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A balanced proper modification of P_3

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1. Introduction

Since a proper modification M of a Kähler manifold is not necessarily Kähler, it is natural to ask what kind of weaker geometrical properties are preserved in this situation. In particular we are interested in the existence of a balanced metric on M, i.e. a metric such that the trace of the torsion of the Chern connection vanishes.

We point out that balanced manifolds are, in some sense, dual to Kählerian ones: in fact, a metric is balanced if its Kähler form is "co-closed" (see Definition 3.1); moreover, balanced n-dimensional manifolds are characterized by means of their $(n-1, n-1)$-currents as well as Kähler manifolds can be characterized by $(1, 1)$-currents (see [Mi] and [HL]).

The classical example of Hironaka ([Hi] or [S]) of a Moishezon non algebraic manifold X is given by a proper modification of $\mathbf{P}_3$; in this note we prove that X is balanced. From a geometrical point of view, the obstruction for X to be Kähler is a curve homologous to zero on the exceptional divisor E: in a dual manner, our proof lies in showing that if E is not homologous to zero then M is balanced. This result is a first confirmation of the conjecture: "Every proper modification of a Kähler manifold is balanced."

About techniques, since positive currents which are (p, p)-components of boundaries are not necessarily closed (see Question on p. 170 in [HL]), we cannot apply the well known results on (locally) flat currents, and therefore we must develope a parallel technique for positive $\partial\bar{\partial}$-closed currents.

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2. Preliminaries

Let M be a connected complex manifold of dimension n; we denote as usually by Ω^p the sheaf of germs of holomorphic p-forms and by $\mathscr{E}^p$ the sheaf of germs of $\mathscr{C}^\infty$ p-forms. Moreover, let $\mathscr{E}^{p,p}(M)_{\mathbf{R}}$ denote the Fréchet space of real (p, p)-forms, while $\mathscr{E}'_{p,p}(M)_{\mathbf{R}}$ denotes its dual space of real currents of bidimension (p, p).

We will say that $T \in \mathscr{E}'_{p,p}(M)_{\mathbf{R}}$ is *the* (p, p)*-component of a boundary* if there exists a current S such that for every $\varphi \in \mathscr{E}^{p,p}(M)$, $T(\varphi) = S(d\varphi)$, i.e. there exists a $(p, p+1)$-current S such that $T = \bar\partial S + \partial \bar S$.

Let V be a complex vector space of dimension n, p an integer and $\sigma_p := i^{p^2} 2^{-p}$. An element of $\Lambda^{p,0}$ that can be expressed as $v_1 \wedge \cdots \wedge v_p$ with $v_j \in V$ is called *simple.*

2.1 DEFINITION. ([AA] and [AB]) $\Omega \in \mathscr{E}^{p,p}(M)_{\mathbf{R}}$ is called *transverse* if $\forall x \in M$, $\forall v \in \Lambda^p(T'_x M)$, $v \neq 0$ and simple, it holds $\Omega_x(\sigma_p^{-1} v \wedge \bar v) > 0$. $T \in \mathscr{E}'_{p,p}(M)_{\mathbf{R}}$ is called *positive* if $T(\Omega) \geq 0$ for every transverse (p, p)-form Ω.

2.2 REMARK. ([H1] p. 325–326) A positive (p, p)-current T has measure coefficients, and moreover, if $\|T\|$ denotes the mass of T, there exists a $\|T\|$-measurable function $\vec T$ such that $\|\vec T_x\| = 1$ for $\|T\|$-a.e. $x \in M$, and

$$T(\varphi) = \int \varphi_x(\vec T_x)\, \|T\| \text{ for all test forms } \varphi.$$

Let us prove now our main technical tool.

2.3 THEOREM. *Let M be a complex n-dimensional manifold, and let T be a positive (p, p)-current on M such that $\partial\bar\partial T = 0$ and the Hausdorff $2p$-measure of* supp T, *$H^{2p}(T)$, vanishes. Then $T = 0$.*

2.4 REMARKS. In this statement, M is not assumed to be compact. If moreover T is closed, the result is well known (see for instance [F]).

Proof of Theorem 2.3. Let $x_0 \in \operatorname{supp} T$; as in [H2] p. 72 we can choose a coordinate neighbourhood $(U, z_1, \ldots, z_n)$ centered at x_0 such that for every increasing multiindex I, $|I| = p$, it holds locally:

(a) if π_I is the projection defined by $\pi_I(z_1, \ldots, z_n) = (z_{i_1}, \ldots, z_{i_p})$, $\operatorname{supp} T \cap \operatorname{Ker} \pi_I = \{x_0\}$

(b) there exist two balls $\Delta'_I \subseteq \mathbf{C}^p$, $\Delta''_I \subseteq \mathbf{C}^{n-p}$ such that $(\Delta'_I \times b\Delta''_I) \cap \operatorname{supp} T = \emptyset$.

