Comment. Math. Helvetici 66 (1991) 580–589 0010-2571/91/040580-10$1.50 + 0.20/0


# The horizontal base locus of multiples of tautological sheaves of projectivised cotangent bundles

TIE LUO

## §0. Introduction

In this paper we are interested in the following problem: Given a smooth projective algebraic variety $X$, let us consider the projectivised cotangent bundle $V$. Let $L$ be the tautological line bundle of the ruled variety $V$. What are the base components of multiples of $L$? The solution of this would have nice applications toward that of a conjecture by Green and Griffiths (see Section 2). It is too general to deal with an arbitrary $X$. In the following we want to treat the situation for minimal algebraic surfaces of general type and make some speculations related to moduli spaces of such surfaces.

The paper comprises two parts. In the first section we are going to establish the basic concepts and try to understand the horizontal base components of multiples of $L$ using some standard techniques in dealing with algebraic surfaces of general type. The main results are Theorems (1.4) and (1.5), in which we give some description of a 'mysterious' number $t_0$ defined by the horizontal base components of multiples of $L$. In the end we also propose some problems of interest. The second section is devoted to an application of the theory developed in the first section to a conjecture of Green and Griffiths. Yau and Lu [L-Y] have proved the similar result (Corollary (2.5)). The point we want to make is that the better understanding of the related base locus would improve the results on the Green–Griffiths conjecture and this can be seen clearly from our approach. The main result in this section is Theorem (2.4), which gives the precise bound for $t_0$, under which the Green–Griffiths conjecture holds.

I would like to thank Professor S. T. Yau for informing me about their preprint, spending time in discussion of the problem and offering me the opportunity of vising Harvard University. The encouragement from Professor J. Kollár is also deeply appreciated. The suggestions from the referee are helpful.

*Notations*

We will follow basically the notations in [H] and we point out the following ones:

| | |
|---|---|
| $\chi(\mathscr{F})$: | the Euler–Poincaré characteristic of some sheaf $\mathscr{F}$. |
| $c_i^j$: | the class of $j$ intersections of the $i$-th Chern class of a variety. |
| $\lvert D\rvert$: | the complete linear system of the divisor $D$ or that of the corresponding line bundle. |
| $K_X$: | the canonical sheaf of a variety $X$. |
| Base locus of $\lvert D\rvert$: | the collection of all base point of $\lvert D\rvert$. |
| Base components of $\lvert D\rvert$: | the support of $F$, where $D$ is decomposed as the moving part $M$ plus the fixed part $F$. |
| $S^m\mathscr{F}$: | the $m$-th symmetic product of a locally free sheaf of finite rank. |
| $\mathbf{C}$: | the ground field over which the varieties are defined and the affine complex curve. |

## §1. Base components

First we want to set up the situation. Let $S$ be a minimal algebraic surface of general type, $\Omega_S$ its cotangent bundle. Let $V$ be the projectivised cotangent bundle. We have the canonical projection:

$$\pi : V = \mathbf{P}(\Omega_S) \rightarrow S.$$

Let $L$ be the tautological line bundle. We are going to work with the following additional assumption:

(1.0.1) $c_1^2 > c_2$ (geometrically this condition guarantees that high multiples of $L$ define birational maps from $V$. In the case $c_1^2 < c_2$ the computation in [S] on the hypersurfaces of $\mathbf{P}^3$ shows that multiples of $L$ have no sections).

LEMMA (1.1). *The Picard group of $V$ equals*

$$\mathbf{Z}L + \pi^* \operatorname{Pic}(S).$$

*Proof.* See [H] Page 170, for example. $\square$

DEFINITION (1.2). An irreducible codimension one subvariety $B$ is said to be a base component of multiples of $L$ if $B$ is contained in the intersection (set theoretically) of the base components of $|iL|$ for all positive $i$. $B$ is vertical if $\pi(B)$ is one dimensional. Otherwise we say $B$ is horizontal.

