Slowly Growing Meromorphic Functions.

Slowly Growing Meromorphic Functions

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1. Introduction

This paper is concerned with one aspect of the NEVANLINNA theory of functions meromorphic in the plane (referred to in the sequel simply as meromorphic functions). We shall assume acquaintance with the standard terminology of the NEVANLINNA theory

$$T(r, f) = T(r),\ m(r, a),\ \delta(a, f),\ N(r, a)\ldots$$

and with NEVANLINNA's fundamental theorems (see e.g. [1]). If $f(z)$ is meromorphic, and in particular if $f(z)$ is an integral function, we define the maximum modulus $M(r, f)$, the minimum modulus $L(r, f)$ and the spherical derivative $\varrho(f(z))$ of $f(z)$ by

$$M(r, f) = M(r) = \max |f(z)| \quad (|z| = r),$$

$$L(r, f) = L(r) = \min |f(z)| \quad (|z| = r),$$

$$\varrho(f(z)) = \frac{|f'(z)|}{1 + |f(z)|^2},$$

respectively.

We shall be concerned with functions which are slowly growing, such growth being measured by $M(r, f)$, $\varrho(f(z))$ or the characteristic function $T(r, f)$. The order λ and lower order μ of $f(z)$ are defined by

$$\lambda = \limsup_{r \to \infty} \frac{\log T(r)}{\log r},$$

$$\mu = \liminf_{r \to \infty} \frac{\log T(r)}{\log r}.$$

Unless specifically stated we assume throughout the paper that the meromorphic functions under consideration are transcendental, i.e. that

$$\log r = o(T(r)) \quad (r \to \infty)$$

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and hence that

$$\limsup_{|z|\to\infty} |z|\,\varrho(f(z)) \geq \tfrac{1}{2}, \tag{1.1}$$

by a well-known result of LEHTO [3].

Analogously to the definition of the NEVANLINNA deficiency of a value a we introduce the VALIRON deficiency of a defined by

$$d(a) = d(a, f) = \limsup_{r\to\infty} \frac{m(r, a)}{T(r)} = 1 - \liminf_{r\to\infty} \frac{N(r, a)}{T(r)}.$$

If $d(a) > 0$ for a particular value of a then that value is said to be VALIRON deficient. A meromorphic function $f(z)$ can have at most countably many values a which are NEVANLINNA deficient, but the best result known for VALIRON deficiencies is due to AHLFORS and FROSTMAN ([5], p. 277). It states that the set of VALIRON deficient values of a meromorphic function has logarithmic capacity zero (in the sense of FROSTMAN). The best possible result is not known. However it cannot be substantially improved since VALIRON has constructed an example of an integral function of order one such that the set of VALIRON deficient values has the power of the continuum ([8], p. 126 and [9]).

2. Suppose $f(z)$ is meromorphic and satisfies

$$T(r) = O((\log r)^2) \quad (r \to \infty). \tag{2.1}$$

For any two complex numbers a, b, we define

$$N(r, a, b) = \max\{N(r, a),\ N(r, b)\}.$$

Then VALIRON has proved the following theorem [7].

Theorem A. *If $f(z)$ satisfies (2.1) then*

$$N(r, a, b) \sim T(r) \quad (r \to \infty). \tag{2.2}$$

In particular if $f(z)$ is an integral function

$$N(r, a) \sim T(r) \sim \log M(r) \quad (r \to \infty)$$

for all finite a.

Thus if $f(z)$ satisfies (2.1) and possesses a NEVANLINNA deficient value a say, (in particular if $f(z)$ is an integral function) then it possesses no other VALIRON deficient values i.e.

$$N(r, b) \sim T(r) \quad (r \to \infty) \quad (b \neq a).$$

We note however that (2.2) does not preclude the possibility that a meromorphic function satisfying (2.1) may have infinitely many VALIRON deficient values. Given any sequence $\{a_k\}_{k=1}^{\infty}$, let $\{\alpha_k\}$ be a sequence in which each a_k appears infinitely often. Consider the expression

$$f(z) = \frac{f_1(z)}{f_2(z)} = \frac{\sum_{k=1}^{\infty} c_k \alpha_k z^{n_k}}{\sum_{k=1}^{\infty} c_k z^{n_k}} . \tag{2.3}$$

We may choose the $\{c_k\}$ so small that $f(z)$ is a meromorphic function satisfying (2.1). Nonetheless, if $\{n_k\}$ is a sufficiently lacunary sequence of integers, $f(z)$ will have each a_k as a VALIRON deficient value.

