Triangle inequalities from the triangle inequality

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1. Introduction

Although the title here may appear somewhat paradoxical, all inequalities of the triangle depend ultimately on the basic triangle inequality $|u| + |v| \geq |u+v|$ where u and v may be complex numbers or vectors. However, many derivations of inequalities of the triangle, including some previous ones of the author, seem to take a tortuous route. Some such examples appear in «Geometric Inequalities» [1] which is the most extensive reference on triangle inequalities. One noteworthy exception is a derivation of Ptolemy's inequality via complex numbers [2]. Here, if A, B, C, D denote any four points in a plane, then

$$AB \cdot CD + AC \cdot BE \geq AD \cdot BC.$$

This result follows immediately by applying the basic triangle inequality to the identity

$$(z_1 - z_4)(z_2 - z_3) + (z_2 - z_4)(z_3 - z_1) + (z_3 - z_4)(z_1 - z_2) = 0$$

for complex numbers. Equivalently, in the usual terms of a triangle $A_1A_2A_3$ and a point P, (a_1R_1, a_2R_2, a_3R_3) are sides of a triangle where $PA_i = R_i$ and $A_1A_2 = a_3$, etc. Consequently, for any valid triangle inequality $\mathrm{I}(a_1, a_2, a_3) \geq 0$, we also have the dual inequality $\mathrm{I}(a_1R_1, a_2R_2, a_3R_3) \geq 0$. Further consequences and extensions of this duality are to be given in a subsequent paper [3].

In this paper, we will exploit this simple direct method by applying the basic triangle inequality to a number of rather simple complex number identities to obtain simpler proofs of some known inequalities as well as obtaining some apparently new inequalities. One disadvantage of the method, however, is that it is not easy in general to obtain simple geometric necessary and sufficient conditions for the equality case.

It is to be noted that quite some time ago, Brill [4], Hayashi [5] and Fujiwara [6] obtained a number of geometrical theorems by simply interpreting a number of complex number identities. In the only exception to obtaining geometric theorems, Hayashi also obtains the elegant triangle inequality

$$a_1 R_2 R_3 + a_2 R_3 R_1 + a_3 R_1 R_2 \geq a_1 a_2 a_3$$

which was conjectured much later by Stolarsky [7] and proved by the author by showing it was a dual inequality, by inversion, to the known polar moment of inertia inequality [8]

$$a_1 R_1^2 + a_2 R_2^2 + a_3 R_3^2 \geq a_1 a_2 a_3 .$$

Subsequently, we will give proofs and extensions of the latter two inequalities.

2. Identities

In this section, we list a number of identities which we will be using. Most of them have been culled from [2, 4, 5]. The summations and products to be indicated are cyclic ones with respect to u, v, w.

$$\sum (v-w)(v+w-u)=0. \tag{1}$$

$$\begin{aligned}\sum vw(v-w) &= \sum u^2(v-w) = \sum u(w^2-v^2) = \sum (w-v)(w^2+v^2)\\ &= \sum (v-w)(v+w)^2 = \sum (v-w)(v+w-2u)^2/9 = \prod (w-v).\end{aligned} \tag{2}$$

$$\{\sum u\}^3 - \sum \{v+w-u\}^3 = 4\sum u(v+w)(v+w-u) = 24\prod u. \tag{3}$$

$$\begin{aligned}\sum vw(v+w) + 2\prod u &= \sum u(v+w)^2 - 4\prod u = \sum u^2(v+w) + 2\prod u\\ &= \sum u(v^2+w^2) + 2\prod u = \sum u(v-w)^2 + 8\prod u\\ &= \{\sum u\}^3/3 - \sum u^3/3 = \prod (v+w).\end{aligned} \tag{4}$$

$$\sum (v-w)^2(v+w-2u) = \prod (2u-v-w). \tag{5}$$

$$\sum (v^3-w^3)=0. \tag{6}$$

$$\sum (v-w)^3(v+w-2u)=0. \tag{7}$$

$$\sum (v-w)(v+w-2u)^3=0. \tag{8}$$

$$\begin{aligned}\sum u(v-w)^3 &= \sum u^3(w-v) = \sum (w-v)(v+w-u)^3/4\\ &= \sum u(v-w)(v+w)^2 = \sum vw(w^2-v^2) = \sum u \cdot \prod (v-w).\end{aligned} \tag{9}$$

$$\sum (v^4-w^4)=0. \tag{10}$$

$$\sum u^2(v^2-w^2)=0. \tag{11}$$

$$\sum u^3(v^2-w^2)=\sum v^2w^2(v-w)=\sum vw\cdot\prod(v-w). \tag{12}$$

$$\begin{aligned}\sum u^3(v-w)^3&=\sum u^3(w-v)(w+v)^2=\sum u^3(v-w)(v^2+w^2)\\&=\sum u(v-w)(v+w-u)^4/16=\prod u(v-w).\end{aligned} \tag{13}$$