Now let us describe the base components of multiples of $L$. Since $S$ is a minimal surface of general type, $K_S$ is a nef and big divisor on $S$. Every divisor on $S$ can be written as a rational combination of $K_S$ with some $E$ where $E \cdot K_S = 0$. Hence every horizontal divisor on $V$ can be written, up to an integral multiple, as

$$L + t\pi^*K_S + s\pi^*E$$

where $s$ and $t$ are rational numbers and $E$ is in Pic $(S)$, by the definition and Lemma (1.1). From now on we assume that $B$ is the horizontal base components of $L$.

LEMMA (1.3). *There is an unique positive number $t_0$ such that each base component $B$ can be decomposed as*

$$B \sim m_0(L - t\pi^*K_S + s\pi^*E),$$

*where $t \le t_0$. In the case* Pic $(S) = \mathbf{Z}$, *we have*

$$B \sim m_0(L - t_0\pi^*K_S)$$

*Proof.* Let us define

$$t_0 = \sup\{t : m(L - t\pi^*K_S + s\pi^*E) > 0$$
$$\text{for some } m \in \mathbf{Z}^+,\ s \in \mathbf{Q} \text{ and } E \cdot K_S = 0\},$$

which is a finite positive real number. The first part of the lemma follows from the definition since $B$ is effective. For the second part we show that $B$ is in fact a multiple of $L - t_0\pi^*K_S$. Assume that $B$ is of the form:

$$m_1(L - t_1\pi^*K_S)$$

since $K_S$ is a generator of Pic $(S)$ over $\mathbf{Q}$ and $E = 0$ in this case. By definition $t_1 \le t_0$ and if $t_1 < t_0$, we rewrite $B$ as

$$B \sim m_1(L - t_0\pi^*K_S) + m_1(t_0 - t_1)\pi^*K_S.$$

The multiples of the first term in the sum are effective. The high multiples of the second term give many sections. It implies that high multiples of $B$ move. This contradicts the assumption that $B$ is fixed. □

Now the question is how to compute $t_0$. It could be a function of the Chern numbers. One may also need more information to obtain it. The only thing we can do at this moment is giving the following bounds.

THEOREM (1.4). *With the notations as above and the assumption of* $\operatorname{Pic}(S) = \mathbf{Z}$, *we have*

$$\frac{1 - \sqrt{\dfrac{4c_1^2 - c_2}{3c_1^2}}}{2} \leq t_0 \leq \frac{1}{2}.$$

*Proof.* To show $t_0 \leq \frac{1}{2}$, we have

$$\begin{aligned} B \in H^0(V, m(L - t_0\pi^*K_S)) &\simeq H^0(S, S^m\Omega_S \otimes (-mt_0K_S)) \\ &\simeq \operatorname{Hom}(mt_0K_S, S^m\Omega_S). \end{aligned}$$

By the semi-stability of $S^m\Omega_S$ we have

$$\frac{\deg(mt_0K_S)}{\operatorname{rank}(mt_0K_S)} \leq \frac{\deg(S^m\Omega_S)}{\operatorname{rank}(S^m\Omega_S)},$$

where the degrees are measured by the nef and big divisor $K_S$. Since $\deg(S^m\Omega_S) = [m(m+1)/2]K_S^2$ (see [B1]) and $\operatorname{rank}(S^m\Omega_S) = m + 1$, this gives

$$mt_0K_S^2 \leq \frac{1}{2}mK_S^2,$$

which implies $t_0 \leq \frac{1}{2}$.

To show

$$\frac{1 - \sqrt{\dfrac{4c_1^2 - c_2}{3c_1^2}}}{2} \leq t_0,$$

we show that for any

$$t < \frac{1 - \sqrt{\dfrac{4c_1^2 - c_2}{3c_1^2}}}{2},$$

high multiples of $B = L - t\pi^* K_S$ in fact move. Let us compute $h^0(mB)$ for $m$ big. We have $h^0(mB) = h^0(S^m \Omega_S \otimes (-mtK_S))$ on $S$. Also $\chi(S^m \Omega_S \otimes (-mtK_S))$ is a polynomial of degree 3 in $m$ with leading coefficient $B^3$, which is positive by the choice of $t$.

