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Convex functionals and generalized harmonic maps into spaces of non positive curvature

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Introduction

The theory of harmonic maps between compact Riemannian manifolds is well developed under the assumption that the image has nonpositive sectional curvature (see e.g. [A1], [A2], [ES], [H], [DO], [D], [C1], [JY3], [La]) and has found important applications (see the introduction of [JY4] for a survey). These applications in turn led Gromov-Schoen [GS] to consider harmonic maps into more general metric spaces. With further applications in mind, a theory of generalized harmonic maps between metric spaces was developed in [J] and [KS]. These latter papers in particular treat the existence of harmonic maps into non locally compact target spaces while the domain still needs to satisfy some compactness (and in [KS] in addition some smoothness and other) properties.

It is one of the purposes of the present paper to abandon all hypotheses on the domain, apart from those structures required to make the definition of a generalized harmonic map meaningful. This definition which is taken from [J] is given at the beginning of §2. (A similar definition was achieved in [KS]). An advantage of this definition is that it puts the well developed theory of Γ-convergence (see [dM]) at our disposal.

The main result of §2 then is the following

THEOREM. *Let X_1, X_2 be metric spaces. Assume that X_2 is complete and nonpositively curved in the sense of Alexandrov (see §1 for the definition, in particular, X_2 is simply connected). Let Γ be a subgroup of the isometry group of X_1, and suppose the measures on X_1 required for defining the energy of a map from X_1 are Γ-equivariant. Let $\rho : \Gamma \to I(X_2)$ be a reductive homomorphism into the isometry group of X_2. If there exists a ρ-equivariant map $f: X_1 \to X_2$ (i.e. $f(\gamma x) = \rho(\gamma) f(x)$ for all $x \in X_1, \gamma \in \Gamma$) of finite energy, then there also exists a ρ-equivariant harmonic map from X_1 to X_2.*

ρ-equivariant maps include, but are more general than maps between quotients of X_1 and X_2. Thus, there are essentially three hypotheses:

(M1) The target X_2 has nonpositive curvature in the sense of Alexandrov.
(M2) ρ is reductive.
(M3) There exists a finite energy map in the class under consideration.

Alexandrov's definition of nonpositive curvature as required in (M1) includes Riemannian manifolds of nonpositive sectional curvature. Other examples of such nonpositively curved spaces that are important for applications are trees and Euclidean buildings. Even in the case of smooth Riemannian manifolds, no general condition other than nonpositive sectional curvature of the image so far has been found that allows to construct a theory of harmonic maps that is strong enough for far reaching geometric applications. Therefore, (M1) seems to be a natural and acceptable assumption, and in applications, it is usually easy to verify.

(M1) is the most important one among the three hypotheses for the present paper. This curvature condition entails certain convexity properties of the distance function that will be crucial for the constructions of §1. In that §, we study convex functionals $F: Y \to \mathbb{R} \cup \{\infty\}$ on a complete metric space Y, and we seek minimizers of such functionals via Moreau–Yosida regularization. This means that for $x \in Y$ and $\lambda > 0$, we put

$$F^\lambda(x) := \inf_{y \in Y} (\lambda F(y) + d^2(x, y))$$

($d(.,.)$ denotes the distance function on Y). If Y is complete and nonpositively curved, this infimum is realized by a unique point $J_\lambda(x) = y_\lambda$. The main theorem of §1 says that if $(y_{\lambda_n})_{n \in \mathbb{N}}$ is bounded for some sequence $\lambda_n \to \infty$, then $(y_\lambda)_{\lambda > 0}$ converges to a minimizer of F as $\lambda \to \infty$. The existence result for harmonic maps then easily follows by letting Y be the space of ρ-equivariant maps from X_1 to X_2 that are locally of class L^2.

(M2) prevents minimizing sequences from escaping to ∞. It is a necessary hypothesis that is usually easy to verify in concrete applications.

(M3) can be much harder to check. The generality attempted in the present paper does not allow to study this hypothesis in more detail. Examples where it has been successfully verified can be found in [JY1], [JY2], [C2], [JZ1], [JZ2]. In other cases, the existence of a finite energy map is open, and this sometimes presents the only obstacle for the application of harmonic maps to a geometric problem.

