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The Axiata mapping problem and symmetry implications on cuspidal conchoids

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1 The Axiata mapping problem

Two rather simple and familiar properties of the real axis are that i) if two real numbers a and b are such that $a + b = 0$ then the points on the axis corresponding to a and b are symmetrically positioned with respect to the 0 position, and ii) if two real numbers a and b are such that $ab = 1$ then the points on the axis corresponding to a and b are not symmetrically positioned with respect to the 1 position. The Axiata mapping problem, as proposed by the author to students for pedagogical purposes, has to do with artificially matching the symmetry property mentioned in ii) by modifying the shape of this real rectilinear axis. This can be done with the use of a mapping. Initially, let $\mathcal{X}$ be the subset of $\mathbb{R}^2$ that corresponds to the points of the x-axis, i.e.:

$$\mathcal{X} = \{(x, 0)|x \in \mathbb{R}\}. \tag{1}$$

The Axiata problem, thus, consists in finding a mapping $A : \mathcal{X} \to \mathbb{R}^2$ such that:

I) $A(1, 0) = (1, 0)$ and $A(-1, 0) = (-1, 0)$.

II) A must be injective.

Die Punkte x und $-x$ auf der reellen Achse liegen symmetrisch zum Ursprung. Aber lässt sich die x-Achse auch so als krumme Kurve in die Ebene legen, dass die Skalenpunkte x und $\frac{1}{x}$ äquidistant zum Punkt 1 liegen? Freilich geht das auf ganz verschiedene Arten. Der Autor der vorliegenden Arbeit schlägt eine besonders natürliche Konstruktion vor, welche als Bildkurven die Konchoide von Nikomedes und die Kardioide miteinander in Verbindung bringt.

III) A must be continuous and well defined, with possible exceptions at points $(-\infty, 0)$, $(0, 0)$ and $(\infty, 0)$.

IV) The point $(0, 0)$ may be eventually excluded from the domain of A.

V) If P_1 and P_2 are the images of the points $(x, 0)$ and $(1/x, 0)$, respectively, with $x \in \mathbb{R} \setminus \{0\}$ and $|x| \leq 1$, then, the corresponding distances d_1 and d_2 from P_1 and P_2 to the point $(1, 0)$ must be the same, and its value must be either $|1-x|$ or $|1-(1/x)|$.

As any mapping $f : \mathcal{X} \to \mathbb{R}^2$ can be expressed equivalently as a mapping from $\mathbb{R}$ to $\mathbb{R}^2$, we may choose to refer to $f(x, 0)$ as $f(x)$, for simplicity. It is not difficult to show that there are infinitely many solutions for the Axiata problem, but two of them emerge from very natural geometrical approaches. Let $A^i(x)$ and $A^e(x)$ denote these two mappings, which are given by:

$$A^i(x) = \begin{cases} (x, 0), & \text{if } |x| \geq 1 \\ \left(x, \sqrt{\frac{1-2x+2x^3-x^4}{x^2}}\right), & \text{if } |x| < 1 \end{cases} \tag{2}$$

$$A^e(x) = \begin{cases} (x, 0), & \text{if } |x| \leq 1 \\ \left(1 + 1/x - 1/x^2, \frac{1}{x}\sqrt{\frac{x^4-2x^3+2x-1}{x^2}}\right), & \text{if } |x| > 1. \end{cases} \tag{3}$$

The indices i and e stand for *internal* and *external* as $A^i(x)$ only maps non-identically points inside the interval $[-1, 1]$, whereas in $A^e(x)$ it only happens with points outside the same interval (see Figures 1 and 2).