On the other hand by Serre's duality we have

$$\begin{aligned} h^2(S^m \Omega_S \otimes (-mtK_S)) &= h^0(S^m \Omega_S \otimes (1 + mt - m)K_S) \\ &\leq h^0(S^m \Omega_S \otimes (-mtK_S)) \end{aligned}$$

since

$$t < \frac{1 - \sqrt{\dfrac{4c_1^2 - c_2}{3c_1^2}}}{2} < \frac{m+1}{2m}$$

for $m$ sufficiently large. When that is so,

$$2h^0(mB) \geq \chi(S^m \Omega_S \otimes (-mtK_S))$$

This shows that $mB$ moves when $m$ gets large. □

For an algebraic surface with larger Picard group, by virtue of Lemma (1.3), we may write a base component $B$ as some multiple of

$$L - t\pi^* K_S + s\pi^* E,$$

where $t \leq t_0$ and $E \cdot K_S = 0$. To determine a lower bound is not easy anymore because of the negative contribution from $E^2$ by the index theorem. But for upper bound we still have

THEOREM (1.5). *With the notations as above, $t_0 \leq \frac{1}{2}$.*

*Proof.* The computation of the first part in Theorem (1.4) goes through verbatim because $E \cdot K_S = 0$. □

It is known that the coarse moduli spaces of algebraic surfaces of general type exist as quasi-projective varieties and unlike the case of curves it may happen that the moduli spaces have different irreducible components, even different connected components. To this aspect we would like to ask the following:

QUESTIONS (1.6). Let $\mathcal{M}$ be the moduli space of algebraic surfaces of general type with fixed Chern numbers. Let $S$ and $S'$ be two surfaces in the same connected component. Are the corresponding values $t_0^S$ and $t_0^{S'}$ equal? It would be interesting if one can find examples in which surfaces from different connected components possess different values. One can also ask the same kind of questions for higher dimensional varieties.

## §2. Applications toward a conjecture of Green–Griffiths

First let us recall a conjecture of Green–Griffiths [G-G]. In the surface case it says that the image of any non-trivial holomorphic map from the affine line $\mathbf{C}$ into an algebraic surface of general type lies in a fixed proper subvariety (i.e; algebraically degenerated). In [G-G] the case of varieties with comparatively high irregularities is studied. The basic idea is that of the use of the Albanese map. Along the same line Grant [CG] made some improvements. Here instead of using the theory of jet bundles in full generality, we simply consider the first order of it, i.e.; the projectivised cotangent bundle. Later we may generalise it to the weighted projectivised jet bundle. We want to quote a result from [L-Y]. The idea behind this goes back to [G-G], which also works in the cases of jet bundles.

LEMMA (2.1). *Let*

$$f:\mathbf{C}\rightarrow S$$

*be a holomorphic map from the affine curve* $\mathbf{C}$. *There is a canonical lifting* $f'$ *of* $f$ *from* $\mathbf{C}$ *to* $V$:

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{f'} & V \\ & & \downarrow{\scriptstyle \pi} \\ & & S \end{array}$$

*And the image of* $\mathbf{C}$ *under* $f'$ *is contained inside the base locus of multiples of* $L$. *Furthermore it lies in the base locus of multiples of the restriction of* $L$ *on* $B$.

*Proof.* See [L-Y]. The Proof is based on Green–Griffith's construction of negatively curved pseudometrics on discs using sections of $|H|$ and $|kL - H|$, where $H$ is a very ample line bundle and $k$ a suitable large positive integer plus a Ahlfors Lemma. □

Since the base locus of lower dimension makes no difficulty in our proof, we may assume that the image of $\mathbf{C}$ under $f'$ lies in some base component $B$. If we could prove, in some cases, the non-existence of $B$, the conjecture then would be confirmed.