This implies that $\pi_I|_{\operatorname{supp} T} : \operatorname{supp} T \cap (\Delta'_I \times \Delta''_I) \to \Delta'_I$ is a proper map.

Fix an I, $|I| = p$, and call $T_I := (\pi_I|_{\operatorname{supp} T})_*(T)$; T_I is a $\partial\bar\partial$-closed (p, p)-current on Δ'_I, hence it can be identified with a $\partial\bar\partial$-closed distribution f_I on Δ'_I; since f_I is pluriharmonic, it is smooth.

But π_I is a Lipschitzian map, and therefore $H^{2p}(\operatorname{supp} f_I) = 0$, i.e. $f_I = 0$. Let

$$\omega := \sigma_1 \sum_\alpha dz_\alpha \wedge d\bar z_\alpha \text{ in } B := \bigcap_I (\Delta'_I \times \Delta''_I).$$

Since T is positive, in order to prove $T = 0$ it is enough to verify that $T(\chi_B(\omega^p/p!)) = 0$. But in B

$$\frac{\omega^p}{p!} = \sum_I \sigma_p \, dz_I \wedge d\bar z_I = \sum_I (\pi_I)^*(\sigma_p \, dz_I \wedge d\bar z_I),$$

therefore

$$T\left(\chi_B \frac{\omega^p}{p!}\right) \leq \sum_I T(\chi_{\Delta'_I \times \Delta''_I}(\pi_I)^*(\sigma_p \, dz_I \wedge d\bar z_I)) = \sum_I \int_{\Delta'_I} f_I \sigma_p \, dz_I \wedge d\bar z_I = 0. \qquad \square$$

3. Hironaka's example

Let us recall the definition of balanced manifold, and the characterization of them by means of positive currents.

3.1 DEFINITION. ([Mi]) A *balanced* manifold M is a compact complex n-dimensional manifold which satisfies one of the following equivalent conditions:

(i) M admits a hermitian metric h such that, if ω is the Kähler form of h, $d\omega^{n-1} = 0$.

(ii) M admits a hermitian metric h such that, if ω is the Kähler form of h and δ is the formal adjoint of d in the metric h, it holds $\delta\omega = 0$.

(iii) M admits a hermitian metric h such that the torsion 1-form τ_h of the canonical hermitian connection of h is zero.

(iv) there are no non trivial positive $(n-1, n-1)$-currents on M which are $(n-1, n-1)$-components of boundaries.

Let us recall also briefly the construction of Hironaka's example, which can be found in his Thesis [Hi] or, for instance, in [S].

3.2 Consider the projective space $\mathbf{P}_3$ with coordinates (x, y, z), and in it the curve C of equation

$$\begin{cases} y^2 = x^2 + x^3 \\ z = 0 \end{cases}$$

In a little ball near zero, blow up one branch of C first, then the other; outside of the origin, just blow up $C - \{O\}$. Then glue together to obtain the compact complex manifold X and the holomorphic map

$$f : X \to \mathbf{P}_3;$$

there exists an analytic subvariety E of X such that

$$f|_{X-E} : X - E \to \mathbf{P}_3 - C \text{ is biholomorphism.}$$

3.3 THEOREM. *The 3-dimensional Hironaka's manifold X described in 3.2 is balanced.*

Proof. Let T be a positive $(2, 2)$-current on X which is the $(2, 2)$-component of a boundary, and let ω be a Kähler form for $\mathbf{P}_3$. $f^*\omega$ is a closed real $(1, 1)$-form which is positive semidefinite, so that from

$$0 = T(f^*\omega) = \int (f^*\omega)_x(\vec{T}_x) \, \|T\|$$

we get $E \supseteq \operatorname{supp} T$.

We need now the following lemma, that we shall prove later.

3.4 LEMMA. *Let $f : X \to \mathbf{P}_3$, C, E, T as before. Then there exists $k \in \mathbf{C}$ such that $T = k[E]$. ($[E]$ denotes the current defined by $[E](\phi) = \int_{\operatorname{Reg} E} \phi$ for all test forms ϕ.)*

The previous lemma tells us in particular that $dT = 0$, and therefore

$$T = \bar{\partial} S^{2,3} + \partial \overline{S^{2,3}}$$

for a $(2, 3)$-current $S^{2,3}$ with $\partial\bar{\partial} S^{2,3} = 0$.