$$\begin{aligned}\sum v^2w^2(v^2-w^2)&=\sum u^4(v^2-w^2)=\sum u^2(w^4-v^4)\\&=\sum(v^2-w^2)(v^2+w^2)^2=\prod(w^2-v^2).\end{aligned} \tag{14}$$

$$\sum u(v-w)(u-z)=-\prod(v-w). \tag{15}$$

$$\sum\frac{(u-b_1)(u-b_2)}{(u-v)(u-w)}\,\frac{1}{u}=\frac{b_1b_2}{uvw}. \tag{16}$$

$$\sum(u'-u)(v-w)=\sum u'(v-w). \tag{17}$$

The next set of identities, which are needed for the next section, give the absolute values of various complex number expressions associated with a general triangle $A_1A_2A_3$. Here, u, v, w are complex numbers from a common origin P to the vertices A_1, A_2, A_3, respectively, and $R_i=PA_i$; $b_j, j=1,2,\ldots$ and z are complex numbers from P to arbitrary points B_j and Z, respectively, with $R_{ij}=\overline{A_iB_j}$, $R_i'=A_iZ$. O, G, H and I denote the circumcenter, centroid, orthocenter and incenter, respectively, of $A_1A_2A_3$. Finally, M_1, etc., R, r, Δ denote the median from A_1, the circumradius, the inradius, the area, respectively, of $A_1A_2A_3$ and m_1, Δ_1, etc., denote the median from P and area of PA_2A_3.

$$|u+v+w|^2=9PG^2=\sum(3R_1^2-a_1^2). \tag{T1}$$

$$|u+v-w|^2=\sum(R_1^2+a_1^2)-2(R_3^2+a_3^2)\equiv D_3^2. \tag{T2}$$

$$|u+v-2w|=2M_3. \tag{T3}$$

$$|v+w|=2m_1. \tag{T4}$$

$$|v^2+w^2|^2=(R_2^2+R_3^2)^2-16\Delta_3^2\equiv F_3^4. \tag{T5}$$

$$|u^2+v^2+w^2|^2=(R_1^2+R_2^2+R_3^2)-16(\Delta_1^2+\Delta_2^2+\Delta_3^2). \tag{T6}$$

$$|vw+wu+uv|^2=3\sum R_2^2R_3^2-\sum a_1^2R_1^2. \tag{T7}$$

$$\left|\sum(v-w)^2\right|^2=2\sum(a_2^2-a_3^2)^2. \tag{T8}$$

$$|v^2+vw+w^2|^2=3(R_2^4+R_2^2R_3^2+R_3^4)-3a_1^2(R_2^2+R_3^2)+a_1^4\equiv E_1^4. \tag{T9}$$

As an indication of the derivations of (T1)-(T9), we illustrate the one for (T9).

$$|v^2+vw+w^2|^2=(v^2+w^2+vw)(\bar{v}^2+\bar{w}^2+\bar{v}\bar{w})$$
$$=R_2^4+R_3^4+R_2^2R_3^2+(R_2^2+R_3^2)(v\bar{w}+\bar{v}w)+(v^2\bar{w}^2+\bar{v}^2w^2).$$

Then, we use

$$v^2\bar{w}^2+\bar{v}^2w^2=(v\bar{w}+\bar{v}w)^2-2R_2^2R_3^2,$$
$$4M_1^2=|v+w|^2=R_2^2+R_3^2+(v\bar{w}+\bar{v}w),$$
$$4M_1^2=2R_2^2+2R_3^2-a_1^2.$$

3. Inequalities

Our numbering of the inequalities here e.g., (2.1), (2.2), ... will indicate that they were derived from identity number (2) by applying $|v|+|w|\geq|v+w|$ in various ways. Also, we will not write down all possible inequalities which can be obtained this way. Also, at the end of some of the inequalities, we will place a symbol C_i to denote that the inequality is sharp for one of the following cases. In general, these have only been established as sufficient conditions since, as was noted in the introduction, the necessary conditions in terms of the invariants of the triangle are not always easy to obtain. Where the condition is both necessary and sufficient, we will use C_i^*.