There are possible variations to the above Axiata functions $A^i(x)$ and $A^e(x)$, since

$$\pm A^i(x), \ \pm A^e(x), \ \pm A^i(x)\,\mathrm{sgn}(x) \quad \text{and} \quad \pm A^e(x)\,\mathrm{sgn}(x-1),$$

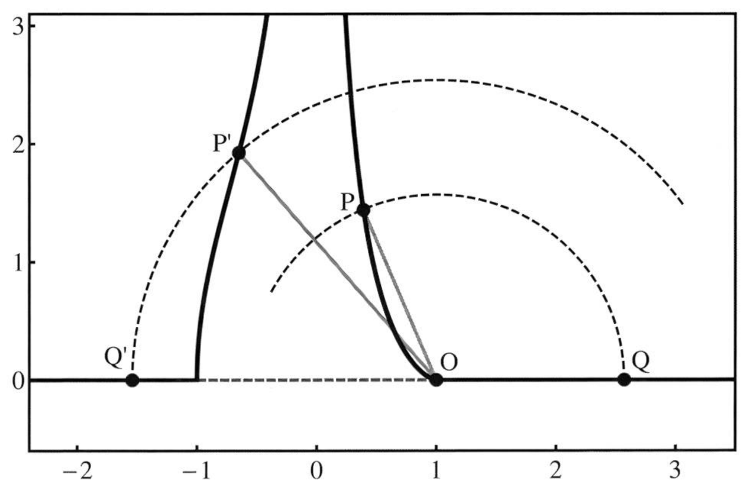

Figure 1 The shape of the Axiata curve A^i. The symmetry condition requires $OP = OQ$ and $OP' = OQ'$ where the abscissas of P and Q are inverses, the same happening with P' and Q'. The solid line of the x-axis was not drawn in the interval $[-1, 1]$ intentionally, because the Axiata is to be considered as the very new shape of this axis.

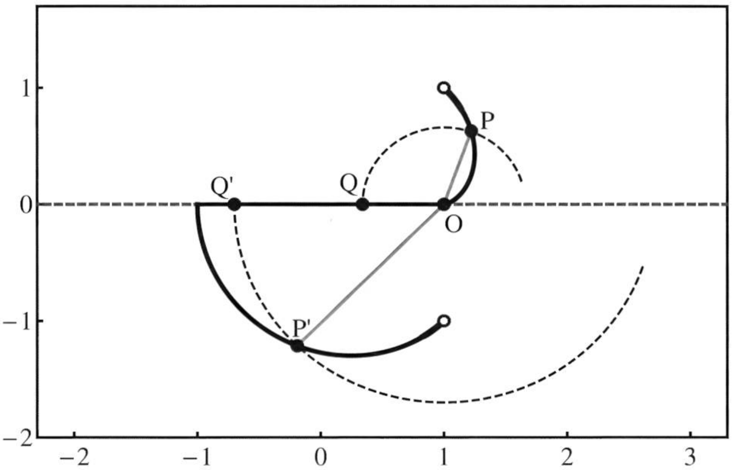

Figure 2 The shape of the Axiata curve A^e. The symmetry condition again requires $OP = OQ$ and $OP' = OQ'$, but now the points on the x-axis with inverted abscissas relatively to those of Q and Q' lie, respectively, on the intersections between the tangent to the dashed circles through P and P' and the x-axis. The solid line of the x-axis was not drawn in the interval $(-\infty, -1) \cup (1, +\infty)$ due to the fact that the Axiata is to be taken as the transformed shape of this axis.

where $\text{sgn}(x)$ is the sign function, still meet the requirements of the problem. Thus, in order to avoid ambiguity, and since these variations are equivalent in most of the aspects considered in the present work, we will choose to define the two Axiatas $A^i(x)$ and $A^e(x)$ as the functions presented in (2) and (3).

The geometrical approaches that lead to (2) and (3) are very natural, as we shall see. In deriving (2), the internal Axiata, the points on the x-axis such that $|x| > 1$ are mapped identically, whereas the points for which $|x| < 1$ are all mapped to points with the same abscissa. The specific position of each point is determined by the symmetry requirement (V), with the distance parameter chosen to be $|1 - (1/x)|$. Thus, for example, the point $(1/3, 0)$ is mapped into $(1/3, r)$, with r real such that the distance from $(1/3, r)$ to $(1, 0)$ is the same as that from $(3, 0)$ to $(1, 0)$. The real number r in this context would correspond to $A^i_y(1/3)$ in (2). The derivation of (3), the external Axiata, is also based on a simple procedure. Now the part of the x-axis which is mapped identically is that in which $|x| < 1$. Every point $(x, 0)$, such that $|x| > 1$, are mapped to the respective point of tangency of a line through $(x, 0)$ on a circle with center at $(1, 0)$ and radius $|1 - (1/x)|$. Therefore, in order to find the point to which the point $(7/3, 0)$, for instance, must be mapped, one has to draw a circle centered at $(1, 0)$ with radius $4/7$ and then trace a line passing through $(7/3, 0)$ which is tangent to the circle. The coordinates of this point of tangency are given by $A^e(7/3)$.