Let $L'$ be the restriction of $L$ on $B$ which is equipped with the reduced structure. We may also assume that $B$ is irreducible. Since the singular locus of $B$ sing $(B)$ is one dimensional and for our purpose we may assume that the image of $\mathbf{C}$ is not totally in sing $(B)$, which is an union of finitely many curves.

The following lemma gives the intersection numbers on $V$ (see also [B2]).

LEMMA (2.2). *The intersection numbers on $V$ are*

$$L^3 = c_1^2 - c_2. \tag{2.2.1}$$

$$L^2 \cdot \pi^* K_S = L \cdot \pi^* K_S^2 = c_1^2. \tag{2.2.2}$$

$$\pi^* K_S^3 = 0. \tag{2.2.3}$$

*Proof.* (2.2.2) and (2.2.3) are clear. It follows from Grothendieck [AG] that

$$L^2 + L \cdot \pi^* c_1 + \pi^* c_2 = 0$$

on $V$. This gives (2.2.1). □

Let $L'$ be the restriction of $L$ on some base component $B$. Following this Lemma we can compute the intersection number of $L'$ on $B$, noticing that $L^2 \cdot \pi^* E = -(L \cdot \pi^* c_1 + \pi^* c^2) \cdot \pi^* E = 0$ if $E \cdot K_S = 0$,

$$(L')^2 = L^2 \cdot B = m_0 L^2 \cdot (L - t\pi^* K_S + s\pi^* E) = m_0(c_1^2 - c_2 - tc_1^2),$$

which is bigger than zero as long as

$$t_0 < \frac{c_1^2 - c_2}{c_1^2},$$

because $t \leq t_0$.

We want to compute $h^0(mL')$ by Riemann-Roch formula on $B$.

LEMMA (2.3). *With the notations as above and assume that $B$ is non-singular, then $h^0(mL') \geq$ a polynomial of degree 2 in $m$ and the leading coefficient is positive.*

*Proof.* First we have $K_V \sim 2\pi^*K_S - 2L$. If $B$ is non-singular, the adjunction formula shows that on $B$

$$K_B \sim 2\pi^*K_S - 2L + m_0(L - t\pi^*K_S + s\pi^*E).$$

Riemann-Roch formula shows $\chi(mL')$ is a polynomial of degree 2 in $m$ with leading coefficient $(L')^2$, which is positive by choice of $t_0$.

Let us compute $h^2(mL')$. By Serre's duality

$$h^2(mL') = h^0_B(2\pi^*K_S - 2L + m_0(L - t\pi^*K_S + s\pi^*E) - mL).$$

I claim that it is bounded ($=0$) when $m$ gets large. For this we have to check the negativity of the intersection number of $2\pi^*K_S - 2L + m_0\ (L - t\pi^*K_S + s\pi^*E) - mL$ with the nef and big divisor $\pi^*K_S$ on $B$, which is

$$m_0((2 - m_0t) - (1 - t)(2 - m_0t + m))c_1^2.$$

It is indeed negative when $m$ gets large. So we have proved

$$h^0(mL') \geq \chi(mL').$$

The conclusion follows immediately. □

So we have

THEOREM (2.4). *If*

$$t_0 < \frac{c_1^2 - c_2}{c_1^2}.$$

*for a minimal algebraic surface $S$ of general type. The Green–Griffiths conjecture holds.*

*Proof.* The image of $\mathbf{C}$ is in $B$ and furthermore in the base locus of $L'$ on $B$, which is an union of finitely many of curves by the above lemma if $B$ is non-singular. For singular $B$, we use Hironaka's embedded resolution of sing ($B$) in $V$ to get $\bar{V}$. As we have explained that we may assume $f'(\mathbf{C})$ is not in sing ($B$), we can lift $f'$ to a map $\bar{f}'$ from $\mathbf{C}$ to $\bar{V}$:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\bar{f}'"] & \bar{V} \arrow[d, "\sigma"] \\ & V \end{tikzcd}$$

