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Mean curvature functions of codimension-one foliations. II

GEN-ICHI OSHIKIRI

Dedicated to Professor Haruo Suzuki on his sixtieth birthday

§0. Introduction

In [11] and [12], Walczak studied the following problem:

PROBLEM. Given a transversely oriented codimension-one foliation F of a closed oriented manifold M. What kind of smooth functions f on M can be represented as a mean curvature function of F with respect to some Riemannian metric of M?

Walczak gave an answer when F is a compact foliation [11]. In [6], the author gave a necessary and sufficient condition on f for arbitrary codimension-one foliations. However, the condition given there is not simple, and it seems difficult to see, in general, if the given function satisfies the condition. In this respect, he gave, in that paper, a conjecture concerning an alternating condition, which is easier to apply for general cases.

In this paper, we prove affirmatively this conjecture. This will be done in Section 2. As an application, in Section 3 we give a topological characterization of codimension-one foliations consisting of constant mean curvature hypersurfaces.

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§1. Preliminary and Result

In this paper, we work in the C^∞-category. In what follows, we always assume that foliations are of codimension-one and transversely oriented, and that the ambient manifolds are closed, connected, oriented, and of dimension $n + 1 \geq 3$, unless otherwise stated.

First we fix a transversely oriented codimension-one foliation F on M. Let g be a Riemannian metric on M. Then there is a unique unit vector field orthogonal to F whose direction coincides with the given transverse orientation. We denote this vector field by N. We give an orientation to F as follows: Let $\{E_1, \ldots, E_n\}$ be an oriented local orthonormal frame for TF. Then the orientation of M given by $\{N, E_1, \ldots, E_n\}$ coincides with the given one of M.

We denote the mean curvature of a leaf L at x with respect to N by $H(x)$, that is,

$$H = \sum_{i=1}^{n} \langle \nabla_{E_i} E_i, N \rangle,$$

where $\langle\ ,\ \rangle$ means $g(\ ,\ )$ and ∇ is the Riemannian connection of (M, g), and $\{E_i\}$ is a local orthonormal frame for TF with $\dim F = n$. We call $H(x)$ the mean curvature function of F with respect to g. We also define an n-norm χ_F on M by

$$\chi_F(V_1, \ldots, V_n) = \det(\langle E_i, V_j \rangle)_{i,j=1,\ldots,n} \qquad \text{for } V_j \in TM,$$

where $\{E_1, \ldots, E_n\}$ is an oriented local orthonormal frame for TF. The restriction $\chi_F|_L$ is the volume element of $(L, g|_L)$ for $L \in F$. We have the following.

PROPOSITION R (Rummler [7]). *$d\chi_F = -H\, dV(M, g) = \mathrm{div}_g(N)\, dV(M, g)$, where $dV(M, g)$ is the volume element of (M, g) and $\mathrm{div}_g(N)$ is the divergence of N with respect to g, i.e.,*

$$\mathrm{div}_g(N) = \sum_{i=1}^{n} \langle \nabla_{E_i} N, E_i \rangle.$$

Let f be a smooth function on M. We call f *admissible* if there is a Riemannian metric g on M so that $-f$ coincides with the mean curvature function of F with respect to g, and set

$$C_{\mathrm{Ad}}(F) = \{f \in C^\infty(M) : \textit{admissible}\}.$$

Note that if $H \equiv 0$, then F is a minimal foliation. If we can find such a g, then we call F *taut*. We also set

$$C_\pm = \{f \in C^\infty(M) : f(x) > 0 > f(y) \text{ for some } x, y \in M\}.$$

Then, by Proposition R, we have $C_{\mathrm{Ad}}(F) \subset C_\pm \cup \{0\}$. From this view point, we have:

THEOREM S1 (Sullivan [10]). *$0 \in C_{\mathrm{Ad}}(F)$, i.e., F is taut if and only if each compact leaf of F is cut out by a closed transversal.*

THEOREM W (Walczak [12], Oshikiri [6]). *F is taut if and only if $C_{\mathrm{Ad}}(F) = C_{\pm} \cup \{0\}$.*

Now recall the set-up of Sullivan [9]. Let $\mathbf{D}_p$ be the space of smooth p-forms on M and $\mathbf{D}_p^*$ be the dual space of $\mathbf{D}_p$, i.e., the space of p-currents. Then we have:

THEOREM SW (Schwartz [8]). $(\mathbf{D}_p^*)^* = \mathbf{D}_p$.