2 Axiata's properties and the connection with cuspidal conchoids

Two common features of these Axiatas are that they have a horizontal tangent at $(1, 0)$ and do not have a tangent at $(-1, 0)$. It is interesting to note that $A^i(x)$ presents a singularity at $x = 0$, which makes the y-axis an asymptote to both the right and left branch of the curve, whereas $A^e(x)$ has a discontinuity at $x = 1$, since $(1, 1)$ and $(1, -1)$ are, respectively, the points where $+\infty$ and $-\infty$ were mapped.

However, the most relevant properties of these mappings $A^i(x)$ and $A^e(x)$, for the purposes of this work, are that, apart from containing parts of a straight line (the x-axis), we can identify in them the presence of certain types of conchoids (see Figure 3). With respect to $A^i(x)$, for any straight line l which passes through the point $(1, 0)$, having a negative or zero slope, there will always be a unitary circle with its center on the y-axis containing, in one of its diametrical lines, the two points of intersection between $A^i(x)$ and l.

From this property it follows directly that the Axiata mapping $A^i(x)$ can be proven to contain a cuspidal member of the family of conchoids of Nicomedes [1]–[9]. Nicomedes, who was a Greek scholar who lived c. II B.C, used these curves to solve important geometrical problems, namely, the trisection of an angle, the duplication of the cube and the construction of a regular heptagon [1]. The geometric construction of a conchoid, in general, requires a given curve $\mathcal{D}$, as the *directrix*; a point O, as the *pole*; and a positive value k, often called *offset*, to serve as the distance parameter. In order to draw points of this conchoid one has to follow this two step procedure:

- Draw a straight line γ through O crossing the curve $\mathcal{D}$ at the point Q.
- Find the points C and C' on the straight line γ such that $CQ = C'Q = k$.

The points C and C' are, then, points of the conchoid with *directrix* $\mathcal{D}$, *offset* k and *pole* O.

The Cartesian equation of the conchoid of Nicomedes, which is, basically, the conchoid of a straight line, is given by:

$$(x - d)^2(x^2 + y^2) - k^2x^2 = 0 \tag{4}$$

where its *directrix* $\mathcal{D}$ is represented by the equation $x = d$, the *pole* O is located at the origin and the distance parameter (*offset*) is k. This conchoid may have three different shapes. For $k < |d|$ the two branches are smooth; for $k = |d|$ there is a cusp in the branch near the pole (Figure 3a); for $k > |d|$ there is a loop also in the branch near the pole.

Concerning the Axiata mapping $A^e(x)$, we may also demonstrate that it contains a cuspidal member of the conchoids of a circle, namely, the cardioid (Figure 3b). A typical equation that describes the conchoid of the circle, with the *pole* located at the origin and belonging to the circle, is:

$$(x^2 + y^2 - ax)^2 - b^2(x^2 + y^2) = 0 \tag{5}$$

where a is the diameter of the circle (the *directrix* of the conchoid) and b is its *offset*. If $b = a$, the cusp shows up yielding the cardioid [4, 6, 10].

In fact, the formal correspondences between (2) and (4) and between (3) and (5) can be established by setting $d = -1$ and $k = 1$ in (4) and $a = 1$ and $b = 1$ in (5). Moreover, regarding the cardioid case, one has to proceed to a reflection with respect to the y-axis. After that, an additional unitary translation to the right is required in both cases.