In the present paper, we do not study regularity questions. Regularity results for generalized harmonic maps have been obtained in [GS] and [KS]. Any such regularity result necessarily needs additional assumptions on the domain that are more special than compatible with the general framework adopted here.

As mentioned above, in §1, we develop a general theory of convex functionals $F: Y \to \mathbb{R} \cup \{\infty\}$ on a complete metric space of nonpositive curvature, and this represents the second main purpose of the second paper. In nonlinear analysis, one often studies convex functionals on Banach spaces, or somewhat more general topological vector spaces. While the functional is nonlinear, the underlying space still has a linear structure. The method of Moreau–Yosida approximation in this context is well presented in [At]. In order to develop an analytic theory that more truely deserves the epithet "nonlinear", we wish to study functionals in spaces that only carry a complete metric, but not necessarily a linear structure. Without further assumptions, however, this might be too general a setting for obtaining strong analytical results. We find that the assumption on the space of nonpositive curvature in the sense of Alexandrov ties in very well with the convexity assumption on the functional, and that the method of Moreau–Yosida approximation can be extended to that setting. Even some resolvent type identities that one might suspect to depend crucially on some linear structure in fact still hold in the present fully nonlinear setting. (While these identities are not needed for our harmonic map results, we still present them here for use in a future paper.)

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1. Convex functionals on spaces of nonpositive curvature

Let Y be a simply connected, complete metric space in which any two points can be connected by a shortest arc. In particular, Y is connected, and its metric is intrinsic (cf. [N]). We also assume that Y has nonpositive curvature in the sense of Alexandrov, that means (cf. [N]) that whenever

$$\gamma : [0, b] \to Y$$

$$\tilde{\gamma} : [0, b] \to \mathbb{R}^2$$

are geometric arcs parametrized by arclength ($\mathbb{R}^2$ is equipped with its Euclidean metric, and so $\tilde{\gamma}$ is a straight line) and $p \in Y$ and $\tilde{p} \in \mathbb{R}^2$ satisfy

$$d(p, \gamma(0)) = d(\tilde{p}, \tilde{\gamma}(0))$$

$$d(p, \gamma(b)) = d(\tilde{p}, \tilde{\gamma}(b))$$

then for all $t \in [0, b]$

$$d(p, \gamma(t)) \le d(\tilde{p}, \tilde{\gamma}(t)) \tag{1.1}$$

where the distances on the left hand sides are taken in Y and those on the right hand sides are the Euclidean ones of $\mathbb{R}^2$.

Nonpositive curvature implies that the shortest connection between any two points is unique. Also for any $x_0 \in Y$,

$$d^2(.\,, x_0)$$

is a convex function.

For $x_1, x_2 \in Y$, we define their mean value $m(x_1, x_2)$ as the unique point on the geodesic arc from x_1 to x_2 that has equal distances to x_1 and x_2.

For $x \in Y$ and $r > 0$, we put

$$B(x, r) := \{y \in Y: d(x, y) \leq r\}.$$

Let $D(F) \subset Y$, and let $F : D(F) \to \mathbb{R}$ be a functional. We say that F is densely defined if $D(F)$ is dense in Y. We say that F is convex if whenever $\gamma : [0, 1] \to Y$ is a geodesic arc parametrized proportionally to arclength, and if $\gamma(0), \gamma(1) \in D(F)$, then also $\gamma(t) \in D(F)$ and

$$F(\gamma(t)) \leq tF(\gamma(0)) + (1-t)F(\gamma(1)) \tag{1.2}$$

for all $t \in [0, 1]$. If F is convex and $A \subset D(F)$, then also the convex hull C_A of A is contained in $D(F)$, see Lemma 2.6 of [J]. We extend any convex functional $F : D(F) \to \mathbb{R}$ to a functional

$$F : Y \to \mathbb{R} \cup \{\infty\}$$

by putting

$$F(y) = \infty \qquad \text{if } y \in Y \backslash D(F).$$

The extension still satisfies the inequality (1.2) characterizing convexity, since if the left hand side takes the value ∞, so does the right hand side. We therefore call any functional $F: Y \to \mathbb{R} \cup \{\infty\}$ convex if it satisfies (1.2) in this extended sense.