Let $x \in M$ and $B = \{e_1, \ldots, e_n\}$ be an oriented basis of $T_x F$. We define a Dirac current $\delta_{x,B}$ by

$$\delta_{x,B}(\phi) = \phi_x(e_1 \wedge \cdots \wedge e_n\} \qquad \text{for } \phi \in \mathbf{D}_n,$$

and set

$C_F =$ the closed convex cone in $\mathbf{D}_n^*$ spanned by Dirac currents $\delta_{x,B}$ for $x \in M$.

PROPOSITION S (Sullivan [9]). *C_F is a compact convex cone in $\mathbf{D}_n$. Here "compact" means that there is a continuous linear functional $L : \mathbf{D}_n^* \to R$ so that the set $L^{-1}(1) \cap C_F$ is compact non-empty.*

We shall call a compact set $L^{-1}(1) \cap C_F$ of the cone C_F the *base* of C_F and denote it by $\mathbf{C}$. Let $d : \mathbf{D}_p \to \mathbf{D}_{p+1}$ be the exterior differentiation and $\partial : \mathbf{D}_{p+1}^* \to \mathbf{D}_p^*$ be the dual of d, i.e., $(d\phi, c) = (\phi, \partial c)$ for $\phi \in \mathbf{D}_p$, $C \in \mathbf{D}_{p+1}^*$. Here $(\ ,\ )$ means the natural coupling $\mathbf{D}_p \times \mathbf{D}_p^* \to R$. Set $B = \partial(\mathbf{D}_{n+1}^*)$ and $Z = \operatorname{Ker} \partial : \mathbf{D}_n^* \to \mathbf{D}_{n+1}^*$.

THEOREM S2 (Sullivan [9]). *There is a canonical one to one correspondence between invariant transversal measures and elements in $Z \cap C_F$.*

The main result in Oshikiri [6] is the following:

THEOREM O. *For $f \in C^\infty(M)$, the following three conditions are equivalent.*
(1) *$f \in C_{\mathrm{Ad}}(F)$.*
(2) *There are an n-form ω and an oriented volume form dV on M so that $d\omega = f\,dV$ and ω is positive on F. Here "positive" means that $\omega_x(e_1 \wedge \cdots \wedge e_n) > 0$ for all oriented bases $\{e_1, \ldots, e_n\}$ of $T_x F$ and $x \in M$.*

(3) *There is an oriented volume form* dV *on* M *so that*
(i) $\int_M f\,dV = 0$, *and*
(ii) $\int_c f\,dV > 0$ *for all* $c \in \partial^{-1}(C_F \cap B - \{0\})$.

Let D be a compact saturated domain of M. We call D a *(+)-foliated compact domain* (abrev. (+)-fcd) if the transverse orientation of F is outward everywhere on ∂D.

The conjecture given in [6] is the following:

CONJECTURE. *Under the hypothesis in Theorem O, the condition* (3*) *below is also equivalent*:
(3*) *There is an oriented volume form* dV *on* M *so that*
(i) $\int_M f\,dV = 0$, *and*
(ii) $\int_D f\,dV > 0$ *for any* (+)*-fcd* D.

In the next section, we shall prove affirmatively this conjecture.

§2. Proof of the equivalence of (3) and (3*)

(3) *implies* (3*). Let D be a non-empty (+)-fcd. It is clear that $D \in \partial^{-1}(C_F \cap B - \{0\})$. Then D must satisfy $\int_D f\,dV > 0$.

(3*) *implies* (3). We proceed the proof as in one of Theorem 7.7 in Harvey–Lawson [3]. Let $c \in \mathbf{D}^*_{n+1}$ be an element in $\partial^{-1}(C_F \cap B - \{0\})$. We must show that $(c, f\,dV) > 0$. Note that the set supp (∂c) consists of compact leaves of F (cf. Harvey–Lawson [3, Theorem 7.2]). We divide the proof in two cases:

Case (a) (when supp (∂c) consists of a finite number of compact leaves). In this case, we can set supp $(\partial c) = \bigcup_{i=1}^{K} L_i$ for a positive integer K. As $\partial c \in B \cap C_F \subset Z \cap C_F$, we can represent ∂c by an invariant transversal measure by Theorem S2. By considering the assumption supp $(\partial c) = \bigcup_{i=1}^{K} L_i$, we have $\partial c = \Sigma_{i=1}^{K} \lambda_i \int_{L_i}$ for some $\lambda_i > 0$. On the other hand, $[\partial c] = \Sigma_{i=1}^{K} \lambda_i [L_i] = 0$ in $H_n(M : R)$, where $[L_i]$ stands for the homology class of L_i in $H_n(M : R)$. Dividing this relation by λ_1 and separating the result into pieces which are linearly independent over Q, we obtain a finite number of relations;

$$\sum_{j=1}^{l_k} n_j^k [L_j^k] = 0 \qquad \text{in } H_n(M : Z), \qquad k = 1, \dots, m.$$

As L_j^k's are oriented, connected and closed hypersurfaces in M, we can assume $n_j^k = 1$ (cf. Conlon–Goodman [2, Lemma 1.1]). This means that for each k there exists a $(+)$-fcd D_k with $\partial D_k = \bigcup_{j=1}^{l_k} L_j^k$. It follows that $\partial(\Sigma_{k=1}^m v_k \int_{D_k} - c) = 0$ as a current for some $v_k > 0$. Thus we get $c = \Sigma_{k=1}^m v_k \int_{D_k} + v \int_M$ for some $v \in R$. Therefore we have $(c, f\,dV) = \Sigma_{k=1}^m v_k \int_{D_k} f\,dV + v \int_M f\,dV > 0$ by the assumption (3*), because D_k's are $(+)$-fcd's.

Case (b) (when supp (∂c) consists of infinitely many compact leaves). We shall reduce this case to the case (a) by showing that there is an $(n+1)$-current T with $\partial T = \partial c - \Sigma_{i=1}^K \lambda_i \int_{L_i}$ and $(T, f\,dV) \geq 0$ for some positive integer K with $\lambda_i > 0$ and L_i being a compact leaf of F. If we could show this, we would have $c = T + c' + v' \int_M$ with supp $(\partial c') = \bigcup_{i=1}^K L_i$. This would complete the proof.

To show this, note that if there are infinitely many compact leaves in a compact manifold, then all but a finite number of them are contained in foliated I-bundles over compact leaves (cf. Hector–Hirsch [4, Chap. V]).

Let $L \times [0, 1]$ be a foliated bundle over a compact leaf L with supp $(\partial c) \cap L \times [0, 1] \neq \emptyset$, $L \times \{0, 1\}$ being compact leaves of F, and the transversal orientation of F being given by the canonical orientation of $[0, 1]$. Note that the restriction of ∂c to $L \times [0, 1]$ is of the form $\int_L \times \mu$, where μ is a Radon measure on $[0, 1]$, that is, for any n-form ω on $L \times [0, 1]$ we have

$$(\partial c \mid L \times [0, 1], \omega) = \int_0^1 \left(\int_{L \times (t)} \omega \right) d\mu(t).$$

At first, we shall show that there exists an $(n+1)$-current S satisfying $(S, f\,dV) \geq 0$ and $\partial S = \int_0^1 (\int_{L \times (t)})\, d\mu(t) - \mu([0, 1]) \int_{L_0}$ for a compact leaf L_0 in $L \times [0, 1]$. We let J be the subset of $[0, 1]$ corresponding to the compact leaves of F in $L \times [0, 1]$, that is, the set $L \times J$ is the union of all compact leaves of $F \mid L \times [0, 1]$. By Harvey–Lawson [3, Theorem 7.7], we know that there is a $(+)$-fcd D with $\partial D = \bigcup_{k=1}^l L_k \supset L \times (0)$. Define $D_t = D \cup L \times [0, t]$ for $t \in J$. We can take $t_0 \in J$ so that

$$\int_0^1 \left(\int_{D_t} f\,dV \right) d\mu(t) - \mu([0, 1]) \int_{D_{t_0}} f\,dV \geq 0.$$

Because, if this were not possible, we would have

$$\int_0^1 \left(\int_{D_t} f\,dV \right) d\mu(t) < \mu([0, 1]) \int_{D_t} f\,dV$$

for all $t \in J$. By integrating this relation again over $[0, 1]$ with respect to μ (note that supp $\mu \subset J$), we would have

$$\mu([0,1]) \int_0^1 \left(\int_{D_t} f\,dV \right) d\mu(t) < \mu([0,1]) \int_0^1 \left(\int_{D_t} f\,dV \right) d\mu(t),$$

a contradiction.