In this paper we show that no restriction weaker than (2.1) permits us to conclude that (2.2) holds. We have

Theorem 1. *Given any continuous function $\varphi(r)$ tending monotonically to infinity as $r \to \infty$, no matter how slowly, and any set of numbers $\{a_k\}_{k=1}^{\infty}$ then there exists an integral function $F(z)$ having each a_k as a VALIRON deficient value and satisfying*

$$\log M(r, F) = O(\varphi(r)(\log r)^2) \quad (r \to \infty). \tag{2.4}$$

Even though no countable set of VALIRON deficient values can restrict the growth of an integral function in any way apart from the negation of (2.1) it may be that an uncountable set of such deficiencies would impose some restriction on the growth. We have not been able to decide this question.

For convenience we split Theorem 1 into the following two theorems.

Theorem 2. *Let $\varphi(r)$ be as in Theorem 1. Then there exists an integral function satisfying (2.4) and such that $d(0, f) = 1$.*

Theorem 3. *Let $f(z)$ be an integral function having 0 as a VALIRON deficient value. Then given any set of numbers $\{a_k\}_{k=1}^{\infty}$ there exists an integral function $F(z)$ having each a_k as a VALIRON deficient value and satisfying*

$$\log M(r, F) = O\left\{\log M\left(\frac{r}{2}, f\right)\right\} \quad (r \to \infty), \tag{2.5}$$

so that

$$T(r, F) = O\{T(r, f)\} \quad (r \to \infty). \tag{2.6}$$

§ 3. Recently OSTROVSKII has shown [6] that if $f(z)$ is meromorphic and of lower order μ, $0 \leq \mu < \frac{1}{2}$ then

$$\limsup_{r\to\infty} \frac{\log^+ L(r)}{T(r)} \geq \pi\mu\,(\operatorname{cosec} \pi\mu)\,(\cos \pi\mu - 1 + \delta(\infty)).$$

In particular if $\mu = 0$

$$\limsup_{r\to\infty} \frac{\log^+ L(r)}{T(r)} \geq \delta(\infty).$$

It is an easy matter to conclude from this result that if a meromorphic function $f(z)$ has lower order $\mu = 0$ then it can have at most one NEVANLINNA deficient value, i.e.

$$\limsup_{r\to\infty} \frac{N(r, a)}{T(r)} = 1$$

for all a with at most one exception; furthermore $f(re^{i\vartheta}) \to a$ uniformly in ϑ as $r \to \infty$ through a suitable sequence $\{r_i\}$. Our function (2.3) shows that such functions can have infinitely many VALIRON deficient values. Moreover by suitable selection of the coefficients in (2.3) we can arrange that the function defined by (2.3) satisfies $T(r, f) = O(\varphi(r) \log r)$ $(r \to \infty)$ for any preassigned monotonic increasing function $\varphi(r)$ tending to infinity as $r \to \infty$.

It is well-known that a meromorphic function $f(z)$ may possess a NEVANLINNA deficient value without that value being an asymptotic value. The above result of OSTROVSKII, however, leads one to conjecture that for slowly growing meromorphic functions (of order zero or lower order zero, say) NEVANLINNA deficient values are asymptotic. We prove this conjecture for function satisfying (2.1).

We need the following concept due to HAYMAN [2]: We call an $\mathcal{E}$-set any countable set of circles not containing the origin and subtending angles at the origin whose sum s is finite.