Case	(a_1,a_2,a_3)	(R_1,R_2,R_3)
C_1	$(2,2,2)$	$\left(\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}},\frac{2}{\sqrt{3}}\right)$
C_2	$(2,2,2)$	$(\sqrt{3},1,1)$
C_3	$(2,2,2)$	$(0,2,2)$
C_4	$(0,2,2)$	$(0,2,2)$
C_5	$(0,2,2)$	$(2,0,0)$
C_6	$(0,6,6)$	$(4,2,2)$
C_7	$(4,2,2)$	$(0,2,2)$
C_8	$(4,2,2)$	$(2,0,4)$
C_9	(a_1,a_2,a_3)	(R,R,R)
C_{10}	(a_1,a_2,a_3)	$r\left(\csc\frac{A_1}{2},\csc\frac{A_2}{2},\csc\frac{A_3}{2}\right)$
C_{11}	(a_1,a_2,a_3)	$R(\lvert\cos A_1\rvert,\lvert\cos A_2\rvert,\lvert\cos A_3\rvert)$
C_{12}	(a_1,a_2,a_3)	$\frac{2}{3}(M_1,M_2,M_3)$
C_{13}	(a_1,a_2,a_3)	$(0,a_3,a_2)$

For cases C_1–C_8, the sides can all be multiplied by a constant factor and/or interchanged. In C_4–C_8, we have degenerate triangles which usually are ruled out. In C_9–C_{12}, we have a general triangle in which the point P coincides with the circumcenter, incenter, orthocenter and centroid, respectively.

We now essentially list the inequalities that follow immediately from our list of identities:

$$(a_1 D_1, a_2 D_2, a_3 D_3) \quad \text{form a triangle}. \tag{1.1}$$

$$a_1 R_2 R_3 + a_2 R_3 R_1 + a_3 R_1 R_2 \geq a_1 a_2 a_3, \qquad \{C_3\}. \tag{2.1}$$

This rather elegant inequality had been conjectured by Stolarsky [7] but had been established previously by Hayashi [5]. Also, the author [3], unaware of the work of Hayashi at the time, established (2.1) by using inversion on a known polar moment of inertia inequality (2.2) (see below). It was also shown that there is equality *iff* C_{13}, C_{11} or C_5. C_5 corresponds to a degenerate triangle which is usually ruled out but these degenerate cases are often useful in making triangle inequality conjectures as well as checking out various inequalities. An extension of (2.1) will be given subsequently.

$$a_1 R_1^2 + a_2 R_2^2 + a_3 R_3^2 \geq a_1 a_2 a_3, \qquad \{C_{10}^*\}. \tag{2.2}$$

$$2\sum a_1 m_1 R_1 \geq a_1 a_2 a_3, \qquad \{C_1\}. \tag{2.3}$$

$$\sum a_1 F_1 \geq a_1 a_2 a_3, \qquad \{C_1\}. \tag{2.4}$$

$$4\sum a_1 m_1^2 \geq a_1 a_2 a_3, \qquad \{C_1\}. \tag{2.5}$$

$$4\sum a_1 M_1^2 \geq 9 a_1 a_2 a_3, \qquad \{C_1^*\}, \tag{2.6}$$

or, equivalently,

$$2\prod (a_2 + a_3) \geq 13 a_1 a_2 a_3 + \sum a_1^3. \tag{2.6$'$}$$

$$27\,\overline{PG}^3 + \sum D_1^3 \geq 24 R_1 R_2 R_3, \qquad \{C_1\}. \tag{3.1}$$

$$\sum m_1 R_1 D_1 \geq 3 R_1 R_2 R_3, \qquad \{C_1\}. \tag{3.2}$$

$$\sum m_1 R_2 R_3 \geq |4 m_1 m_2 m_3 - R_1 R_2 R_3|. \tag{4.1}$$

$$\sum R_1 m_1^2 \geq |2 m_1 m_2 m_3 - R_1 R_2 R_3|, \qquad \{C_4, C_8\}. \tag{4.2}$$

$$\sum m_1 R_1^2 \geq |4 m_1 m_2 m_3 - R_1 R_2 R_3|, \qquad \{C_4, C_8\}. \tag{4.3}$$

$$\sum R_1 a_1^2 \geq 8 |m_1 m_2 m_3 - R_1 R_2 R_3|, \quad \{C_4, C_8\}. \tag{4.4}$$

$$2\sum m_1 R_2 R_3 + \sum R_1 a_1^2 \geq 6 R_1 R_2 R_3, \quad \{C_5\}. \tag{4.5}$$

$$2\sum R_1 m_1^2 + \sum m_1 R_1^2 \geq 3 R_1 R_2 R_3, \quad \{C_{12}^*\}. \tag{4.6}$$

$$4\sum R_1 m_1^2 + \sum R_1 a_1^2 \geq 12 R_1 R_2 R_3, \quad \{C_1\}. \tag{4.7}$$