The current S we want is given by $\int_0^1 (\int_{D_t}) \,d\mu(t) - \mu([0,1]) \int_{D_{t_0}}$. Note that $L_0 = L \times (t_0)$ and the surplus terms corresponding to $\int_{L_k}$, $L_k \neq L \times (0)$, are cancelled by definition of S.

Since supp (∂c) is compact, we can construct the desired $(n+1)$-current T in a finite number of such steps.

§3. Foliations of constant mean curvature

Let (M, F, g) be a codimension-one foliation of a Riemannian manifold. We call F a foliation of constant mean curvature if the mean curvature function H of the foliation is constant on each leaf of F (cf. §1). We call a codimension-one foliation *tense*, if we can find a Riemannian metric so that the foliation becomes of constant mean curvature with respect to this metric. In this section, we give a topological characterization of tense foliations.

We say that a compact leaf L_0 is *contained in a continuous family* if there is a compact saturated set D which contains L_0 and is diffeomorphic to $L_0 \times [0, 1]$ so that the induced foliation on D by F corresponds to $L_0 \times \{t\}$, $t \in [0, 1]$. Denote by $C(F)$ the union of all compact leaves which are contained in continuous families. Let D be a non-empty compact saturated domain of M. We call D a *$(-)$-foliated compact domain* (abrev. $(-)$-fcd) if the transverse orientation of F is inward everywhere on ∂D (cf. §2, $(+)$-fcd).

Now, our characterization is the following:

THEOREM. *Let (M, F) be a transversely oriented codimension-one foliation of a connected, closed, and oriented manifold M with* dim $M \geq 3$. *Then F is tense if and only if each connected component of $M - C(F)$ does not contain $(+)$-fcd and $(-)$-fcd simultaneously.*

Proof. Assume F is tense. Then there is a Riemannian metric g on M so that each leaf L of F is a hypersurface of constant mean curvature. We denote the mean curvature of F by H. As the set $\{x \in M \mid dH_x \neq 0\}$ consists of compact leaves of F (cf. Barbosa–Kenmotsu–Oshikiri [1]), the mean curvature function H is constant on each connected component of $M - C(F)$. If a connected component X of

$M - C(F)$ contains both $(+)$-fcd C_+ and $(-)$-fcd C_-, then by assumption and Proposition R,

$$-\int_{C_+} H\,dV = (C_+, d\chi_F) = (\partial C_+, \chi_F) = \mathrm{Vol}\,(\partial C_+) > 0.$$

On the other hand, by the same argument, we get

$$-\int_{C_-} H\,dV = -\mathrm{Vol}\,(\partial C_-) < 0.$$

As H has the same value on both C_+ and C_-, this is impossible.

Now we show the converse. To do this, we need the following lemma.

LEMMA (see Oshikiri [5, Lemma 5]). *Let L be a closed n-dimensional manifold and $I = [0, 10]$ be the closed interval between* 0 *and* 10. *Assume that there are Riemannian metrics g_0 on $L \times [0, 4]$ and g_1 on $L \times [6, 10]$ such that each $L \times \{t\}$, $t \in [0, 4]$, is a hypersurface of constant mean curvature c, where c is a non-positive constant independent of $t \in [0, 4]$, and g_1 on $L \times [6, 10]$ is a Riemannian product of a Riemannian metric of L and the standard metric on I induced from R. Here we give a transverse orientation to F induced by the canonical orientation of R, and consider the mean curvature function of F with respect to this orientation. Then there is a Riemannian metric g on $L \times [0, 10]$ such that $g \mid L \times [0, 1] = g_0$, $g \mid L \times [9, 10] = g_1$, and each $L \times \{t\}$, $t \in I$, is a hypersurface of constant mean curvature with respect to g.*

First, we assume that the set $M - C(F)$ consists of a finite number of connected components. Let D be a connected component of $M - C(F)$. Set

$$D' = D \cup \partial D \times [0, 1]/\partial D \sim \partial D \times \{0\},$$

and

$$F' = F \cup \{\partial D \times \{t\};\ t \in [0, 1]\}.$$

We may assume $D' \subset M$ and $F' = F \mid D'$. By the lemma, we have only to show that on each D' there exists a Riemannian metric g so that each leaf of F' has a constant mean curvature with respect to g.