Theorem 4. *Let $f(z)$ be meromorphic and satisfy* (2.1). *Suppose* $\delta(a, f) > 0$ *for some* a. *Then*

$$\liminf \frac{-\log |f(z) - a|}{T(r)} \geq \delta(a) \tag{3.1}$$

uniformly in ϑ as $z = re^{i\vartheta}$ tends to infinity outside an $\mathcal{E}$-set. In particular a is an asymptotic value of $f(z)$. If $a = \infty$ (3.1) *is to be understood as*

$$\liminf \frac{\log |f(z)|}{T(r)} \geq \delta(\infty).$$

The meromorphic function given in § 2 shows that for functions satisfying (2.1) VALIRON deficient values need not be asymptotic.

§ 4. For transcendental functions LEHTO has shown [3] that the inequality (1.1) is sharp. We shall say that a (transcendental) meromorphic function belongs to the class $\mathcal{S}$ if

$$\varrho(f(z)) = O\left(\frac{1}{|z|}\right) \quad (|z| \to \infty).$$

This class of functions was introduced by LEHTO and VIRTANEN [4]. They showed (loc. cit.) that if $f(z) \in \mathcal{S}$ then $f(z)$ satisfies (2.1) and $f(z)$ possesses no asymptotic values. Since, by IVERSEN's theorem PICARD exceptional values are asymptotic values, LEHTO and VIRTANEN concluded that such an $f(z)$ can have no PICARD exceptional values. In view of Theorem 4 we have

Theorem 5. *If $f(z) \in \mathcal{S}$ then $f(z)$ possesses no NEVANLINNA deficient values. In other words, if a meromorphic function $f(z)$ possesses a NEVANLINNA deficient value then*

$$\limsup_{|z| \to \infty} |z| \varrho(f(z)) = +\infty. \tag{4.1}$$

We shall give a direct proof of this theorem, i.e. one which is not based on LEHTO and VIRTANEN's theorem.

Remarks. a) It is an open question whether a function $f(z) \in \mathcal{S}$ can possess a VALIRON deficient value.

b) In view of LEHTO's theorem ([3], Theorem 3) relating the growth of the spherical derivative to the set in which PICARD's theorem holds we conclude immediately from (4.1) that a meromorphic function possessing a NEVANLINNA deficient value has a direction of JULIA.

§ 5. To prove Theorem 2 we shall construct an integral function of the form

$$f(z) = \prod_{\nu=1}^{\infty} \left(1 + \frac{z}{b_\nu}\right)^{c_\nu} \tag{5.1}$$

where the $\{b_\nu\}$ and $\{c_\nu\}$ are suitable lacunary sequences, $b_\nu > 0$, $c_\nu > 0$, the c_ν being integers. We may assume without loss of generality that $\varphi(r) = O(\log r)$ $(r \to \infty)$. We define the sequences $\{b_\nu\}$ and $\{c_\nu\}$ inductively as follows: let $b_1 = c_1 = 1$, and if b_i and c_i are defined for $i = 1, 2, \ldots, \nu - 1$ $(\nu > 1)$ define b_ν by

$$\varphi(b_\nu) = \left(\sum_{i=1}^{\nu-1} c_i\right)^2. \tag{5.2}$$

Since $\varphi(r)$ is a continuous monotonic increasing function of r (5.2) determines b_ν uniquely and $b_\nu > b_{\nu-1}$. We let

$$c_\nu = [\varphi(b_\nu) \log b_\nu],$$

where $[x]$ denotes the integral part of x.

For a given r let ν denote the integer such that $b_\nu \leq r < b_{\nu+1}$. Then

$$\begin{aligned} n(r, 0) = n(r) = n(b_\nu) &= \sum_{i=1}^{\nu} c_i = c_\nu + [\varphi(b_\nu)]^{\frac{1}{2}} \\ &= O(\varphi(b_\nu) \log b_\nu) \quad (\nu \to \infty) \\ &= O(\varphi(r) \log r) \qquad (r \to \infty). \end{aligned}$$

