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1. I have been asked by colleagues to write down that proof of the fundamental and classical Theorem of Beckman, Quarles [1] that I have presented in a beginners course on Geometric Transformations for students already familiar with the basic methods of Linear Algebra. The proof in question, which is already sketched in a more general context in [2], is a mixture of ideas of Beckman, Quarles [1], Schröder [5], Benz [2] up to some new details. In this connection we also refer to Parhomenko and Modenov [4] and to their proof of the Theorem in question.
Let $\mathbb{R}^n$ $(1 < n < \infty)$ be equipped with the usual scalar product

$$a\,b := \sum_{i=1}^{n} \alpha_i \beta_i$$

for $a = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ and $b = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n$.
Then

$$\| a - b \| := \sqrt{(a-b)^2}$$

is called the distance of $a, b \in \mathbb{R}^n$.
Theorem of Beckman and Quarles: Suppose $k > 0$ to be a fixed real number and suppose f to be a mapping of $\mathbb{R}^n$ $(1 < n < \infty)$ into itself such that

$$\| p - q \| = k \text{ implies } \| f(p) - f(q) \| = k$$

for all $p, q \in \mathbb{R}^n$. Then f is an isometry of $\mathbb{R}^n$ and hence a bijective linear mapping up to a translation.
In section 2 we shall collect some simple facts which are useful later on. Those elementary facts could be presented in a course far ahead the proof of the theorem in question, possibly in the form of exercises for the students.
The proof itself will be given in sections 3 and 4. It might be noticed that the original theorem in [1] was formulated for multivalued transformations f. This is however no substantial generalization as was pointed out in [3] in the case of Lorentz transformations of $\mathbb{R}^n$.

2. Throughout this note exactly the elements of $\mathbb{R}^n$ $(1 < n < \infty)$ are called points.
1) Suppose that a, m, b are points such that

$$\| m - a \| = \| b - m \| = \tfrac{1}{2} \| b - a \| .$$

Then $m = \frac{1}{2}(a + b)$.

Proof: Putting $\varrho := \| m - a \|$, $a' := m - a$, $b' := b - m$ we have $(b-a)^2 + (a' - b')^2 = (a' + b')^2 + (a' - b')^2 = 4\varrho^2$ and hence $(a' - b')^2 = 0$.

2) A set of n distinct points of $\mathbb{R}^n$ which are pairwise of distance $\beta > 0$ will be called a β-set. Suppose that α, β are positive real numbers with

$$\gamma(\alpha, \beta) := 4\,\alpha^2 - 2\,\beta^2\left(1 - \frac{1}{n}\right) > 0$$

and suppose that P is a β-set. Then there exist exactly two distinct points in $\mathbb{R}^n$ which have distance α from all $p \in P$. Those two points will be called the α-associated points of P. Their distance is $\sqrt{\gamma(\alpha, \beta)}$.

Proof: a) Let $P = \{p_1, \dots, p_n\}$ be a β-set. Then for $i, j \in \{1, 2, \dots, n-1\}$ with $i \neq j$ we have

$$(p_i - p_n)(p_j - p_n) = \tfrac{1}{2}\beta^2,$$

because of $\beta^2 = (p_i - p_j)^2 = ((p_i - p_n) - (p_j - p_n))^2$. Define $\lambda_r := \dfrac{\beta}{\sqrt{2\,r\,(r+1)}}$ for $r = 1, 2, \dots$ and $e_1, \dots, e_{n-1}$ by $(1+s)\,\lambda_s\,e_s := (p_s - p_n) - \sum_{r=1}^{s-1} \lambda_r\,e_r$ for $s = 1, \dots, n-1$. Obviously, $e_1^2 = 1$. We now prove

$$e_i\,e_j = \begin{cases} 1 & \text{for} \quad i = j \le n-1 \\ 0 & \phantom{\text{for}} \quad i < j \le n-1 \end{cases}$$

by induction along the sequence

$$(1,1), (1,2), (2,2), (1,3), (2,3), (3,3), \dots, (n-1, n-1) \quad \text{for} \quad (i, j):$$

Step $(i, i) \to (1, i+1)$: Here we have

$$\begin{aligned} \tfrac{1}{2}\beta^2 &= (p_1 - p_n)(p_{i+1} - p_n) = 2\,\lambda_1\,e_1\left(\sum_{r=1}^{i} \lambda_r\,e_r + (2+i)\,\lambda_{i+1}\,e_{i+1}\right) \\ &= 2\,\lambda_1^2 + 2\,(2+i)\,\lambda_1\,\lambda_{i+1}\,e_1\,e_{i+1}, \end{aligned}$$

and hence $e_1\,e_{i+1} = 0$, because of $\frac{1}{2}\beta^2 = 2\,\lambda_1^2$.
Step $(i-1, j) \to (i, j)$ in case $i < j$: Here we have