Let W be the double of D' with the foliation $\tilde{F}$ induced by F'. We regard W as the union $D \cup \partial D \times [-1, 1] \cup (-D)$, where $-D$ is the set D with the reversed orientation, and D' corresponds to $D \cup \partial D \times [-1, 0]$ and $-D'$ to $\partial D \times [0, 1] \cup (-D)$. Let θ be the natural involution of W. We may assume that $\theta(D) = -D$ and $\theta(x, t) = (x, -t)$ for $(x, t) \in \partial D \times [-1, 1]$. First we choose a Riemannian metric g on W so that θ is an isometry.

Case (a) (when D contains a $(+)$-fcd). Note that, in this case, by assumption, D does not contain any $(-)$-fcd. Now define a smooth function f on W as follows: First choose a smooth monotonic decreasing function h on $[-1, 1]$ with $h(t) = -h(-t)$ for $t \in [-1, 1]$, $h(t) = 1$ near $t = -1$, $h(t) = 0$ near $t = 0$, and $-1 \leq h \leq 1$. Lift this h onto $\partial D \times [-1, 1]$ naturally, and denote it by f. Define $f(x) = 1$ on D, and $f(x) = -1$ on $-D$. Note that $f \circ \theta = -f$.

Then, by definition, we have $\int_W f\, dV(M, g) = 0$. We show that $f \in C_{\mathrm{Ad}}(F)$ by showing that $dV(M, g)$ and f satisfiy condition (3*).

Let A be a $(+)$-fcd in W. We must show $\int_A f\, dV(M, g) > 0$. To do this, we assume $\int_A f\, dV(M, g) \leq 0$, and derive a contradiction. If $\int_A f\, dV(M, g) \leq 0$, then the set $\theta(A \cap (-D)) - (A \cap D)$ is a non-empty $(-)$-fcd in D. In fact, as $f \geq 0$ on $D \cup \partial D \times [-1, 0]$ and $f \circ \theta = -f$, if $A \cap D'$ contains $\theta(A \cap (-D'))$ properly, then $\int_A f\, dV(M, g) > 0$, which contradicts our assumption. If $A \cap D' = \theta(A \cap (-D'))$, then A cannot be a $(+)$-fcd. Thus $A \cap D'$ cannot contain $\theta(A \cap (-D'))$. The situation is similar when D' is replaced by D. Thus $A \cap D$ cannot contain $\theta(A \cap (-D))$, and $\theta(A \cap (-D)) - (A \cap D) \neq \varnothing$. As the transverse orientation of F on $\theta(\partial A \cap (-D))$ is reversed by θ, the set $\theta(A \cap (-D)) - (A \cap D)$ is a non-empty $(-)$-fcd in D. This contradicts our assumption.

Case (b) (when D contains a $(-)$-fcd). The proof is the same as in the case (a).

Case (c) (when D does not contain either $(+)$-fcd or $(-)$-fcd). In this case, W does not contain either $(+)$-fcd or $(-)$-fcd. In fact, if W contains a $(+)$-fcd or $(-)$-fcd X, then, by assumption, both sets $X \cap D$ and $X \cap (-D)$ are non-empty. Thus $\theta(X \cap (-D)) - (X \cap D)$ or $(X \cap D) - \theta(X \cap (-D))$ must be a non-empty $(+)$-fcd or $(-)$-fcd in D, which is a contradiction. Hence by condition (3*), we have $0 \in C_{\mathrm{Ad}}(\tilde{F})$. This means, by Theorem S, that $\tilde{F}$ is taut. Thus we can find a Riemannian metric g on W so that $\tilde{F}$ is a minimal foliation.

Now consider the case when the set $M - C(F)$ consists of infinitely many connected components. In this case, all but a finite number of them are of the form of foliated I-bundles over compact leaves. This fact follows from a local study near compact leaves (see Hector–Hirsch [4, Chap. V, 2.2]). This enables us to reduce this case to the finite one, by considering as a set D in the finite case the union of a connected component and suitable foliated I-bundles with common boundaries. This does not affect the proof.

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