuous (a fortiori bounded) on E. By way of verification of 3.2 (ii)-(iv) we will show that

$$f^* (T_{k_n}) < \infty \text{ for every } n \in N, \tag{5.14}$$

$$\lim_{n\to\infty} T_{k_n} = 0 \text{ in } E, \tag{5.15}$$

$$\lim_{n\to\infty} f_n(T_{k_n}) = \infty. \tag{5.16}$$

Regarding (5.14), we have

$$f_m(T_{k_n}) = | u_m * T_{k_n} v_m(e) | = | u_m * v_m * k_n(e) | \leqq \| u_m * v_m \|_\infty \| \check{k}_n \|_1$$

which, by the first clause of (5.13), does not exceed $\| \check{k}_n \|_1$. Hence $f^* (T_{k_n}) \leqq \| \check{k} \|_1$, which is finite since $k_n \in C_c(G)$.

As to (5.15), the Riesz-Thorin convexity theorem ([11], Volume II, p. 95) shows that for $(p, q) \in I$ satisfying $\frac{1}{p} + \frac{1}{q} \geqq 1$ one has

$$\| T_{k_n} \|_{p,q} \leqq \| T_{k_n} \|_{2,2}^{\alpha} \| T_{k_n} \|_{1,s}^{1-\alpha}, \tag{5.17}$$

where $1/p = \alpha/2 + (1-\alpha)/1$, $1/q = \alpha/2 + (1-\alpha)/s$, so that $\alpha = 2/p' \in (0, 1]$ and $s \in [1, \infty]$. On combining the first clause of (5.1), (5.8), (5.9) and (5.17), we see that

$$\lim_{n\to\infty} \| T_{k_n} \|_{p,q} = 0 \tag{5.18}$$

for every pair $(p, q) \in I$ satisfying $1/p + 1/q \geqq 1$. If, on the other hand, $(p, q) \in I$ and $1/p + 1/q < 1$, a similar argument gives

$$\| T_{k_n} \|_{p,q} \leqq \| T_{k_n} \|_{2,2}^{\alpha} \| T_{k_n} \|_{s,\infty}^{1-\alpha} \tag{5.19}$$

where $1/p = \alpha/2 + (1-\alpha)/s$ and $1/q = \alpha/2$, so that $\alpha = 2/q \in (0, 1)$ and $s \in (1, \infty]$. On combining the second clause of (5.1), (5.8), (5.9) and the fact that Δ is bounded away from zero on U, (5.18) appears once more. The verification of (5.15) is thus complete.

The definition of f_n combines with (5.7) and (5.13) to yield (5.16).

Appeal to 3.2 provides a construction for a continuum of sequences $(\omega_n) \in l_+^1(N)$ for each of which the series (5.12) converges normally in E to a sum T satisfying

$$f^*(T) = \infty. \tag{5.20}$$

This entails that, for every $(p, q) \in I$, $T \in L_p^q(G)$ and the series (5.12) is normally convergent in $L_p^q(G)$ to the sum T. Since supp $T_{k_n} \subseteq U$ for every n, it is simple to verify that sup $T \subseteq \bar{U}$. It remains to show that T is not a measure. However, were T to be the measure μ, it would be the case that supp $\mu \subseteq \bar{U}$ and so, using the first clause of (5.7), that

$$\begin{aligned} f_n(T) &= | u_n * Tv_n(e) | = | u_n * v_n * \mu(e) | \\ &= | \textstyle\int_G (u_n * v_n)^\vee \Delta^{-1} d\mu | \\ &\leqq \textstyle\int_G \Delta^{-1} d | \mu |. \end{aligned}$$

Since μ has a compact support, this inequality would lead to a contradiction of (5.20). Thus T cannot be a measure.

(2) Finally, when G is unimodular, everything remains valid when right multipliers replace left multipliers throughout: this can be seen by either repeating the entire argument ab initio, or by deriving it from the result already obtained by making use of the properties of the adjoint discussed in 5.3.

5.8 THE NON-UNIMODULAR CASE. (i) If G is non-unimodular, there can be no full analogue of Theorem 5.7 applying to right multipliers. This is so because in this case there exist no non-trivial right (p, q)-multipliers when $p \neq q$.

To see this, suppose that T is a right (p, q)-multiplier and that $p \neq q$. For $f \in C_c(G)$ and $a \in G$ we then have

$$\| \rho_a Tf \|_q = \| T\rho_a f \|_q \leqq \| T \|_{p,q} \| \rho_a f \|_p = \| T \|_{p,q} \Delta(a)^{1/p} \| f \|_p$$

and

$$\| \rho_a Tf \|_q = \Delta(a)^{1/q} \| Tf \|_q.$$

Hence

$$\| Tf \|_q \leqq \Delta(a)^{1/p - 1/q} \| T \|_{p,q} \| f \|_p.$$

Since G is non-unimodular and $p \neq q$,

$$\inf_{a \in G} \Delta(a)^{1/p - 1/q} = 0,$$

and we infer that $T = 0$.

(ii) In spite of (i) immediately above, there is a partial analogue taking the following form.

Assume that there exists a sequence (h_n) satisfying (5.6), where now $\| h_n \|_{2,2}$ is defined to mean

$$\sup \{ \| h_n * f \|_2 : f \in C_c(G) , \| f \|_2 \leqq 1 \}.$$

Then modification of the proof of Theorem 5.7 will lead to the construction of operators T which are right multipliers of type (p, p) for every $p \in (1, \infty)$, have supports contained in $\bar{U}$, and are not of the form $f \mapsto \mu * f$ for any measure μ.

§ 6. (p, q)-*multipliers whose transforms are not measures*

6.1 Introduction. Throughout this section we suppose that G is a locally compact Abelian (= LCA) group with dual group Γ, both groups being additively written. We begin by slightly modifying the form of the definition of (p, q)-multipliers, so rendering it possible to make certain statements about their Fourier transforms without attempting a general definition of such transforms. To this end, let F denote the set of functions on G which belong to $\bigcap \{L^p(G) : 1 \leqq p \leqq \infty\}$ and which possess Fourier transforms with compact supports, and denote by $L_p^q(G)$ the set of continuous linear operators from F, equipped with the $L^p(G)$-norm, into $L^q(G)$ which commute with translations. As before, equip $L_p^q(G)$ with the $(L^p(G), L^q(G))$ operator norm. It is easy to specify a natural isometry between $L_p^q(G)$ as defined above and $L_p^q(G)$ as defined in § 5, and so we speak of the elements of $L_p^q(G)$ as (p, q)-multipliers on G.

When T is a (p, q)-multiplier in this sense, we say that its *Fourier transform* $\hat{T}$ *is a measure* μ if and only if there exists a measure μ on Γ such that

$$h * Tg\,(0) = \int_\Gamma \hat{h}\hat{g}\,d\mu \tag{6.1}$$

for all $g, h \in F$, where $\hat{u}$ denotes the Fourier transform of u. Similarly, if Ω is an open subset of Γ, we shall write $\hat{T} = \mu$ on Ω if and only if (6.1) holds for all g, $h \in F$ such that supp $\hat{g} \subseteq \Omega$. If Σ is a closed subset of Γ, we shall write supp $\hat{T} \subseteq \Sigma$ if and only if $\hat{T} = 0$ on Γ/Σ.