DEFINITION 1. Let $F : Y \to \mathbb{R} \cup \{\infty\}$. For $\lambda > 0$, the Moreau–Yosida approximation F^λ of F is defined as

$$F^\lambda(x) := \inf_{y \in Y} (\lambda F(y) + d^2(x, y)). \tag{1.3}$$

LEMMA 1. *Let $F : Y \to \mathbb{R} \cup \{\infty\}$ be a function, $\lambda, \mu > 0$. We then have the resolvent equation*

$$\frac{1}{\mu}\left(\frac{1}{\lambda}F^{\lambda}\right)^{\mu} = \frac{1}{\lambda+\mu}F^{\lambda+\mu} \tag{1.4}$$

Proof.

$$\begin{aligned}\frac{1}{\mu}\left(\frac{1}{\lambda}F^{\lambda}\right)^{\mu}(x) &= \inf_{y \in Y}\left(\frac{1}{\lambda}F^{\lambda}(y) + \frac{1}{\mu}d^2(x, y)\right)\\ &= \inf_{y \in Y}\left(\inf_{z \in Y}\left(F(z) + \frac{1}{\lambda}d^2(y, z)\right) + \frac{1}{\mu}d^2(x, y)\right).\end{aligned}$$

Now for each $z \in Y$,

$$\inf_{y \in Y}\left(\frac{1}{\lambda}d^2(y, z) + \frac{1}{\mu}d^2(x, y)\right)$$

is realized by a unique point y_0, namely the point on the geodesic arc from x to z with

$$d(x, y_0) = \frac{\mu}{\lambda+\mu}d(x, z), \qquad d(z, y_0) = \frac{\lambda}{\lambda+\mu}d(x, z).$$

Thus

$$\frac{1}{\lambda}d^2(y_0, z) + \frac{1}{\mu}d^2(x, y_0) = \frac{1}{\lambda+\mu}d^2(x, z),$$

and

$$\frac{1}{\mu}\left(\frac{1}{\lambda}F^{\lambda}\right)^{\mu}(x) = \inf_{z \in Y}\left(F(z) + \frac{1}{\lambda+\mu}d^2(x, z)\right) = \frac{1}{\lambda+\mu}F^{\lambda+\mu}(x). \qquad \text{q.e.d.}$$

LEMMA 2. *We assume that F is convex, $\not\equiv \infty$, and lower semicontinuous. For every $x \in Y$ and $\lambda > 0$, there exists a unique $y_\lambda \in Y$ with*

$$\lambda F(y_\lambda) + d^2(x, y_\lambda) = F^{\lambda}(x).$$

We write $y_\lambda = J_\lambda(x)$.

Proof. We have to show that the infimum in (1.3) is realized by a unique y_λ. Uniqueness is easy: If there were two different such points y_λ^1, y_λ^2, we could take their mean value y_λ^0. By convexity of F

$$F(y_\lambda^0) \leq \frac{1}{2}(F(y_\lambda^1) + F(y_\lambda^2))$$

and by nonpositive curvature of Y,

$$d^2(x, y_\lambda^0) < \frac{1}{2}(d^2(x, y_\lambda^1) + d^2(x, y_\lambda^2)),$$

and hence

$$\lambda F(y_\lambda^0) + d^2(x, y_\lambda^0) < \lambda F(y_\lambda^1) + d^2(x, y_\lambda^1) = \lambda F(y_\lambda^2) + d^2(x, y_\lambda^2)$$

contradicting the minimizing property of y_λ^1 and y_λ^2. In order to prepare the existence proof, we observe that for any two points $y_1, y_2 \in Y$, their mean value $y_0 = m(y_1, y_2)$ is the midpoint of the geodesic arc connecting y_1 and y_2, and it satisfies

$$d^2(x, y_0) \leq \frac{1}{2}(d^2(x, y_1) + d^2(x, y_2)) - \frac{1}{4} d^2(y_1, y_2) \tag{1.5}$$

because that inequality holds for the Euclidean metric and Y has nonpositive curvature. We now let $(y_n)_{n \in \mathbb{N}}$ be a minimizing sequence, i.e.

$$\lambda F(y_n) + d^2(x, y_n) \longrightarrow \inf_{y \in Y} (\lambda F(y) + d^2(x, y)) =: \kappa_\lambda. \tag{1.6}$$