Thus the infinite product (5.1) converges for all finite z; i.e. $f(z)$ is an integral function. Moreover $f(z)$ has genus zeros and real negative zeros and so

$$\begin{aligned} \log M(r, f) = \log f(r) &= \int_0^\infty \log\left(1 + \frac{r}{t}\right) dn(t) \\ &= r \int_0^\infty \frac{n(t)\, dt}{t(t+r)} \qquad (5.3) \\ &< \int_0^r \frac{n(t)\, dt}{t} + r \int_r^\infty \frac{n(t)\, dt}{t^2} \\ &< O(1)\varphi(r) \int_1^r \frac{\log t\, dt}{t} + O(r) \int_r^\infty \frac{(\log t)^2\, dt}{t}, \end{aligned}$$

since we assumed that $\varphi(r) = O(\log r)$ $(r \to \infty)$. Thus

$$\begin{aligned} \log M(r, f) &= O(\varphi(r)(\log r)^2) + O((\log r)^2) \\ &= O(\varphi(r)(\log r)^2) \quad (r \to \infty) \end{aligned}$$

as required.

§ 6. We now show that $d(0, f) = 1$, i.e. that

$$\liminf_{r \to \infty} \frac{N(r, 0)}{T(r)} = 0. \qquad (6.1)$$

We have

$$\begin{aligned} N(b_\nu, 0) &= \int_0^{b_\nu} \frac{n(t)\, dt}{t} < n(b_{\nu-1}) \int_1^{b_\nu} \frac{dt}{t} = \left(\sum_{i=1}^{\nu-1} c_i\right) \log b_\nu \\ &= [\varphi(b_\nu)]^{\frac{1}{2}} \log b_\nu. \qquad (6.2) \end{aligned}$$

Also, by a well-known inequality for integral functions (see [1], p. 18),

$$T(r, f) \geq \tfrac{1}{3} \log M\left(\frac{r}{2}, f\right)$$

for all $r > 0$. Hence, by (5.3),

$$T(r, f) \geq \tfrac{1}{6} r \int_0^\infty \frac{n(t)\, dt}{t\left(t + \frac{r}{2}\right)} .$$

Thus, since the integrand is positive, we have for all $r > 0$

$$T(r, f) \geq \tfrac{1}{6} r \int_{b_\nu}^{2b_\nu} \frac{n(t)\, dt}{t\left(t + \frac{r}{2}\right)} .$$

But for $b_\nu \leq r \leq 2b_\nu$, $n(r) = n(b_\nu) = \sum_{i=1}^{\nu} c_i > c_\nu = [\varphi(b_\nu) \log b_\nu]$. Hence

$$T(r, f) \geq \tfrac{1}{6} (\varphi(b_\nu) \log b_\nu + O(1))\, r \int_{b_\nu}^{2b_\nu} \frac{dt}{t\left(t + \frac{r}{2}\right)} .$$

Now put $r = b_\nu$ and make the substitution $t = b_\nu u$. We obtain

$$T(b_\nu, f) \geq \tfrac{1}{6} (\varphi(b_\nu) \log b_\nu + O(1)) \int_1^2 \frac{dt}{t\,(t + \frac{1}{2})} . \tag{6.3}$$

From (6.2) and (6.3) we conclude, since $\varphi(r)$ tends to infinity, that

$$\lim_{\nu \to \infty} \frac{N(b_\nu, 0)}{T(b_\nu)} = 0$$

and so (6.1) holds. This completes the proof of Theorem 2.

§ 7. In this section we prove Theorem 3. We assume without loss of generality that $f(0) = 1$, since otherwise we can consider $\dfrac{f(z)}{A z^n}$ with suitable A and n in place of $f(z)$. If $f(z) = \sum_0^\infty a_n z^n$ set

$$s_n(z) = \sum_{k=0}^{n} a_k z^k, \quad \sigma_n(z) = \sum_{k=1}^{n} a_k z^k .$$

Let $\{\alpha_k\}_1^\infty$ be an infinite sequence in which each a_k occurs infinitely often. Then we shall prove that there is a sequence $\{n_k\}_1^\infty$ of positive integers, a sequence

