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A 3-Fold Vector Product in R^8

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An r-fold vector product in R^n is, according to Eckmann [2], a continuous map

$$X : R^{rn} = R^n \times R^n \times \ldots \times R^n \to R^n$$

satisfying (i) $X(a_1, a_2, \ldots, a_r) \cdot a_i = 0, \; 1 \leqq i \leqq n,$

(ii) $\| X(a_1, a_2, \ldots, a_r) \|^2 = |a_i \cdot a_j|.$

It can be deduced from [2] and [4] that R^n admits an r-fold vector product in precisely the following cases:

a) $r = 1$ and n is even,

b) $r = n - 1$,

c) $r = 2$ and $n = 3, 7$,

d) $r = 3$ and $n = 8$.

Explicit formulas for these vector products are known in all but the last case, and the purpose of this note is to give a simple formula for the last case, taking advantage of the Cayley numbers. Such a formula was useful to the author in examining homotopy classes of vector fields on S^7 (cf. [5]).

1. The Cayley Numbers

We take the Cayley numbers C to be the 8-dimensional algebra over the reals having basis vectors $e_1, e_2, \ldots, e_8$ and suitably defined products. In particular, $e_1 = 1$ is the unit and for $i, j > 1$ one has $e_i e_j = \begin{cases} -1, & i = j \\ -e_j e_i, & i \neq j. \end{cases}$ The subspace of C orthogonal to e_1 is called the pure Cayley numbers. Conjugation is defined by writing any Cayley number x as $x = r e_1 + x'$, where x' is pure, and setting $\bar{x} = r e_1 - x'$. The Cayley numbers inherit the norm and scalar product of R^8, and

$$x\bar{x} = \bar{x}x = x \cdot x = |x|^2.$$

Lemma 1.1. For any CAYLEY numbers a, b, one has $\bar{\bar{a}} = a$, $\overline{ab} = \bar{b}\bar{a}$, and $a\bar{b} + b\bar{a} = 2(a \cdot b)$.

Proof: By linearity it suffices to prove these when a, b are the units $e_1, \dots, e_8$, and these cases follow from the above remarks on multiplication.

Corollary: $\bar{a}b + \bar{b}a = 2(\bar{a} \cdot \bar{b}) = 2(a \cdot b)$.

The multiplication in C can be explicitly defined by regarding C as a 2-dimensional algebra over the quaternions H, with basis 1, k, and setting

$$(q_1 + q_2 k)(q_1' + q_2' k) = (q_1 q_1' - \bar{q}_2' q_2) + (q_2' q_1 + q_2 \bar{q}_1')k.$$

We shall need this formula to prove the next lemma.

Lemma 1.2. For any CAYLEY numbers a, b, c,

$$a(\bar{b}c) + b(\bar{a}c) = 2(a \cdot b)c.$$

Proof: Let $a = a_1 + a_2 k$, $b = b_1 + b_2 k$, $c = c_1 + c_2 k$, where a_1, a_2, b_1, b_2, c_1, c_2 are quaternions. By direct computation,

$$\begin{aligned} a(\bar{b}c) + b(\bar{a}c) = {} & (b_1\bar{a}_1 + a_1\bar{b}_1)c_1 + c_1(\bar{a}_2 b_2 + \bar{b}_2 a_2) + \\ & + [c_2(\bar{a}_1 b_1 + \bar{b}_1 a_1) + (b_2\bar{a}_2 + a_2\bar{b}_2)c_2]k. \end{aligned}$$

Applying Lemma 1.1 and its corollary, this equals

$$\begin{aligned} & 2[(b_1 \cdot a_1)c_1 + c_1(a_2 \cdot b_2) + (c_2(a_1 \cdot b_1) + (b_2 \cdot a_2)c_2)k] \\ & = 2(a_1 \cdot b_1 + a_2 \cdot b_2)(c_1 + c_2 k) \\ & = 2(a \cdot b)c. \end{aligned}$$

According to [1], the subalgebra of C generated by any two elements (and their conjugates) is associative. In particular, for any CAYLEY numbers a, b, c, the product $a(bc)a$ is well defined, and according to [3], we have $a(bc)a = (ab)(ac)$.

Lemma 1.3. For any CAYLEY numbers a, b, c, one has

(i) $a \cdot (a(\bar{b}c)) = |a|^2 (b \cdot c)$,

(ii) $c \cdot (a(\bar{b}c)) = |c|^2 (a \cdot b)$, and

(iii) $b \cdot (a(\bar{b}c)) = -|b|^2 (c \cdot a) + 2(a \cdot b)(b \cdot c)$.

