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Étale descent for Hochschild and cyclic homology

CHARLES A. WEIBEL[1] AND SUSAN C. GELLER[2]

Abstract. If B is an étale extension of a k-algebra A, we prove for Hochschild homology that $HH_*(B) \cong HH_*(A) \otimes_A B$. For Galois descent with group G there is a similar result for cyclic homology: $HC_*(A) \cong HC_*(B)^G$ if $\mathbf{Q} \subseteq A$. In the process of proving these results we give a localization result for Hochschild homology without any flatness assumption. We then extend the definition of Hochschild homology to all schemes and show that Hochschild homology satisfies cohomological descent for the Zariski, Nisnevich and étale topologies. We extend the definition of cyclic homology to finite-dimensional noetherian schemes and show that cyclic homology satisfies cohomological descent for the Zariski and Nisnevich topologies, as well as for the étale topology over $\mathbf{Q}$. Finally, we apply these results to complete the computation of the algebraic K-theory of seminormal curves in characteristic zero.

This paper studies three related topics: étale descent for Hochschild homology and cyclic homology, and the algebraic K-theory of seminormal curves. Our results are simplest for Hochschild homology. Let k be a commutative ring with identity, A a k-algebra, M an A-bimodule, and $R = A \otimes_k A^{\mathrm{op}}$. Then the Hochschild homology of A and M is $HH_*(A; M) = \mathrm{Tor}_*^{R/k}(A, M)$ as in [Mac, IX.8]. We write $HH_*(A)$ for the Hochschild homology $HH_*(A; A)$. If A is commutative, it is well-known that $HH_*(A)$ is a graded A-module.

ÉTALE DESCENT THEOREM (0.1). *Let $A \subset B$ be an étale extension of commutative k-algebras. Then*

$$HH_*(B) \cong HH_*(A) \otimes_A B.$$

For example, any localization $B = S^{-1}A$ is étale over A, so (0.1) yields the formula $HH_*(S^{-1}A) \cong S^{-1}HH_*(A)$ without the extra hypothesis of [Bl], [Bry], and [GRW] that A be flat over k. This formula also holds when A is noncommutative; see §1. Another immediate consequence of (0.1), which breaks up the long exact sequences in *loc. cit.* as well as removing the spurious hypotheses (i) A flat over k and (ii) $B_1 = S^{-1}A$, is the following.

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COROLLARY 0.2. *Suppose that* $X = \operatorname{Spec}(A)$ *is covered by affine open subsets* $U_1 = \operatorname{Spec}(B_1)$ *and* $U_2 = \operatorname{Spec}(B_2)$. *Then there is a short exact sequence*

$$0 \to HH_n(A) \to HH_n(B_1) \times HH_n(B_2) \to HH_n(B_1 \otimes_A B_2) \to 0.$$

Indeed, B_1, B_2, and $B_1 \otimes B_2$ are étale over A, so this arises from (0.1) upon tensoring $HH_n(A)$ with the exact sequence $0 \to A \to B_1 \times B_2 \to B_1 \otimes_A B_2 \to 0$.

The other typical étale extension is the Galois extension ([KO]). In this case we can also say something about cyclic homology. (See 2.2 and 3.2 below.)

GALOIS DESCENT THEOREM (0.3). *Let* $A \subset B$ *be a Galois extension of commutative* k*-algebras with Galois group* G. *Then* G *acts on* $HH_*(B)$ *and*

$$HH_*(A) \cong HH_*(B)^G.$$

In addition, if A *contains a field of characteristic zero, then the action of* G *on the cyclic homology* $HC_*(B)$ *is such that*

$$HC_*(A) \cong HC_*(B)^G.$$

Let us explain why we call (0.1) a "descent theorem." There are two relevant notions of descent in the literature. One is the notion of faithfully flat descent ([TDTE], [KO]). Let F be any functor from k-algebras to abelian groups, and $A \subseteq B$ be a faithfully flat extension. We say that *F satisfies naïve descent* for $A \subseteq B$ if the augmented Amitsur complex

$$0 \to F(A) \xrightarrow{\varepsilon} F(B) \to F(B \otimes_A B) \to F(B \otimes_A B \otimes_A B) \to \cdots$$

is exact. The HH_n satisfy naïve descent for all faithfully flat étale extensions by (0.1), and the HC_n satisfy naïve descent for all Galois extensions by (0.3).