The fact that $f'(\mathbf{C})$ is in $B$ implies that $\bar{f}'(\mathbf{C})$ is in $\bar{B}$ which is the proper transformation of $B$ in $\bar{V}$. Let $\bar{L} = \sigma^*L$, $\bar{L}'$ the restriction of $\sigma^*L$ on $\bar{B}$ and $K_{\bar{V}} = \sigma^*K_V + F$, where $F$ is a positive integral combination of exceptional divisors. Now use Riemann-Roch formula on $\bar{B}$ exactly as we did in Lemma (2.3), $\bar{L}^2 \cdot \bar{B} = L^2 \cdot B$ and $K_{\bar{V}} \cdot \bar{B} \cdot \sigma^*\pi^*K_S = K_V \cdot B \cdot \pi^*K_S$, we conclude that $h^0(m\bar{L}')$ is superior to a polynomial of degree 2 in $m$ with positive leading coefficient. Since $f'(\mathbf{C})$ is in the base locus of $L'$, $\bar{f}'(\mathbf{C})$ is in that of $\bar{L}'$, which is again a union of finitely many curves. We are done. □

COROLLARY (2.5). *If $S$ is a minimal algebraic surface of general type with $c_1^2 > 2c_2$, then the Green–Griffiths conjecture holds.*

*Proof.* In this case, recall Theorem (1.5) and Lemma (2.3), we have

$$\frac{c_1^2 - c_2}{c_1^2} > \frac{1}{2} \geq t_0.$$

We are in the situation of Theorem 2.4. □

COROLLARY (2.6). *Let $S$ be a minimal algebraic surface of general type with $c_1^2 = 2c_2$ and that $K_S$ is ample. The Green–Griffiths conjecture holds for $S$.*

*Proof.* In this case we have the stability of $S^m\Omega_S$ with respect to $K_S$. □

REMARK. By constructing double covers of the above surfaces ramified along $2mK_S$ for sufficiently large $m$, one can compute that the ratios of $c_1^2$ over $c_2$ on the double covers are going to be closer to $\frac{1}{2}$ as $m$ becomes bigger while the irregularities remain the same. These give new examples for which the Green–Griffiths' conjecture is true although both known methods do not apply.

REFERENCES

[B1] F. V. Bogomolov, *Families of Curves on a Surface of General Type*, Dolk. Akad. Nauk SSSR *236* (1977), 1041–1044.

[B2] F. V. Bogomolov, *Holomorphic Tensors and Vector Bundles on Projective Varieites*, Math. USSR Ivz. *13* (1979), 499–555.

[H] R. HARTSHORNE, *Algebraic Geometry*, Springer-Verlag New York 1977.

[AG] A. GROTHENDIECK, *La Théorie des Class de Chern*, Bull. Soc. Math. France *86* (1958), 137–154.

[CG1] C. GRANT, *Entire Holomorphic Curves in Surfaces*, Duke Math. Jour. *53* (1986), 345–358.

[CG2] C. GRANT, *Hyperbolicity of Surfaces Modulo Rational and Elliptic Curves*, Pacific Jour. of Math. *139* (1989), 241–249.

[G-G] M. GREEN and P. GRIFFITHS, *Two Applications of Algebraic Geometry to Entire Holomorphic Mappings*, The Chern Symposium (1979), 41–74.

[L-Y] S. LU and S. T. YAU, *On Hyperbolicity and the Green-Griffiths Conjecture for Surfaces*, preprint, 1989.

[L] T. LUO, *Riemann-Roch Type Inequalities for Big and Nef Divisors*, Amer. Jour. of Math. *111* (1989), 457–487.

[M] Y. MIYAOKA, *On the Chern Numbers of Surfaces of General Type*, Inventiones Math. (1977), 225–237.

[S] F. SAKAI, *Symmetric Power of the Cotangent Bundle and Classification of Algebraic Varieties*, Proc. Copenhagen Summer Meeting in Algebraic Geometry (1977), 545–563.

*Department of Mathematics*
*University of Texas*
*Arlington, TX 76019, USA*

Received July 7, 1990; November 19, 1990