Proof of (i): $2a\cdot(a(\bar{b}c)) = a\overline{(a(\bar{b}c))} + (a(\bar{b}c))\bar{a} = a((\bar{c}b)\bar{a}) + (a(\bar{b}c))\bar{a}$. Since a, $\bar{a}$, $\bar{c}b$, and $\overline{\bar{c}b} = \bar{b}c$ generate an associative subalgebra, this equals

$$a(\bar{c}b)\bar{a} + a(\bar{b}c)\bar{a} = a(\bar{c}b + \bar{b}c)\bar{a} = 2a(b\cdot c)\bar{a} = 2|a|^2(b\cdot c).$$

Proof of (ii): $2c\cdot(a(\bar{b}c)) = c((\bar{c}b)\bar{a}) + (a(\bar{b}c))\bar{c} = 2r$, say, where r must be a real number. Then making use of the preceding results without specific mention, we have

$$\begin{aligned} rc &= c((\bar{c}b)\bar{a})c + (a(\bar{b}c))|c|^2 \\ &= (c(\bar{c}b))(\bar{a}c) + (a(\bar{b}c))|c|^2 \\ &= |c|^2(b(\bar{a}c) + a(\bar{b}c)) \\ &= 2|c|^2(a\cdot b)c. \end{aligned}$$

Proof of (iii): $2b\cdot(a(\bar{b}c)) = b((\bar{c}b)\bar{a}) + (a(\bar{b}c))\bar{b}$

$$\begin{aligned} &= b[(-\bar{b}c + 2b\cdot c)\bar{a}] + [a(-\bar{c}b + 2b\cdot c)]\bar{b} \\ &= 2(b\cdot c)(b\bar{a} + a\bar{b}) - b((\bar{b}c)\bar{a}) - (a(\bar{c}b))\bar{b} \\ &= 4(b\cdot c)(a\cdot b) - 2r, \text{ for some real number } r. \end{aligned}$$

Computing rb as in (ii), one finds $r = |b|^2(c\cdot a)$.

2. Applications to Vector Products

A 2-fold vector product in R^7 is obtained (cf. [2]) by identifying R^7 with the pure Cayley numbers, and for any two vectors b, c in R^7 putting $X_2(b, c) = bc + b\cdot c$. We shall now define a 3-fold vector product in R^8, which in a certain sense generalizes X_2.

Theorem 2.1. Let a, b, c be vectors in R^8, regarded as the Cayley numbers. Then

$$X_3(a, b, c) = -a(\bar{b}c) + a(b\cdot c) - b(c\cdot a) + c(a\cdot b)$$

defines a 3-fold vector product in R^8.

Proof: Lemma 1.3 will be frequently used without specific mention, and we write $X = X_3(a, b, c)$. The continuity of X_3 is obvious. Thus $a\cdot X = -$ $-|a|^2(b\cdot c) + |a|^2(b\cdot c) - (a\cdot b)(c\cdot a) + (a\cdot c)(a\cdot b) = 0$, and the proofs that $b\cdot X = 0$, $c\cdot X = 0$, are entirely similar. Finally, writing $d = (a(\bar{b}c))$, we have

$$\begin{aligned}|X|^2 = X\cdot X &= |d|^2 + |a|^2(b\cdot c)^2 + |b|^2(a\cdot c)^2 + |c|^2(a\cdot b)^2 - 2(a\cdot d)(b\cdot c)\\ &\quad + 2(b\cdot d)(c\cdot a) - 2(c\cdot d)(a\cdot b) + (-2+2-2)(a\cdot b)(b\cdot c)(c\cdot a)\\ &= |a|^2|b|^2|c|^2 + |a|^2(b\cdot c)^2 + |b|^2(a\cdot c)^2 + |c|^2(a\cdot b)^2\\ &\quad - 2|a|^2(b\cdot c)^2 - 2|b|^2(a\cdot c)^2 + 4(a\cdot b)(b\cdot c)(c\cdot a) - 2|c|^2(a\cdot b)^2\\ &\quad - 2(a\cdot b)(b\cdot c)(c\cdot a)\\ &= |a|^2|b|^2|c|^2 - |a|^2(b\cdot c)^2 - |b|^2(a\cdot c)^2 - |c|^2(a\cdot b)^2 + 2(a\cdot b)(b\cdot c)(c\cdot a)\\ &= \begin{vmatrix} |a|^2 & a\cdot b & a\cdot c\\ b\cdot a & |b|^2 & b\cdot c\\ c\cdot a & c\cdot b & |c|^2\end{vmatrix}.\end{aligned}$$

When restricted to the STIEFEL manifold $V_{7,2}$, the 2-fold product X_2 gives a cross section $X_2 : V_{7,2} \to V_{7,3}$. Similarly, X_3 gives a cross section $X_3 : V_{8,3} \to V_{8,4}$. Then the fact that $X_3(e_1, b, c) = bc + b\cdot c = X_2(b, c)$, where $(e_1, b, c) \in V_{8,3}$ (and hence b, c are pure), implies that the following diagram commutes:

$$\begin{tikzcd} V_{7,2} \arrow[r, "i"] \arrow[d, "X_2"] & V_{8,3} \arrow[d, "X_3"] \\ V_{7,3} \arrow[r, "i'"] & V_{8,4} \end{tikzcd}.$$

Further details on these vector products can be found in [5].

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