$$\begin{aligned} \tfrac{1}{2}\beta^2 &= (p_i - p_n)(p_j - p_n) = \left(\sum_{r=1}^{i-1} \lambda_r\,e_r + (1+i)\,\lambda_i\,e_i\right)\left(\sum_{r=1}^{j-1} \lambda_r\,e_r + (1+j)\,\lambda_j\,e_j\right) \\ &= \sum_{r=1}^{i-1} \lambda_r^2 + (1+i)\,\lambda_i^2 + (1+i)(1+j)\,\lambda_i\,\lambda_j\,e_i\,e_j, \end{aligned}$$

and hence $e_i\,e_j = 0$, because of $\frac{1}{2}\beta^2 = \sum_{r=1}^{i-1} \lambda_r^2 + (1+i)\,\lambda_i^2$ by observing

$$\lambda_r^2 = \frac{\beta^2}{2}\left(\frac{1}{r} - \frac{1}{r+1}\right).$$

Step $(i-1, i) \rightarrow (i, i)$: We finally have

$$\beta^2 = (p_i - p_n)^2 = \left(\sum_{r=1}^{i-1} \lambda_r e_r + (1+i)\,\lambda_i e_i\right)^2 = \sum_{r=1}^{i-1} \lambda_r^2 + (1+i)^2 \lambda_i^2 e_i^2,$$

and hence $e_i^2 = 1$.

b) Suppose now that $q \in \mathbb{R}^n$ has distance α from all $p_s \in P$. This implies

$$(q - p_n)(p_s - p_n) = \tfrac{1}{2}\beta^2 \quad \text{for all } s = 1, \ldots, n-1,$$

because of $\alpha^2 = (q - p_s)^2 = ((q - p_n) - (p_s - p_n))^2$.

Put $q - p_n := \sum_{r=1}^{n} \mu_r e_r$, $\mu_r \in \mathbb{R}$, by extending $\{e_1, \ldots, e_{n-1}\}$ of part a) to an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. We get the equation

$$\tfrac{1}{2}\beta^2 = (q - p_n)(p_s - p_n) = \sum_{r=1}^{s-1} \mu_r \lambda_r + (1+s)\,\mu_s \lambda_s \quad \text{for } s = 1, \ldots, n-1.$$

The case $s = 1$ leads to $\mu_1 = \lambda_1$, and having already $\mu_i = \lambda_i$ for $i \in \{1, \ldots, s-1\}$, $s < n$, we also get $\mu_s = \lambda_s$ by comparing the equation above with

$$\tfrac{1}{2}\beta^2 = \sum_{r=1}^{s-1} \lambda_r^2 + (1+s)\,\lambda_s^2.$$

Hence $q - p_n = \sum_{r=1}^{n-1} \lambda_r e_r + \mu_n e_n$. Now $(q - p_n)^2 = \alpha^2$ leads to

$$\mu_n^2 = \alpha^2 - \sum_{r=1}^{n-1} \lambda_r^2 = \alpha^2 - \frac{\beta^2}{2}\left(1 - \frac{1}{n}\right) = \frac{1}{4}\gamma(\alpha, \beta).$$

There are exactly two solutions q, namely the points

$$q_i = p_n + \sum_{r=1}^{n-1} \lambda_r e_r \pm \tfrac{1}{2}\sqrt{\gamma(\alpha, \beta)} \cdot e_n, \quad i = 1, 2,$$

which are in fact of distance α from all $p \in P$. Obviously, $(q_1 - q_2)^2 = \gamma(\alpha, \beta)$.

3) Again suppose that α, β are positive real numbers with $\gamma(\alpha, \beta) > 0$. Let x, y be points of distance $\sqrt{\gamma(\alpha, \beta)}$. Then there exists a β-set P such that x, y are the α-associated points of P.

Proof: Define $e_n := \dfrac{y - x}{\sqrt{\gamma(\alpha, \beta)}}$ and extend $\{e_n\}$ to an orthonormal basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$. If p_n is an arbitrary point of $\mathbb{R}^n$, then $P = \{p_1, \ldots, p_n\}$ with

$$p_s - p_n := \sum_{r=1}^{s-1} \lambda_r e_r + (1+s)\,\lambda_s e_s \quad \text{for } s = 1, \ldots, n-1$$

is a β-set by using the earlier defined λ_r. If we now take the special point

$$p_n := \frac{x+y}{2} - \sum_{r=1}^{n-1} \lambda_r e_r ,$$

then the α-associated points of P are given by (see part b) of 2))

$$q_i = p_n + \sum_{r=1}^{n-1} \lambda_r e_r + \tfrac{1}{2} \sqrt{\gamma(\alpha,\beta)}\, e_n = \frac{x+y}{2} \pm \frac{y-x}{2} = \begin{cases} y \\ x \end{cases} .$$