$\{\varrho_k\}_1^\infty$ of real numbers with $\varrho_k \to \infty$ $(k \to \infty)$ and a sequence $\{R_k\}_1^\infty$ of real numbers with $R_k \to \infty$ $(k \to \infty)$ such that the following is true. Let $F_0(z) \equiv 1$ and define

$$F_k(z) = F_{k-1}(z)\, s_{n_k}\left(\frac{z}{R_k}\right) - \alpha_k\, \sigma_{n_k}\left(\frac{z}{R_k}\right) \quad (k \geq 1).$$

Then $F_k(z) \to F(z)$ where $F(z)$ is an integral function satisfying (2.5) and (2.6), and

$$\liminf_{k \to \infty} \frac{m\left(\varrho_k R_k, \dfrac{1}{F - \alpha_k}\right)}{T(\varrho_k R_k, F)} > 0. \tag{7.1}$$

Since each a_k occurs infinitely often among the α_k it follows from (7.1) that each a_k is a VALIRON deficient value of $F(z)$.

We now assume that

$$n_1, \ldots, n_{k-1}; \quad 1 \leq \varrho_1 < \varrho_2 < \ldots < \varrho_{k-1}$$

and $R_1 < R_2 < \ldots < R_{k-1}$ have been defined. In the following it will be shown how n_k, ϱ_k and R_k are chosen.

Since $F_{k-1}(z)$ is a polynomial in z and $f(z)$ is transcendental it follows that for all large R,

$$M(r, F_{k-1}) < M\left(\frac{r}{2}, f\right) \quad (r \geq R). \tag{7.2}$$

Also for all large R,

$$|\alpha_{k+1}| + |\alpha_k| < M\left(\frac{r}{2}, f\right) \quad (r \geq R). \tag{7.3}$$

If we define, with $n \geq 1$,

$$g(z) = F_{k-1}(z)\, s_n\left(\frac{z}{R}\right) - \alpha_k\, \sigma_n\left(\frac{z}{R}\right),$$

then

$$g(z) - F_{k-1}(z) = \sigma_n\left(\frac{z}{R}\right)\{F_{k-1}(z) - \alpha_k\}$$

and so, for $|z| \leq \varrho_{k-1} R_{k-1} \leq R$,

$$|g(z) - F_{k-1}(z)| < K\, \frac{\varrho_{k-1} R_{k-1}}{R}\, \{M(\varrho_{k-1} R_{k-1}, F_{k-1}) + |\alpha_k|\},$$

where K depends only on $f(z)$. Hence, by taking R large enough, we can ensure that

$$|g(z) - F_{k-1}(z)| < \frac{1}{2^k} e^{-T(\varrho_{k-1}, f)} \quad (|z| \le \varrho_{k-1} R_{k-1}). \tag{7.4}$$

We now choose as R_k a value of $R > R_{k-1} + 1$ for which (7.2), (7.3) and (7.4) are satisfied.

Since 0 is a VALIRON deficient value of $f(z)$ there is a sequence of $\varrho \to \infty$, which we denote by σ, and constants $1 > \eta > 0$, $\delta > 0$ so that

$$|f(\varrho e^{i\vartheta})| < e^{-\eta T(\varrho, f)} \tag{7.5}$$

with $\varrho \epsilon \sigma$ for a set of ϑ in $[0, 2\pi]$, depending on ϱ, of measure at least δ. On the other hand, since $F_{k-1}(z) - \alpha_k$ is a polynomial, for all sufficiently large ϱ we have

$$|F_{k-1}(z)| + |\alpha_k| < e^{\frac{\eta}{2} T(\varrho, f)} \quad (|z| = \varrho R_k). \tag{7.6}$$

We choose as ϱ_k a value $\varrho \epsilon \sigma$ such that $\varrho > \varrho_{k-1}$ and (7.5) and (7.6) are satisfied with $\varrho = \varrho_k$.