$\partial S^{2,3}$ becomes a holomorphic 2-form on X, hence the equality

$$\int \partial S^{2,3} \wedge \overline{\partial S^{2,3}} \wedge f^*\omega = \int d(S^{2,3} \wedge \overline{\partial S^{2,3}} \wedge f^*\omega) = 0$$

proves that $\partial S^{2,3}$ vanishes, and we get

$$T = d(S^{2,3} + \overline{S^{2,3}}).$$

If k were different from zero, we should conclude that the exceptional set E is homologous to zero, which is not the case (for a proof of this fact, use a Mayer-Vietoris argument, as is done for example in [Mü]). In this way we conclude $T = 0$, i.e. X is balanced. □

Proof of Lemma 3.4. Since $\dim E = 2$, for every $x \in \operatorname{Reg} E$ we can choose a coordinate neighbourhood (U, z_1, z_2, z_3) of x such that

$$E \cap U = \{z \in U / z_3 = 0\}.$$

As in ([HL] Lemma 32), we get in U

$$T = \sigma_1 \sum_{\alpha,\beta=1}^{3} t_{\alpha\bar\beta}(z)\, dz_\alpha \wedge d\bar z_\beta$$

where the measure $t_{\alpha\bar\beta}$ can be written as

$$t_{\alpha\bar\beta}(z) = r_{\alpha\bar\beta}(z_1, z_2) \otimes \delta_0(z_3).$$

Since T is a $(2, 2)$-component of a boundary, $0 = \partial\bar\partial T =$

$$\sigma_1 \sum_{\alpha<\delta,\beta<\gamma} (-\partial_\delta\, \partial_{\bar\gamma} t_{\alpha\bar\beta} + \partial_\delta\, \partial_{\bar\beta} t_{\alpha\bar\gamma} + \partial_\alpha\, \partial_{\bar\gamma} t_{\delta\bar\beta} - \partial_\alpha\, \partial_{\bar\beta} t_{\delta\bar\gamma})\, dz_\alpha \wedge dz_\delta \wedge d\bar z_\beta \wedge d\bar z_\gamma.$$

For $\gamma = \delta = 3$ we get

$$\begin{cases} r_{\alpha\bar\beta} = 0 & \text{for } \alpha, \beta \in \{1, 2\} \\ r_{\alpha\bar 3} & \text{is holomorphic for } \alpha \in \{1, 2\} \\ r_{3\bar 3} & \text{is pluriharmonic} \end{cases}$$

As $(r_{\alpha\bar\beta})$ is a non negative matrix, it follows $r_{\alpha\bar 3} = 0$ for $\alpha \in \{1, 2\}$ and therefore

$$T = \sigma_1(r_{3\bar 3}(z_1, z_2) \otimes \delta_0(z_3))\, dz_3 \wedge d\bar z_3 \text{ in } U.$$

In this way we have proved that there exists a non negative pluriharmonic map $h : \operatorname{Reg} E \to \mathbf{R}$ such that

$$T|_{X - \operatorname{Sing} E} = h[E]|_{X - \operatorname{Sing} E}.$$

The next step is to prove that h is constant.

For every $x_0 \in C$, $x_0 \neq O$, $h|_{f^{-1}(x_0)}$ is harmonic and therefore constant, so that we can project h to a non negative harmonic map $h_1 : C - \{O\} \to \mathbf{R}$.

The following parametrisation of C

$$\begin{cases} x(t) = t^2 - 1 \\ y(t) = t(t^2 - 1) \end{cases}, \quad z = 0$$

induces biholomorphisms

$$C - \{O\} \cong \mathbf{P}_1 - \{a, b\} \cong \mathbf{C} - \{O\}$$

which allow us to use the following remark (see [B] p. 402–403): "Let $B(O, b)$ be the ball of center O and radius $b > 0$ in $\mathbf{C}$, and let $g : B(O, b) - \{O\} \to \mathbf{R}$ be harmonic and non negative. Then there exist a constant A and a holomorphic function F in $B(O, b)$ such that

$$g(z) = \operatorname{Re} F(z) + A \log |z| \text{ for } z \in B(O, b) - \{O\}."$$

Let $h_1(z) = \operatorname{Re} F(z) + A \log |z|$; F and A are independent on the radius of the chosen ball and moreover, replacing z by $1/w$, we get $F(z) = \text{constant}$. Since h_1 is non negative, it must be $A = 0$, hence h_1 is constant and h is constant too.

Now, the two currents T and $h[E]$ are $\partial\bar{\partial}$-closed and coincide on $X - \operatorname{Sing} E$; if we prove that $T - h[E]$ is positive, the statement will follow from Theorem 2.3.

To do this, let Ω be a transverse $(2, 2)$-form, let $x \in \operatorname{Sing} E$, let K be a compact neighbourhood of x, let $\{V_\varepsilon\}$ be a fundamental family of neighbourhoods of $K \cap \operatorname{Sing} E$. Since

$$h \int_{E \cap (K - V_\varepsilon)} \Omega = T(\chi_{K - V_\varepsilon} \Omega) \uparrow T(\chi_{K - \operatorname{Sing} E} \Omega)$$

we get

$$h \int_{E \cap K} \Omega \leq T(\chi_{K - \operatorname{Sing} E} \Omega) \leq T(\chi_K \Omega). \qquad \square$$

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