Our original proof of (0.1) used the theory of faithfully flat descent; after hearing it, Brylinski ([Brylet]) sent a simpler argument to us, and we give a modification of his argument in Section 2. Another proof of (0.1) has been found independently by C. Kassel and A. Sletsjøe [KS] using Harrison homology, at least when $\mathbf{Q} \subseteq k$ and A is flat over k.

In Section 3 we show that the individual HC_n do not satisfy naïve descent for all faithfully flat étale extensions, but that instead there is a fourth quadrant descent spectral sequence

$$E_2^{pq} = H^p(B/A, HC_{-q}) \Rightarrow HC_{-p-q}(A)$$

converging in the usual good cases. (See 3.4 below.) The E_2^{pq} terms are the Amitsur cohomology groups of $F = HC_{-q}$, *i.e.*, the cohomology of the Amitsur complex.

The second notion of descent in the literature is cohomological descent (for presheaves of chain complexes) in the sense of Grothendieck ([Hart]), Thomason ([AKTEC]), *et al.* In Section 4, we extend the definition of Hochschild homology to schemes over k, and show that HH_* satisfies cohomological descent for the Zariski, Nisnevich, and étale topologies. We also extend the definition of cyclic homology to finite-dimensional noetherian schemes over k. (This restriction arises from our insistence that $HC_*(\mathrm{Spec}(A))$ be the same as $HC_*(A)$.) Then we show that HC_* satisfies cohomological descent for the Zariski and Nisnevich topologies, and for the étale topology over $\mathbf{Q}$. The main result needed for this is the following description of the Hochschild homology sheaves.

COROLLARY 0.4 (HOCHSCHILD SHEAF). *Let X be a scheme over k, and $\mathcal{HH}_n$ be the Zariski sheafification of the presheaf $U \mapsto HH_n(\Gamma(U, \mathcal{O}_X))$ on the big Zariski site of X. Then*

(i) *$\mathcal{HH}_n$ is a quasicoherent sheaf on X;*

(ii) *$\mathcal{HH}_n$ is also a sheaf for the étale topology on X;*

(iii) *$H^i_{\mathrm{et}}(X; \mathcal{HH}_n) \cong H^i_{\mathrm{Zar}}(X; \mathcal{HH}_n)$ for all i; and,*

(iv) *if X is affine, i.e., $X = \mathrm{Spec}(A)$, then $H^0(X, \mathcal{HH}_n) \cong HH_n(A)$ and $H^i(X; \mathcal{HH}_n) = 0$ for $i \neq 0$.*

Proof. Our étale descent theorem (0.1) implies that $\mathcal{HH}_n$ is an étale sheaf and that it is quasicoherent. Parts (iii) and (iv) follow from [M, III.3.8]. $\square$

In §5 we complete the calculation of the algebraic K-theory of an arbitrary seminormal curve over a field l of characteristic zero. This calculation was begun in [GRW], and our desire to finish the calculation was the original motivation for this paper. In [GRW, §6, 8] the problem was reduced to the computation of the K-theory of the affine "linear" seminormal curves $\mathrm{Spec}(A)$, $A = l \oplus t(\Pi l_i)[t]$, where Πl_i is a finite product of finite field extensions of l. This reduction follows from the fact that every seminormal curve singularity has this analytic type ([D]). Therefore the following calculation completes our program.

THEOREM 0.5. *Let $l_1, l_2, \ldots, l_r$ be finite extension fields of a field l of characteristic zero. Let $A = l \oplus t(\Pi l_i)[t]$. Then:*

$$K_n(A) \cong K_n(l) \oplus V_n,$$

and

$$HC_n^{\mathbf{Q}}(A) \cong HC_n^{\mathbf{Q}}(l) \oplus V_{n+1} \oplus (A/l \otimes_l \Omega_l^n).$$

Here $V_n = 0$ *if* $n < 2$ *and, for* $n \geq 2$:

$$V_n = \coprod_{c(b,n)} l \oplus \coprod_{c(b,n-1)} \Omega_l \oplus \cdots \oplus \coprod_{c(b,2)} \Omega_l^{n-2},$$

where $b = -1 + \Sigma \dim_l (l_i)$, *and the combinatorial numbers* $c(b, i)$ *are given in* [GRW, 3.13]; $c(b, 2) = (b^2 + b)/2$ *and* $c(b, n)$ *is approximately* b^n/n *for large* n.