We now choose n_k so that

$$|f(\varrho_k e^{i\vartheta}) - s_{n_k}(\varrho_k e^{i\vartheta})| < e^{-\eta T(\varrho_k, f)} \quad (0 \le \vartheta \le 2\pi). \tag{7.7}$$

Hence with $n = n_k$, $R = R_k$ in $g(z)$ we obtain $F_k(z)$, and, from (7.5), (7.6) and (7.7), for $z = \varrho_k R_k e^{i\vartheta}$ and ϑ in a set of measure at least δ contained in $[0, 2\pi]$,

$$|F_k(z) - \alpha_k| = \left| s_{n_k}\left(\frac{z}{R_k}\right) \right| |F_{k-1}(z) - \alpha_k| < 2 e^{-\frac{\eta}{2} T(\varrho_k, f)}.$$

To sum up, assuming $n_1, \ldots, n_{k-1}$; $\varrho_1, \ldots, \varrho_{k-1}$ and $R_1, \ldots, R_{k-1}$ to be defined we can choose n_k, ϱ_k, R_k so that $\varrho_k > \varrho_{k-1} \ge 1$ and $\varrho_k \to \infty (k \to \infty)$, $R_k > R_{k-1}$ and $R_k \to \infty$ $(k \to \infty)$ and

i) $$M(r, F_{k-1}) < M\left(\frac{r}{2}, f\right) \quad (r \ge R_k),$$

ii) $$|\alpha_k| + |\alpha_{k+1}| < M\left(\frac{r}{2}, f\right) \quad (r \ge R_k),$$

iii) $$|F_k(z) - F_{k-1}(z)| \le \frac{1}{2^k} e^{-T(\varrho_{k-1}, f)} \quad (|z| \le \varrho_{k-1} R_{k-1}),$$

iv) $$|F_k(z) - \alpha_k| < 2e^{-\frac{\eta}{2}T(\varrho_k, f)}$$

for $z = \varrho_k R_k e^{i\vartheta}$ and ϑ belonging to a set of measure at least δ in $[0, 2\pi]$,

v) $T(\varrho_k R_k, F_k) < KT(\varrho_k, f)$ (from (7.6) and (7.7) and the definition of F_k).

Hence, by induction, there are sequences $\{n_k\}_1^\infty$, $\{\varrho_k\}_1^\infty$ and $\{R_k\}_1^\infty$ for which i)–v) are satisfied and $\varrho_k \to \infty$ $(k \to \infty)$, $R_k \to \infty$ $(k \to \infty)$.

We shall now show that $F_k(z) \to F(z)$ $(k \to \infty)$ where $F(z)$ is an integral function satisfying (2.5), (2.6) and (7.1). From iii), together with $\varrho_k \to \infty$, $R_k \to \infty (k \to \infty)$, it is clear that $\sum_{k=0}^{\infty} \{F_{k+1}(z) - F_k(z)\}$ converges uniformly in $|z| \le R$ for any fixed R and so $F_k(z) \to F(z)$ $(k \to \infty)$, where $F(z)$ is an integral function.

Lemma 1. *We have*

$$|F_k(z)| \le KM^2\left(\frac{r}{2}, f\right) \quad (r = |z| \ge R_k), \tag{7.8}$$

$$|F_k(z)| \le KM^3\left(\frac{r}{2}, f\right) \quad (r = |z| \ge R_{k-1}), \tag{7.9}$$

where K is a constant depending only on $f(z)$.

Proof. We have

$$|F_k(z)| \le |F_{k-1}(z)| \left|s_{n_k}\left(\frac{z}{R_k}\right)\right| + |\alpha_k| \left|\sigma_{n_k}\left(\frac{z}{R_k}\right)\right| \tag{7.10}$$

$$\le M\left(\frac{r}{2}, f\right)\left\{\left|s_{n_k}\left(\frac{z}{R_k}\right)\right| + \left|\sigma_{n_k}\left(\frac{z}{R_k}\right)\right|\right\} \quad (|z| \ge R_k) \tag{7.11}$$

from i) and ii). Also if $R_k > 3$ then, for all r,

$$\left|s_{n_k}\left(\frac{z}{R_k}\right)\right| \le KM\left(\frac{r}{2}, f\right), \qquad \left|\sigma_{n_k}\left(\frac{z}{R_k}\right)\right| \le KM\left(\frac{r}{2}, f\right) \tag{7.12}$$

by a straightforward application of CAUCHY's inequality and PARSEVAL's Theorem. From (7.11) and (7.12) we get (7.8).