We claim that y_n is a Cauchy sequence. For $l, k \in \mathbb{N}$ we let $y_{k,l} := m(y_k, y_l)$. Using the convexity of F as in the uniqueness argument and the stronger version of convexity for $d^2(x, .)$, (1.5), we obtain

$$\lambda F(y_{k,l}) + d^2(x, y_{k,l})$$
$$\leq \frac{1}{2}(\lambda F(y_k) + d^2(x, y_k)) + \frac{1}{2}(\lambda F(y_l) + d^2(x, y_l)) - \frac{1}{4} d^2(y_k, y_l).$$

By definition of κ_λ, the left hand side cannot be smaller than κ_λ, and so we obtain from (1.6) that $d^2(y_k, y_l) \to 0$ as $k, l \to \infty$, and $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence indeed. Since the distance function is continuous and F is assumed to be lower semicontinuous, the limit of (y_n) then is the desired y_λ. q.e.d.

COROLLARY 1. *Let $F: Y \to \mathbb{R} \cup \{\infty\}$ be convex and lower semicontinuous, $x \in Y, y_\lambda = J_\lambda(x)$ for some $\lambda > 0$. For $0 \leq s \leq 1$, we define $y_{\lambda,s}$ as follows: Let $\gamma: [0, 1] \to Y$ be the geodesic arc with $\gamma(0) = x, \gamma(1) = y_\lambda$, parametrized proportionally to arclength, and put*

$$y_{\lambda,s} := \gamma(s).$$

Then

$$J_{(1-s)\lambda}(y_{\lambda,s}) = y_\lambda. \tag{1.7}$$

Proof. Given the uniqueness result of Lemma 2, this follows from the proof of Lemma 1. q.e.d.

LEMMA 3. *Let Y_λ be as in Lemma 2. Assume that F is densely defined. For $\lambda \to 0$, we have*

$$y_\lambda \longrightarrow x.$$

Proof. Since F is densely defined, for every $\delta > 0$ there exists $x_\delta \in B(x, \delta)$ with $F(x_\delta) < \infty$. Then

$$\lim_{\lambda \to 0} (\lambda F(x_\delta) + d^2(x, x_\delta)) \leq \delta^2,$$

and consequently

$$\limsup_{\lambda \to 0} \kappa_\lambda \leq 0.$$

Let us now assume that there exists a sequence $\lambda_n \to 0$ for $n \to \infty$ with

$$d^2(x, y_{\lambda_n}) \geq \alpha > 0 \qquad \text{for all } n.$$

Then

$$\limsup_{n \to \infty} (\lambda_n F(y_{\lambda_n}) + d^2(x, y_{\lambda_n})) \leq 0,$$

and hence

$$F(y_{\lambda_n}) \longrightarrow -\infty \quad \text{as} \quad n \longrightarrow \infty.$$

But then

$$F(y_1) + d^2(x, y_1) \leq F(y_{\lambda_n}) + d^2(x, y_{\lambda_n}) \longrightarrow -\infty \qquad \text{as} \quad n \longrightarrow \infty,$$

which is impossible. The claim follows. q.e.d.

Remark. If F is not necessarily densely defined, the result still holds for all x in the closure of $D(F)$.

THEOREM 1. *Let $F: Y \to \mathbb{R} \cup \{\infty\}$ be convex and lower semicontinuous, and assume that F is not identically ∞. For $x \in Y$, let $y_\lambda = J_\lambda(x)$ as in Lemma 2. If $(y_{\lambda_n})_{n \in \mathbb{N}}$ is bounded for some sequence $\lambda_n \to \infty$, then $(y_\lambda)_{\lambda > 0}$ converges to a minimizer of F as $\lambda \to \infty$.*

Proof. Since (y_{λ_n}) is bounded, it is a minimizing sequence for F, i.e.

$$F(y_{\lambda_n}) \longrightarrow \inf_{y \in Y} F(y),$$

because y_{λ_n} minimizes

$$F(y) + \frac{1}{\lambda_n} d^2(x, y).$$

We now show that

$$d^2(x, y_\lambda)$$

is monotonically increasing in λ. Indeed, let $0 < \mu_1 < \mu_2$. Then by definition of y_{μ_1}

$$F(y_{\mu_2}) + \frac{1}{\mu_1} d^2(x, y_{\mu_2}) \geq F(y_{\mu_1}) + \frac{1}{\mu_1} d^2(x, y_{\mu_1}),$$