§1. Localization without flatness

The goal of this section is to prove two localization results without the customary hypothesis that A be flat over k. Proposition 1.1 was proven with the additional hypothesis that A be flat over k in [GRW, A.3], [Bl] and [Bry]. We shall write $HH_*(A; M)$ for the Hochschild homology of an A-bimodule M.

PROPOSITION 1.1 (LOCALIZATION FOR HOCHSCHILD HOMOLOGY). *Let* C *be the center of a* k*-algebra* A. *Then for every multiplicatively closed set* S *in* C,

$$S^{-1}HH_*(A; A) \cong HH_*(A; S^{-1}A) \cong HH_*(S^{-1}A; S^{-1}A).$$

In order to prove this result, we need to recall the notion of relative torsion products, $\mathrm{Tor}_*^{R/k}(M, N)$, from [Mac, IX.8]. Here R is a k-algebra, M is a right R-module, and N is a left R-module. The reason for our interest in $\mathrm{Tor}_*^{R/k}$ is the fact [Mac, X.1.4, p. 280] that if we take $R = A \otimes_k A^{\mathrm{op}}$ and consider an A-bimodule N as a left R-module, then

$$HH_*(A; N) \cong \mathrm{Tor}_*^{R/k}(A, N).$$

DEFINITION 1.2. [Mac, IX.8, p. 273] An R-module P is called a *relatively projective* R/k*-module* if it has the projective lifting property relative to the class of "k-split epis", *i.e.*, the R-module epimorphisms $L \twoheadrightarrow M$ which have k-module splittings.

For example, if V is a k-module, then $P = R \otimes_k V$ is easily seen to be relatively projective. In fact, it follows from the standard proof of [Mac, IX.8.4] that every relatively projective R/k-module P is a direct summand of some $R \otimes_k V$ (take $V = P$, for example).

By [Mac, IX.8.5] we may compute $\mathrm{Tor}_*^{R/k}(M, N)$ by forming any k-split resolution $P_\bullet$ of M by relatively projective R/k-modules and taking the homology of the complex $P_\bullet \otimes_R N$. Equivalently, we may compute $\mathrm{Tor}_*^{R/k}(M, N)$ by forming any k-split resolution $P_\bullet$ of N by relatively projective R/k-modules and taking the homology of the complex $M \otimes_R P_\bullet$.

DEFINITION 1.3. Call M a *relatively flat R/k-module* if $\mathrm{Tor}_*^{R/k}(M, N) = 0$ for $* \neq 0$ and for all left R-modules N.

The usual homological yoga (see [Mac, p. 276]) shows that we can compute relative torsion products using a k-split resolution of M (or N) by relatively flat R/k-modules.

LEMMA 1.4. *Let V be a right R-module and S a flat R-algebra. Then $P = V \otimes_k S$ is a relatively flat R/k-module.*

Proof. (Cf. [Mac, IX.8.3].) We have to see that $P \otimes_R$ sends a k-split exact sequence $N_\bullet$ of left R-modules to an exact sequence. Since k is commutative, by k-linearity we have:

$$P \otimes_R N = V \otimes_k S \otimes_R N_\bullet \cong S \otimes_R N_\bullet \otimes_k V.$$

Now $N_\bullet \otimes_k V$ is exact because $N_\bullet$ is split exact as a sequence of k-modules. Applying $S \otimes_R$ after that retains exactness because S is flat in the usual sense. □

COROLLARY 1.5 (cf. [BX, 6.6]). *Let B be a flat A-algebra. Then $T = B \otimes_k B^{\mathrm{op}}$ is flat over $R = A \otimes_k A^{\mathrm{op}}$. For every right T-module M and left R-module N,*

$$\mathrm{Tor}_*^{T/k}(M, T \otimes_R N) \cong \mathrm{Tor}_*^{R/k}(M, N).$$

Proof. Since $(B \otimes_k B^{\mathrm{op}}) \otimes_{A \otimes_k A^{\mathrm{op}}} M \cong B \otimes_A M \otimes_A B$, flatness of T over R is immediate. To prove the second statement, find a k-split resolution $P_\bullet$ of M with $P_i = V_i \otimes_k T$ for some k-modules V_i. For example, $P_\bullet$ could be the bar resolution $\beta(M) = M \otimes_T \beta(T)$ of [Mac, IX.8.2]. Then $\mathrm{Tor}_*^{T/k}(M, T \otimes_R N)$ is the homology of the complex $P_\bullet \otimes_T (T \otimes_R N)$. But $P_\bullet \otimes_T (T \otimes_R N) \cong P_\bullet \otimes_R N$, whose homology is $\mathrm{Tor}_*^{R/k}(M, N)$ since $P_\bullet$ is a k-split resolution of the R-module M by relatively flat R/k-modules. □