If we now apply ii) and (7.8) with $k-1$ in place of k to (7.10) we obtain

$$|F_k(z)| \leq KM^2\left(\frac{r}{2}, f\right)\left\{\left|s_{n_k}\left(\frac{z}{R_k}\right)\right| + \left|\sigma_{n_k}\left(\frac{z}{R_k}\right)\right|\right\} \quad (|z| \geq R_{k-1}). \tag{7.13}$$

From (7.12) and (7.13) we get (7.9).

Now consider $R_k \leq |z| \leq R_{k+1}$ and

$$F(z) = F_{k+1}(z) + \sum_{\nu=k+1}^{\infty} \{F_{\nu+1}(z) - F_\nu(z)\}. \tag{7.14}$$

From (7.9) we have

$$|F_{k+1}(z)| \leq KM^3\left(\frac{r}{2}, f\right) \quad (R_k \leq |z| \leq R_{k+1}). \tag{7.15}$$

From iii) we have

$$\sum_{\nu=k+1}^{\infty} |F_{\nu+1}(z) - F_\nu(z)| = O(1) \quad (R_k \leq |z| \leq R_{k+1}) \tag{7.16}$$

From (7.14), (7.15) and (7.16) it follows that $F(z)$ satisfies (2.5) and hence (2.6).

Consider now $F(z) - \alpha_k$ on $|z| = \varrho_k R_k$. We have

$$F(z) - \alpha_k = F_k(z) - \alpha_k + \sum_{\nu=k}^{\infty} \{F_{\nu+1}(z) - F_\nu(z)\}.$$

From iii),

$$\left|\sum_{\nu=k}^{\infty} \{F_{\nu+1}(z) - F_\nu(z)\}\right| \leq \frac{1}{2^{k-1}} e^{-T(\varrho_k, f)} \quad (|z| = \varrho_k R_k) \tag{7.17}$$

since $T(r, f)$ is an increasing function of r, From iv),

$$|F_k(z) - \alpha_k| < 2e^{-\frac{\eta}{2}T(\varrho_k, f)} \tag{7.18}$$

when $z = \varrho_k R_k e^{i\vartheta}$ and ϑ belongs to a set of measure at least δ in $[0, 2\pi]$. Hence, from (7.17) and (7.18),

$$|F(z) - \alpha_k| < 3e^{-\frac{\eta}{2}T(\varrho_k, f)} \tag{7.19}$$

when $z = \varrho_k R_k e^{i\vartheta}$ and ϑ belongs to a set of measure at least δ in $[0, 2\pi]$. Consequently

$$m\left(\varrho_k R_k, \frac{1}{F-\alpha_k}\right) \geq \frac{\delta\eta}{4\pi} T(\varrho_k, f) + O(1). \tag{7.20}$$

Since, from v),

$$T(\varrho_k R_k, F_k) < K T(\varrho_k, f)$$

it follows that, when account is taken of (7.17),

$$T(\varrho_k R_k, F) < K T(\varrho_k, f). \tag{7.21}$$

From (7.20) and (7.21) we obtain (7.1).

This completes the proof of the theorem.

§ 8. In the proof of Theorem 4 we need the following result of HAYMAN [2]:

Theorem B. *If an integral function* $\psi(z)$ *satisfies*

$$\log M(r, \psi) = O((\log r)^2) \quad (r \to \infty) \tag{8.1}$$

then

$$\log |\psi(z)| \sim \log M(r, \psi)$$

uniformly in ϑ *as* $z = re^{i\vartheta} \to \infty$ *outside an* $\mathcal{E}$*-set.*

We note the following two facts about $\mathcal{E}$-sets.

a) The union of two $\mathcal{E}$-sets is again an $\mathcal{E}$-set.

b) Given any $\mathcal{E}$-set then for almost all fixed ϑ and $r > r_0(\vartheta)$, $z = re^{i\vartheta}$ lies outside the $\mathcal{E}$-set.