hence

$$F(y_{\mu_2}) + \frac{1}{\mu_2} d^2(x, y_{\mu_2}) \geq F(y_{\mu_1}) + \frac{1}{\mu_2} d^2(x, y_{\mu_1}) + \left(\frac{1}{\mu_1} - \frac{1}{\mu_2}\right)(d^2(x, y_{\mu_1}) - d^2(x, y_{\mu_2}))$$

which is compatible with the definition of y_{μ_2} only if

$$d^2(x, y_{\mu_1}) \leq d^2(x, y_{\mu_2}),$$

and monotonicity follows. The monotonicity then implies that

$$d^2(x, y_\lambda)$$

is bounded independently of λ since it is assumed to be bounded for the sequence $\lambda_n \to \infty$. This monotonicity also implies that

$$F(y_\lambda)$$

monotonically decreases towards

$$\inf_{y \in Y} F(y)$$

for $\lambda \to \infty$. Namely, from the definition of y_λ,

$$F(y_\lambda) = \inf_{y \in B(x, d(x, y_\lambda))} F(y),$$

and $F(y_\lambda)$ indeed decreases since $d(x, y_\lambda)$ increases as $\lambda \to \infty$.

We now show that $(y_\lambda)_{\lambda > 0}$ satisfies the Cauchy property, i.e. for every $\epsilon > 0$ there exists λ_0 such that for all $\lambda, \mu \geq \lambda_0$,

$$d^2(y_\lambda, y_\mu) < \epsilon.$$

For that purpose, we choose λ_0 so large that for $\lambda, \mu \geq \lambda_0$

$$|d^2(x, y_\lambda) - d^2(x, y_\mu)| < \frac{\epsilon}{2}$$

which is possible by the preceding monotonicity and boundedness results. We may also assume

$$F(y_,) \geq F(y_\mu).$$

We let y' be the mean value of y_λ and y_μ. Then by convexity of F and nonpositive curvature of Y,

$$F(y') + \frac{1}{\lambda} d^2(x, y')$$

$$\leq F(y_\lambda) + \frac{1}{\lambda}\left(\frac{1}{2} d^2(x, y_\lambda) + \frac{1}{2} d^2(x, y_\mu) - \frac{1}{4} d^2(y_\lambda, y_\mu)\right)$$

$$< F(y_\lambda) + \frac{1}{\lambda}\left(d^2(x, y_\lambda) + \frac{\epsilon}{4} - \frac{1}{4} d^2(y_\lambda, y_\mu)\right).$$

This, however, is compatible with the definition of y_λ only if

$$d^2(y_\lambda, y_\mu) < \epsilon.$$

Since Y is complete, (y_λ) converges for $\lambda \to \infty$ towards some $\bar{y} \in Y$ which then minimizes F because $F(y_\lambda)$ decreases to $\inf_{y \in Y} F(y)$ and F is lower semicontinuous. q.e.d.

We now establish a stabilizing property of the Moreau–Yosida approximates.

LEMMA 4. *Let $F : Y \to \mathbb{R} \cup \{\infty\}$ be convex and lower semicontinuous. Then for any $x_1, x_2 \in Y, \lambda > 0$*

$$d(J_\lambda(x_1), J_\lambda(x_2)) \leq d(x_1, x_2).$$

Proof. We put $y_i := J_\lambda(x_i)$, $i = 1, 2$, and we let $\gamma : [0, 1] \to Y$ be the geodesic arc from y_1 to y_2, parametrized proportionally to arclength. Since $F(y_1)$ and $F(y_2)$ have to be finite and F is convex, the restriction of F to γ is a bounded convex function. It then assumes its minimum at some point $y_0 \in \gamma$. If y_0 is an interior point of γ, then y_i has to be the point on γ closest to x_i, because otherwise we would decrease both the values of F and of $d^2(\,.\,, x_i)$ by moving on γ closer to y_0, contradicting the definition of y_i. In that case, however, it is an easy consequence of nonpositive curvature that

$$d(x_1, x_2) \geq d(y_1, y_2).$$

If y_0 is an endpoint of γ, say y_1, we assume

$$d(y_1, y_2) > d(x_1, x_2) \tag{1.8}$$

and shall reach a contradiction.