Proof of 1.1. (cf. proof of [GRW, A.3].) Let $R = A \otimes_k A^{\mathrm{op}}$ and $T = S^{-1}A \otimes_k S^{-1}A^{\mathrm{op}}$. Then by Corollary 1.5 with $M = S^{-1}A$ and $N = A$,

$$\begin{aligned} HH_*(S^{-1}A; S^{-1}A) &\cong \operatorname{Tor}_*^{T/k}(S^{-1}A, S^{-1}A) \cong \operatorname{Tor}_*^{R/k}(S^{-1}A, A) \\ &\cong \operatorname{Tor}_*^{R/k}(A, S^{-1}A) \cong HH_*(A; S^{-1}A). \end{aligned}$$

Finally, let $P_{\bullet}$ be a k-split, relatively projective resolution of the left R-module A. Since localization is exact, we have

$$\begin{aligned} S^{-1}H_*(A; A) &\cong S^{-1}\operatorname{Tor}_*^{R/k}(A, A) \cong S^{-1}H_*(A \otimes_R P_{\bullet}) \\ &\cong H_*(S^{-1}A \otimes_R P_{\bullet}) \cong \operatorname{Tor}_*^{R/k}(S^{-1}A, A). \quad \square \end{aligned}$$

Another localization result that we need in order to prove étale descent for Hochschild homology concerns localizations of relative Tors, and is similar to Proposition 1.1 (cf. [BX, 6.5 and 6.6]).

PROPOSITION 1.6. *Let C be the center of a k-algebra A. Let M be a right A module and N be a left A-module. Then for every multiplicatively closed set S in C,*

$$S^{-1}\operatorname{Tor}_*^{A/k}(M, N) \cong \operatorname{Tor}_*^{A/k}(M, S^{-1}N) \cong \operatorname{Tor}_*^{S^{-1}A/k}(S^{-1}M, S^{-1}N).$$

Proof. Let $P_{\bullet}$ be a relatively flat $S^{-1}A/k$ resolution of $S^{-1}N$. Then $P_{\bullet}$ is a relatively flat A/k resolution of $S^{-1}N$. Thus

$$\begin{aligned} \operatorname{Tor}_*^{A/k}(M, S^{-1}N) &\cong H_*(M \otimes_A P_{\bullet}) \cong H_*(M \otimes_A S^{-1}A \otimes_{S^{-1}A} P_{\bullet}) \\ &\cong H_*(S^{-1}M \otimes_{S^{-1}A} P_{\bullet}) \cong \operatorname{Tor}_*^{S^{-1}A/k}(S^{-1}M, S^{-1}N). \end{aligned}$$

Now let $P_{\bullet}$ be a relatively flat k-split resolution of M. Then

$$\begin{aligned} S^{-1}\operatorname{Tor}_*^{A/k}(M, N) &\cong H_*(P_{\bullet} \otimes_A N) \otimes_A S^{-1}A \cong H_*(P_{\bullet} \otimes_A S^{-1}N) \\ &\cong \operatorname{Tor}_*^{A/k}(M, S^{-1}N). \quad \square \end{aligned}$$

§2. Étale descent for Hochschild homology

The following proof of the Étale Descent Theorem (0.1) is an adaptation of a proof by J.-L. Brylinski ([Brylet]) under the assumption that A is flat over k. We are using his elegant approach rather than our original descent-theoretic proof which was straightforward but unenlightening. We are most grateful to him for communicating it to us.

Recall that if $A \to B$ is a map of commutative k-algebras, then there is a natural map $HH_*(A) \otimes_A B \to HH_*(B)$. Our descent result will be that this map is an isomorphism when B is étale over A. We first give a related result for flat extensions.