If $f(z)$ satisfies the hypotheses of Theorem 4 then it is easy to see that we may write $f(z) = (f_1(z)) / (f_2(z))$ where $f_1(z)$ and $f_2(z)$ are integral functions having no zeros in common and both satisfying (8.1). Thus using (a) above we see that

$$\log |f(z)| = \log |f_1(z)| - \log |f_2(z)| = (1 + o(1)) (\log M(r, f_1) - \log M(r, f_2)) \tag{8.2}$$

uniformly as $z = re^{i\vartheta} \to \infty$ outside on $\mathcal{E}$-set.

By performing a bilinear transformation if necessary we may assume that infinity is the deficient value of $f(z)$ i.e. $\delta(\infty) > 0$. The functions $f(z)$, $f_1(z)$ and $f_2(z)$ all satisfy the conditions of Theorem A. Thus

$$\log M(r, f_j) \sim N(r, 0, f_j) \quad (r \to \infty)\,(j = 1, 2)$$

and moreover since $\delta(\infty, f) > 0$

$$N(r, 0, f) \sim T(r, f) \quad (r \to \infty)$$

$$N(r, \infty, f) < (1 - \delta(\infty, f) + o(1))\, T(r, f) \quad (r > r_0).$$

But $N(r, 0, f) = N(r, 0, f_1)$ and $N(r, \infty, f) = N(r, 0, f_2)$. Thus, by substituting into (8.2) we obtain, as $z = re^{i\vartheta} \to \infty$ outside an $\mathcal{E}$-set,

$$\begin{aligned} \log |f(z)| &= (1 + o(1))\, (N(r, 0, f) - N(r, \infty, f)) \\ &> (1 + o(1))\, T(r, f) - (1 - \delta(\infty, f) + o(1))\, T(r, f) \\ &= (\delta(\infty, f) + o(1))\, T(r, f). \end{aligned}$$

Thus we obtain

$$\liminf \frac{\log |f(z)|}{T(r)} \geq \delta(\infty, f)$$

uniformly in ϑ as $z = re^{i\vartheta}$ tends to infinity outside an $\mathcal{E}$-set as required.

By using the property (*b*) above and the fact that the characteristic $T(r)$ is unbounded we have that $f(re^{i\vartheta}) \to a$ as $(r \to \infty)$ for almost all fixed $\vartheta\,(0 \leq \vartheta < 2\pi)$. Thus a is a fortiori an asymptotic value.

To prove Theorem 5 we suppose again that $\delta(\infty, f) > 0$ and that for all z

$$|z|\, \varrho(f(z)) < K.$$

Then, since $f(z)$ is transcendental and satisfies (2.1) it must have infinitely many zeros. Suppose we are given $\varepsilon > 0$. Then there exists a sequence $z_\nu = R_\nu e^{i\vartheta_\nu}$ of zeros of $f(z)$ such that each disc $|z - z_\nu| < \varepsilon R_\nu$ contains a point z'_ν where $|f(z'_\nu)| > 1$. Otherwise the set outside of which (3.1) holds would need to subtend angles at the origin whose sum diverges (i.e. would not be an $\mathcal{E}$-set). Now consider the images of $f(z_\nu)$ and $f(z'_\nu)$ on the RIEMANN sphere. We have

$$\int_{z_\nu}^{z'_\nu} \varrho(f(z))\, |dz| > \frac{\pi}{4}$$

for any path joining z_ν to z'_ν. Thus

$$\frac{\pi}{4} \leq K \int_{z_\nu}^{z'_\nu} \frac{|dz|}{|z|} \leq K \log (1 + \varepsilon)$$

by the definition of z'_ν. This holds for all $\varepsilon > 0$, and a contradiction follows on letting $\varepsilon \to 0$. This completes the proof of Theorem 5.

Note added in proof. Mr. D. SHEA has pointed out to us that the result of Theorem 4 is stated in § 5 of [7]. However, although VALIRON may have seen how to prove Theorem 4 from the appropriate result of § 1 of [7], it is clear that Theorem 4 does not follow directly from this result as VALIRON seems to assert.

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