Since y_1 is the minimum of F on γ and since F is convex, we have for $0 < t < 1$

$$F(\gamma(t)) - F(\gamma_1) \leq F(\gamma_2) - F(\gamma(1-t)) \tag{1.9}$$

(recall $y_1 = \gamma(0), y_2 = \gamma(1)$). A consequence of Reshetnyak's quadrilateral comparison theorems ([Re]), namely Formula (2.1v) of [KS], implies

$$\begin{aligned} d^2(\gamma(t), x_1) + d^2(\gamma(1-t), x_2) \leq d^2(x_1, y_1) &+ d^2(x_2, y_2) \\ &+ td^2(x_1, x_2) - td^2(y_1, y_2) \\ &+ 2t^2 d^2(y_1, y_2) \\ &- t(d(x_1, x_2) + d(y_1, y_2))^2. \end{aligned} \tag{1.10}$$

For sufficiently small $t > 0$, we then conclude from our assumption (1.8)

$$d^2(\gamma(1-t), x_2) - d^2(y_2, x_2) < d^2(y_1, x_1) - d^2(\gamma(t), x_1). \tag{1.11}$$

From (1.9), (1.11), we obtain for such $t > 0$

$$\begin{aligned} \lambda F(\gamma(1-t)) + d^2(\gamma(1-t), x_2) &< \lambda F(y_2) + d^2(y_2, x_2) \\ &\quad + \{\lambda F(y_1) + d^2(y_1, x_1) - (\lambda F(\gamma(t)) + d^2(\gamma(t), x_1)\} \\ &\leq \lambda F(y_2) + d^2(y_2, x_2) \end{aligned}$$

by definition of $y_1 = J_\lambda(x_1)$. This, however, contradicts the definition of $y_2 = J_\lambda(x_2)$. Thus, (1.8) cannot hold. q.e.d.

2. Existence of harmonic maps between metric spaces

We recall the definition of equilibrium maps of [J] (a related, though less general construction was given in [KS]). Let X_1 and X_2 be metric spaces with metrics indiscriminately denoted by $d(. , .)$. We assume that X_2 is complete. Let a measure μ as well as a family of measures μ_x^ϵ depending on $x \in X_1$ and $\epsilon > 0$ be given on X_1. A typical example is

$$\mu_x^\epsilon = \mu \llcorner \mathring{B}(x, \epsilon). \tag{2.1}$$

For $f \in L^2_{\mathrm{loc}}(X_1, X_2)$, we define

$$E_\epsilon(f) := \int \frac{\int d^2(f)x), f(y))\, d\mu_x^\epsilon(y)}{\int d^2(x, y)\, d\mu_x^\epsilon(y)}\, d\mu(x)$$

and

$$E(f) := \Gamma - \operatorname*{limit}_{\epsilon \to 0} E_\epsilon(f)$$

whenever this Γ-limit in the sense of de Giorgi (cf. [dM]) exists. Provided we are willing to restrict ourselves to some fixed subsequence $\epsilon_n \to 0$, this limit indeed exists under quite general assumptions. See for example, by Thm. 8.5 of [dM] if $L^2(X_1, X_2)$ as a metric space has a countable base. If X_1 is a finite dimensional Riemannian manifold, with μ the volume form and μ_x^ϵ as in (2.1), the existence of $E(f)$ as the Γ-limit can also be verified in an elementary manner, see [J; §1], for any complete metric space X_2. It also follows from a general result ([dM; Thm. 6.8]) that E as a Γ-limit is lower semicontinuous on $L^2(X_1, X_2)$.

For applications, it is important to consider the case of ρ-equivariant maps, where $\rho : \Gamma \to I(X_2)$ is a homomorphism from some subgroup Γ of the isometry group $I(X_1)$ of X_1 into the isometry group $I(X_2)$ of X_2. $f : X_1 \to X_2$ here is called ρ-equivariant if

$$f(\gamma x) = \rho(\gamma) f(x) \qquad \text{for all } x \in X_1, \gamma \in \Gamma.$$

Typical examples are the lifts of maps between compact quotients of X_1 and X_2.

The following definition was suggested by Scot Adams:

A subgroup $G \subset I(X_2)$ is called reductive if there exists a complete totally geodesic subspace X of X_2 that is stabilized by G with the following property: Whenever there is an unbounded sequence $(p_n)_{n \in \mathbb{N}}$ in X with

$$d(p_n, \gamma p_n) \leq \text{const.}$$

for all $\gamma \in G$ (with a constant that is allowed to depend on γ, but not on n), then G stabilizes a finite-dimensional complete, flat, totally geodesic subspace of X. ρ is called reductive if $\rho(\Gamma)$ is.