3 Discussion and conclusion

In this work, the Axiata mapping problem has been introduced as an investigation on the possible shapes of the real axis which present the property of equidistance of inverses with relation to the $x = 1$ position. Two of its solutions were presented as well as some of its

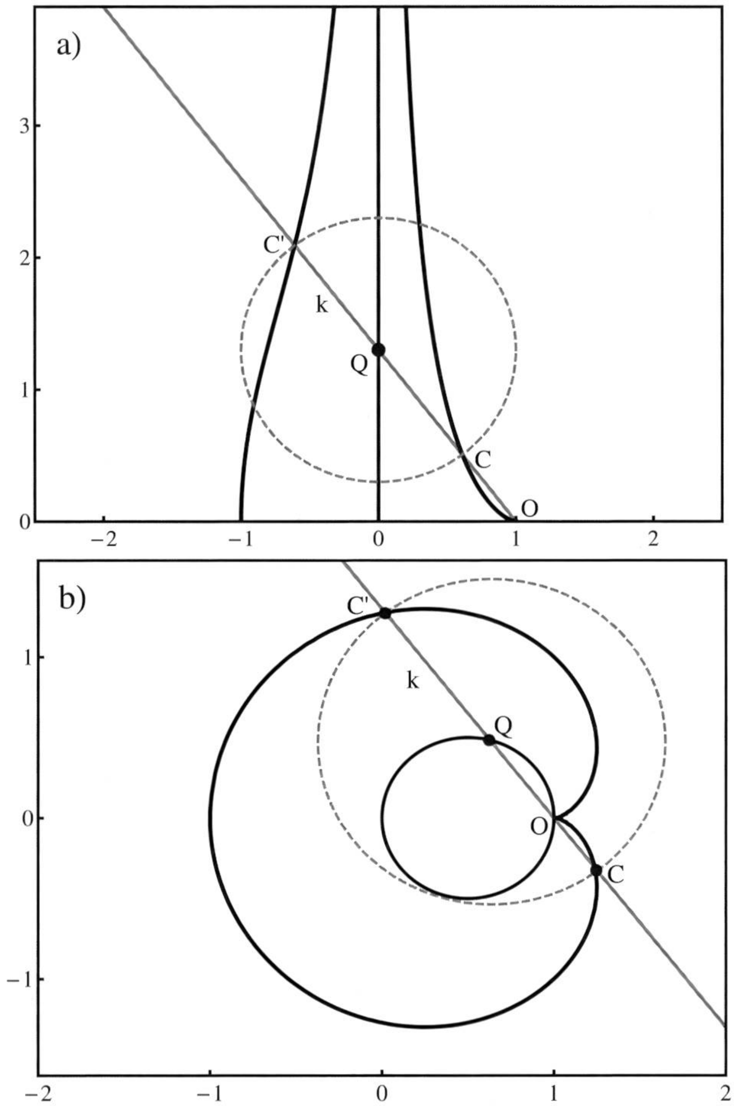

Figure 3 The defining properties of the conchoid of Nicomedes and the cardioid are present in the Axiatas $A^i(x)$ and $A^e(x)$ respectively. In both a) and b) we have $CQ = C'Q = k$, for any straight line passing through O.

features. It was observed that one Axiata, $A^i(x)$, can be recognized as containing a singular (cuspidal) member of the family of conchoids of Nicomedes, with parameters $k = 1$ and $d = -1$ and the other, $A^e(x)$, contains another cuspidal conchoid, the cardioid, associated with parameters $a = 1$ and $b = 1$. It was pointed out that these two Axiata mappings were obtained from simple geometrical approaches. One of them preserves the rectilinear shape of the x-axis in the interval $[-1, 1]$ by identically mapping this region, while the other preserves its complement. In $A^i(x)$ each point of the interval $[-1, 1]$ was mapped into a point with the same abscissa, whereas in $A^e(x)$, any given point $(x, 0)$ lying outside the interval $[-1, 1]$ were mapped into the tangency point between a straight line passing through $(x, 0)$ and a circle with radius $|1 - (1/x)|$ centered at $(1, 0)$. Considering typical definitions found in the literature, the poles of both conchoids are placed at the origin of the Cartesian system. In the Axiata problem the poles naturally appear at $(1, 0)$, as a consequence of the symmetry condition. The connection presented in this work between the Axiata mapping problem and the cuspidal conchoids of the straight line (Nicomedes) and of the circle (cardioid) can be understood as resulting in alternative algebraic methods of construction of these specific curves based on a new symmetry requirement.

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