$$\sum M_1 a_1^2 \geq 4 \prod M_1, \quad \{C_1^*\}. \tag{5.1}$$

$$(a_1 E_1^2, a_2 E_2^2, a_3 E_3^2) \quad \text{form a triangle}. \tag{6.1}$$

$$(a_1^3 M_1, a_2^3 M_2, a_3^3 M_3) \quad \text{form a triangle}. \tag{7.1}$$

$$(a_1 M_1^3, a_2 M_2^3, a_3 M_3^3) \quad \text{form a triangle}. \tag{8.1}$$

The latter two inequalities are duals, i.e., if F $(a_1, a_2, a_3, M_1, M_2, M_3) \geq 0$ is a valid inequality, so is F $(M_1, M_2, M_3, \tfrac{3}{4}a_1, \tfrac{3}{4}a_2, \tfrac{3}{4}a_3) \geq 0$.

$$\sum R_1 a_1^3 \geq 3 a_1 a_2 a_3 \overline{PG}, \quad \{C_5\}. \tag{9.1}$$

$$\sum R_1^3 a_1 \geq 3 a_1 a_2 a_3 \overline{PG}, \quad \{C_5\}. \tag{9.2}$$

$$4\sum a_1 R_1 m_1^2 \geq 3 a_1 a_2 a_3 \overline{PG}, \quad \{C_5\}. \tag{9.3}$$

$$(a_1 m_1 F_1^2, a_2 m_2 F_2^2, a_3 m_3 F_3^2) \quad \text{form a triangle}. \tag{10.1}$$

$$(a_1 m_1 R_1^2, a_2 m_2 R_2^2, a_3 m_3 R_3^2) \quad \text{form a triangle}. \tag{11.1}$$

$$\sum a_1 R_2^2 R_3^2 \geq R_1 R_2 R_3 \{\sum (3 R_2^2 R_3^2 - a_1^2 R_1^2)\}^{1/2}, \quad \{C_5\}. \tag{12.1}$$

$$4\sum a_1 m_1^2 R_1^3 \geq \prod a_1 R_1, \quad \{C_1\}. \tag{13.1}$$

$$\sum a_1 R_1 D_1^4 \geq 16 \prod a_1 R_1, \quad \{C_1\}. \tag{13.2}$$

$$\sum a_1 m_1 R_2^2 R_3^2 \geq 4 \prod a_1 m_1, \quad \{C_1\}. \tag{14.1}$$

$$\sum a_1 m_1 R_1^4 \geq 4 \prod a_1 m_1, \quad \{C_1\}. \tag{14.2}$$

$$\sum a_1 R_1 R_1' \geq a_1 a_2 a_3. \tag{15.1}$$

This is an extension of the polar moment of inertia inequality (2.2), and there is equality if the two points P and Z lie within or on the triangle and are isogonal conjugates [9]. For example, if P is the incenter of the triangle, then Z coincides with

P; if P is the circumcenter, then Z is the orthocenter. Also, P and Z are isogonal conjugates if they are the foci of any inscribed ellipse of the given triangle.

$$a_1 R_{11} R_{12} R_2 R_3 + a_2 R_{21} R_{22} R_3 R_1 + a_3 R_{31} R_{33} R_1 R_2 \geq |b_1|\,|b_2|\, a_1 a_2 a_3 . \tag{16.1}$$

This is a variation of Hayashi's inequality (2.1) and reduces to it if b_1 and b_2 coincide with the circumcenter and P is on the circumcircle.
Our last inequality here is a simple proof of a result of Tweedie [10], i.e., if $A_1A_2A_3$ and $A'_1A'_2A'_3$ denote two directly similar triangles in the plane, then

$$(a_1\overline{A_1A'_1}, a_2\overline{A_2A'_2}, a_3\overline{A_3A'_3}) \qquad \text{form a triangle}. \tag{17.1}$$

Using (17) with (u, v, w) and (u', v', w') as complex numbers representing the vertices of the two triangles, we have by similarity that $v-w=\lambda(v'-w')$, $w-u=\lambda(w'-u')$, $u-v=\lambda(u'-v')$. Whence, $\sum u'(v-w)=0$ and then

$$(v'-v)(w-u)+(w'-w)(u-v)=-(u'-u)(v-w) \tag{17'}$$

and the result follows by taking absolute values of both sides. A simple synthetic proof of (17.1) was also given by Pinkerton [10], p. 27.

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Vierseitige Flächen in Geradenanordnungen

Durch Anordnungen $A(n)$ von $n \geqq 4$ Geraden werden die projektive Ebene P und die euklidische Ebene E in einfach zusammenhängende Gebiete (Flächen) zerlegt. In P treten dabei nach [1], S. 29, maximal