THEOREM 2. *Let X_2 be a complete metric space of nonpositive curvature in the sense of Alexandrov. Let Γ be a subgroup of the isometry group of the metric space X_1, and assume that the measures μ and μ_x^ϵ are Γ-equivariant ($\gamma_* \mu_x^\epsilon = \mu_{\gamma x}^\epsilon$ for all x, γ, and μ induces a measure μ_Γ on X_1/Γ). Let*

$$\rho : \Gamma \to I(X_2)$$

be a reductive homomorphism. Assume that there exists some ρ-equivariant $f: X_1 \to X_2$ with

$$E^\rho(f) < \infty,$$

with $E^\rho(f) := \Gamma - \mathrm{limit}_{\epsilon \to 0} E^\rho_\epsilon(f)$ (or perhaps for a subsequence $\epsilon_n \to 0$) and

$$E^\rho_\epsilon(f) = \int \frac{\int d^2(f(x), f(y))\, d\mu^\epsilon_x(y)}{\int d^2(x, y)\, d\mu^\epsilon_x(y)}\, d\mu_\Gamma(x).$$

Then there exists a ρ-equivariant equilibrium map, i.e. a map that minimizes E^ρ in the class of ρ-equivariant maps.

Proof. We define Y as the space of ρ-equivariant maps $f: X_1 \to X_2$ of class L^2 on X_1/Γ. Y then is a complete metric space, with metric given by

$$d^2(f, g) = \int_{X_1/\Gamma} d^2(f(x), g(x))\, d\mu_\Gamma(x).$$

Since X_2 has nonpositive curvature, so does Y. We then apply the results of §1 to $F = E^\rho$. We choose $f_0 \in Y$ and put for $n \in \mathbb{N}$

$$f_n := J_n(f_0) \qquad \text{(Moreau–Yosida approximation).}$$

Since E^ρ is not identically ∞, we have

$$E^\rho(f_n) < \infty.$$

We have for $\gamma \in \Gamma$

$$d^2(f_n, f_n \circ \gamma) = \int_{X_1/\Gamma} d^2(f_n(x), \rho(\gamma) f_n(x))\, d\mu_\Gamma(x).$$

Since by equivariance $J_n(f_0 \circ \gamma) = f_n \circ \gamma$, Lemma 4 implies

$$d^2(f_n, f_n \circ \gamma) \le d^2(f_0, f_0 \circ \gamma),$$

and this quantity is bounded for each γ independently of n. Thus, if f_n is unbounded, i.e. if $f_n(x)$ is unbounded on a set of positive μ_Γ measure, the reductivity assumption implies that $\rho(\Gamma)$ stabilizes a finite dimensional totally geodesic flat subspace L.

We can then search for the desired minimizer among ρ-equivariant maps from X_1 into L. Since the metric of L is Euclidean, its group of translations is commutative, and we may therefore conjugate with suitable translations, in order to construct a bounded minimizing sequence, without destroying the ρ-equivariance. Therefore, we obtain in any case a bounded minimizing sequence.

Since X_2 has nonpositive curvature, d^2 is convex, and consequently, for $\epsilon > 0$, E_ϵ is a convex functional on Y. Therefore, also

$$E : Y \to \bar{\mathbb{R}}$$

is convex by Thm. 11.1 of [dM]. By Thm. 1, we then obtain the existence of a minimizer of E. q.e.d.

Remark. Thm. 2 includes the case of infinite dimensional or other non locally compact domains. Of course, for such domains, one needs to check whether measures appropriate for the definition of harmonic maps exist. The standard examples where such measures do exist are Wiener measures or other Gaussian type measures, e.g. on loop spaces.

Our methods can also be used to treat some generalizations of harmonic maps that have been considered in Riemannian geometry. Let, for simplicity, M and N be compact Riemannian manifolds. N having nonpositive sectional curvature. Let $\sigma : M \to \mathbb{R}$ be a positive function, and for a Sobolev map $f : M \to N$, we consider

$$I(f) = \int_M |df(x)^2|\sigma(x)\, d\,\mathrm{vol}_M(x),$$

with expressions defined through the Riemannian metrics in the standard manner. It then follows from Thm. 1 and the arguments of Thm. 2 that I assumes a minimum in a given homotopy class (expressed through an equivariance condition for the lifts to universal covers).