3. Proposition: Let $\varrho > 0$ be a fixed real number and let $N > 2$ be a fixed integer. Suppose that $f\colon \mathbb{R}^n \to \mathbb{R}^n$ $(1 < n < \infty)$ is a mapping such that

α) $\|x-y\| = \varrho$ implies $\|f(x)-f(y)\| \le \varrho$,

β) $\|x-y\| = N\varrho$ implies $\|f(x)-f(y)\| = N\varrho$

for all $x, y \in \mathbb{R}^n$. Then $\|x-y\| = \|f(x)-f(y)\|$ holds true for all $x, y \in \mathbb{R}^n$.

Proof: a) Distances ϱ and 2ϱ are preserved under f: Having points x, y with $\|x-y\| = \varrho$ define $z := 2y - x$ and having points x, z with $\|x-z\| = 2\varrho$ define $y := \frac{1}{2}(x+z)$. Put $p_\lambda := x + \frac{\lambda}{2}(z-x)$ for $\lambda = 0, 1, \dots, N$. Observe $\|f(p_0)-f(p_N)\| = N\varrho$ and $\|f(p_\lambda)-f(p_{\lambda+1})\| \le \varrho$ for $\lambda = 0, 1, \dots, N-1$ because of $\|p_0 - p_N\| = N\varrho$ and $\|p_\lambda - p_{\lambda+1}\| = \varrho$. The triangle inequality yields

$$\begin{aligned} N\varrho &= \|f(p_0)-f(p_N)\| \le \|f(p_0)-f(p_2)\| + \sum_{\lambda=2}^{N-1} \|f(p_\lambda)-f(p_{\lambda+1})\| \le \\ &\le \sum_{\lambda=0}^{N-1} \|f(p_\lambda)-f(p_{\lambda+1})\| \le N\varrho \end{aligned}$$

and hence $\|f(p_\lambda)-f(p_{\lambda+1})\| = \varrho$ $(\lambda = 0, 1, \dots, N-1)$ and

$$\|f(p_0)-f(p_2)\| = \|f(p_0)-f(p_1)\| + \|f(p_1)-f(p_2)\| .$$

Because of $p_0 = x$, $p_1 = y$, $p_2 = z$ we thus have

$$\|f(x)-f(z)\| = 2\varrho \quad \text{and} \quad \|f(x)-f(y)\| = \varrho .$$

b) Suppose that $\|x-y\| = \varrho$ for $x, y \in \mathbb{R}^n$. Then

$$f(x+\lambda(y-x)) = f(x) + \lambda(f(y)-f(x)) \tag{1}$$

holds true for all $\lambda = 0, 1, 2, \dots$: Put $p_\lambda := x + \lambda(y-x)$ for $\lambda = 0, 1, 2, \dots$ and observe

$$\|p_\lambda - p_{\lambda-1}\| = \varrho = \|p_{\lambda+1} - p_\lambda\| \quad \text{and} \quad \|p_{\lambda+1} - p_{\lambda-1}\| = 2\varrho$$

for $\lambda = 1, 2, \ldots$ Since distances ϱ and 2ϱ are preserved we get

$$\varrho = \| f(p_\lambda) - f(p_{\lambda-1}) \| = \| f(p_{\lambda+1}) - f(p_\lambda) \| = \tfrac{1}{2} \| f(p_{\lambda+1}) - f(p_{\lambda-1}) \|$$

and hence (compare 1) in section 2) $f(p_\lambda) = \frac{1}{2}[f(p_{\lambda-1}) + f(p_{\lambda+1})]$. This proves (1) by induction since (1) is trivial in cases $\lambda = 0$ and $\lambda = 1$.

c) Let λ, μ be positive integers and suppose that $\| x - y \| = \dfrac{\lambda \varrho}{\mu}$ for $x, y \in \mathbb{R}^n$. Then $\| f(x) - f(y) \| = \dfrac{\lambda \varrho}{\mu}$ holds true: Because of $n > 1$ and $2\lambda\varrho > \| x - y \|$ there exists a point $z \in \mathbb{R}^n$ with $\| z - x \| = \lambda \varrho = \| z - y \|$. With such a fixed z define a, b by

$$x = z + \lambda (a - z), \quad y = z + \lambda (b - z) \tag{2}$$

and put

$$x' := z + \mu (a - z), \quad y' = z + \mu (b - z). \tag{3}$$

Since $\| a - z \| = \varrho = \| b - z \|$ we hence have the corresponding formulas to (2), (3) for the images because of b). Now

$$\| x' - y' \| = \varrho = \| f(x') - f(y') \| = \mu \| f(a) - f(b) \|$$

and

$$\| f(x) - f(y) \| = \lambda \| f(a) - f(b) \| \quad \text{imply} \quad \| f(x) - f(y) \| = \frac{\lambda \varrho}{\mu}.$$

d) Let r, s be positive rational numbers and let x, y be points such that $r\varrho < \| x - y \| < s\varrho$. Then $r\varrho \leq \| f(x) - f(y) \| \leq s\varrho$: Since $n > 1$ and $s\varrho > \| x - y \|$ there exists a point z with $\| z - x \| = \dfrac{s\varrho}{2} = \| z - y \|$. Now c) implies $\| f(z) - f(x) \| = \dfrac{s\varrho}{2} = \| f(z) - f(y) \|$ and hence $\| f(x) - f(y) \| \leq \| f(x) - f(z) \| + \| f(z) - f(y) \| = s\varrho$.

Put $w := x + \dfrac{s\varrho}{\| x - y \|}(y - x)$ and observe $\| w - x \| = s\varrho$ and

$$\| w - y \| = \left(\frac{s\varrho}{\| x - y \|} - 1 \right) \| y - x \| = s\varrho - \| y - x \| < (s - r)\varrho.$$

Hence $\| f(w) - f(x) \| = s\varrho$ by c) and $\| f(w) - f(y) \| \leq (s - r)\varrho$ by the already proved part of d). This implies

$$\| f(x) - f(y) \| \geq \| f(x) - f(w) \| - \| f(y) - f(w) \| \geq s\varrho - (s - r)\varrho = r\varrho.$$

4. Throughout this section let $k > 0$ be a fixed real number and f be a mapping of $\mathbb{R}^n$ $(1 < n < \infty)$ into itself such that distance k is preserved under f, i.e. $\| x - y \| = k$ implies $\| f(x) - f(y) \| = k$ for all $x, y \in \mathbb{R}^n$.

Lemma: Suppose that α, β are positive real numbers such that $\gamma(\alpha, \beta) > 0$ (compare section 2). Suppose moreover that f preserves distances α and β and that x, y are points with $\| x - y \| = \varepsilon := \sqrt{\gamma(a, \beta)}$. Then $\| f(x) - f(y) \| \in \{0, \varepsilon\}$ and in case $2\varepsilon > \alpha$ we even have $\| f(x) - f(y) \| = \varepsilon$.

Proof: This is trivial for $\varepsilon = \alpha$ since distance α is preserved. So assume $\varepsilon \neq \alpha$. Let P be a β-set such that x, y are the α-associated points of P (compare 3) of section 2). It is $P' := f(P)$ also a β-set since distance β is preserved. If we denote the α-associated points of P' by x', y' we get $f(x)$, $f(y) \in \{x', y'\}$ since distance α is also preserved under f and since the α-associated points of P' are uniquely determined. This implies $\| f(x) - f(y) \| \in \{0, \| x' - y' \|\} = \{0, \varepsilon\}$ according to 2) in section 2. Assume now $2\varepsilon > \alpha$. We have to show that $f(x) \neq f(y)$. Assume $f(x) = f(y)$ and take a $z \in \mathbb{R}^n$ with $\| z - x \| = \varepsilon$ and $\| y - z \| = \alpha$ which exists since $n > 1$ and $2\varepsilon > \alpha$. The already proved part of the lemma yields $\| f(x) - f(z) \| \in \{0, \varepsilon\}$, i.e. $\| f(y) - f(z) \| \in \{0, \varepsilon\}$ because of $f(x) = f(y)$. Hence $\alpha = \| y - z \| = \| f(y) - f(z) \| \in \{0, \varepsilon\}$. This contradicts $\varepsilon \neq \alpha > 0$.

We note the following three consequences of our Lemma:

a) Putting $\alpha = k = \beta$ we realize that distance $\sqrt{\gamma(\alpha, \beta)} = k \sqrt{2\left(1 + \frac{1}{n}\right)}$ is preserved.

b) Putting $\alpha = \beta = k \sqrt{2\left(1 + \frac{1}{n}\right)}$ we realize that distance $\sqrt{\gamma(\alpha, \beta)} = (n+1) \cdot \frac{2k}{n}$ is preserved.

c) Put $\alpha = k$ and $\beta = k \sqrt{2\left(1 + \frac{1}{n}\right)}$. Then $\| x - y \| = \sqrt{\gamma(\alpha, \beta)} = \frac{2k}{n}$ implies

$$\| f(x) - f(y) \| \in \left\{0, \frac{2k}{n}\right\}, \text{ i.e. } \| f(x) - f(y) \| \leq \frac{2k}{n} \text{ for all } x, y \in \mathbb{R}^n.$$

If we now take $\varrho := \frac{2k}{n}$ in the Proposition of section 3 and $N := n + 1$ we realize that f is an isometry according to c), b) and $n > 1$.

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