Likewise, we can treat, for compact M again, variational problems of the type

$$H(f) := \int g_{ij}(x, f(x))\gamma^{\alpha\beta} \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\beta} \sqrt{\det(\gamma_{\mu\nu})}\, dx$$

(in local coordinates, $(\gamma_{\alpha\beta})$ is the metric of M, $(\gamma^{\alpha\beta}) = (\gamma_{\alpha\beta})^{-1}$), if for each $x \in M$, $g_{ij}(x, \cdot)$defines a Riemannian metric of nonpositive sectional curvature. We thus consider harmonic sections of fibre bundles, with nonnegatively curved fibres. The existence of energy minimizing sections in this context has been shown by Kourouma [K], and our methods reproduce his result (obtained by a different method).

REFERENCES

[A1] AL'BER, S. I., *On n-dimensional problems in the calculus of variations in the large*, Sov. Math. Dokl. *5* (1964), 700–804.

[A2] AL'BER, S. I., *Spaces of mappings into a manifold with negative curvature*, Sov. Math. Dokl. *9* (1967), 6–9.

[At] ATTOUCH, H., *Variational convergence for functions and operators*, Pitman, 1984.

[C1] CORLETTE, K., *Flat G-Bundles with canonical metrics*, J. Diff. Geom. *28* (1988), 361–382.

[C2] CORLETTE, K., *Archimedean superrigidity and hyperbolic geometry*, Ann. Math. *135* (1992), 165–182.

[D] DONALDSON, S., *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. *55* (1987), 127–131.

[dM] DAL MASO, G., *An introduction to Γ-convergence*, Birkhäuser, 1993.

[DO] DIEDERICH, K. and OHSAWA, T., *Harmonic mappings and disk bundles over compact Kähler manifolds*, Publ. Res. Inst. Math. Sci. *21* (1985), 819–833.

[ES] EELLS, J. and SAMPSON, J., *Harmonic mappings of Riemannian manifolds*, Am. J. Math. *85* (1964), 109–160.

[GS] GROMOV, M. and SCHOEN, R., *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Publ. Math. IHES *76* (1992), 165–246.

[H] HARTMAN, P., *On homotopic harmonic maps*, Can. J. Math. *19* (1967), 673–687.

[J] JOST, J., *Equilibrium maps between metric spaces*, Calc. Var. *2* (1994), 173–204.

[JY1] JOST, J. and YAU, S. T., *The strong rigidity of locally symmetric complex manifolds of rank one and finite volume*, Math. Ann. *271* (1985), 143–152.

[JY2] JOST, J. and YAU, S. T., *On the rigidity of certain discrete groups and algebraic varieties*, Math. Ann. *278*, (1987) 481–496.

[JY3] JOST, J. and YAU, S. T., *Harmonic maps and group representations*, in: B. Lawson and K. Tenenblat (eds.), *Differential Geometry and Minimal Submanifolds*, Longman Scientific, 1991, pp. 241–260.

[JY4] JOST, J. and YAU, S. T., *Harmonic maps and superrigidity*, Proc. Sym. Pure Math. *54*, Part I (1993), 245–280.

[JZ1] JOST, J. and ZUO, K., *Harmonic maps and $Sl(r, \mathbb{C})$-representations of π_1 of quasi projective manifolds*, J. Alg. Geom., to appear.

[JZ2] JOST, J. and ZUO, K., *Harmonic maps into Tits buildings and factorization of non rigid and non arithmetic representations of π_1 of algebraic varieties.*

[KS] KOREVAAR, N. and SCHOEN, R., *Sobolev spaces and harmonic maps for metric space targets*, Comm. Anal. Geom. *1* (1993), 561–569.

[K] KOUROUMA, M., *Harmonic sections of Riemannian fiber bundles.*

[La] LABOURIE, F., *Existence d'applications harmoniques tordues à valeurs dans les variétés à courbure négative*, Proc. AMS *111* (1991), 877–882.

[N] NIKOLAEV, I., *Synthetic methods in Riemannian geometry*, Lecture Notes.

[Re] RESHETNYAK, Y. G., *Nonexpanding maps in a space of curvature no greater than K*, Siberian Math. Journ. *9* (1968), 683–689.

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