THEOREM 2.1. *Let $A \subseteq B$ be a flat extension of commutative k-algebras. Then,*

$$HH_*(A) \otimes_A B \cong HH_*(B; B \otimes_A B).$$

Proof. Let $R = A \otimes A^{\mathrm{op}}$ and $T = B \otimes B^{\mathrm{op}}$. Let $P_{\bullet}$ be a k-split, relatively projective resolution of the left R-module A. Then

$$\begin{aligned} HH_*(A) \otimes_A B &\cong \mathrm{Tor}_*^{R/k}(A, A) \otimes_A B \cong B \otimes_A H_*(A \otimes_R P_{\bullet}) \\ &\cong H_*(B \otimes_A A \otimes_R P_{\bullet}) \qquad \text{since } B \text{ is flat over } A \\ &\cong H_*(B \otimes_R P_{\bullet}) \cong \mathrm{Tor}_*^{R/k}(B, A) \\ &\cong \mathrm{Tor}_*^{T/k}(B, T \otimes_R A) \qquad \text{by 1.5} \\ &\cong \mathrm{Tor}_*^{T/k}(B, B \otimes_A B) \cong HH_*(B; B \otimes_A B). \quad \square \end{aligned}$$

Proof of Étale Descent (0.1). Since B is an étale extension of A, it is unramified over A. Thus $B \otimes_A B \cong B \times C$ for some étale extension C of B, and

$$HH_*(B; B \otimes_A B) \cong HH_*(B) \oplus HH_*(B; C).$$

By Theorem 2.1 we need only show that $HH_*(B; C) = 0$.

Let $T = B \otimes_k B$, and let m be a maximal ideal of T. Since T surjects onto the ring $B \otimes_A B \cong B \times C$, either B_m or C_m is zero. Hence

$$\begin{aligned} HH_*(B; C) \otimes_T T_m &\cong \mathrm{Tor}_*^{T/k}(B, C) \otimes_T T_m \\ &\cong \mathrm{Tor}_*^{T_m/k}(B_m, C_m) \qquad \text{by 1.6} \\ &= 0 \end{aligned}$$

Since $HH_*(B; C)$ localized at every maximal ideal of T is zero, $HH_*(B; C) = 0$. $\square$

EXAMPLE 2.2 (GALOIS DESCENT). Galois extensions are a special class of étale extensions. Recall ([KO, p. 46]) that if G is a finite group of A-automorphisms of a faithfully flat A-algebra B, then B/A is a *Galois extension with group G* if and only if $B \otimes_A B \cong \Pi_{g \in G} B$ by the map whose g^{th} coordinate is $b \otimes c \mapsto b \cdot g(c)$. Note that $A \cong B^G$ follows from this definition and faithfully flat descent. G also acts on $HH_*(B)$, and the isomorphism $HH_*(B) \cong HH_*(A) \otimes_A B$ is G-equivariant. Therefore,

$$HH_*(B)^G \cong [HH_*(A) \otimes_A B]^G \cong HH_*(A) \otimes_A B^G \cong HH_*(A).$$

EXAMPLE 2.3 (FAITHFULLY FLAT ÉTALE DESCENT). If B/A is not only étale but also faithfully flat, then (0.1) implies that the augmented Amitsur complex

$$0 \to HH_*(A) \xrightarrow{\varepsilon} HH_*(B) \to HH_*(B \otimes_A B) \to HH_*(B \otimes_A B \otimes_A B) \to \cdots$$

is exact ([A], [TDTE, I, p. 18], [KO, p. 30], [M, p. 16]). This is called "naïve descent" in the introduction and "faithfully flat étale descent" in the literature.

As a special case, suppose that $\mathscr{U} = \{\operatorname{Spec}(B_1), \ldots, \operatorname{Spec}(B_n)\}$ is a cover of $\operatorname{Spec}(A)$ by affine open subsets. Then $B = \Pi B_i$ is faithfully flat and étale over A, and the Amitsur complex computes Čech cohomology of the cover $\mathscr{U}$ ([TDTE, I, p. 14], [M, p. 97]):

$$\check{H}^m(\mathscr{U}, \mathscr{H}\mathscr{H}_*) = \begin{cases} HH_*(A) & \text{if } m = 0 \\ 0 & \text{otherwise} \end{cases}.$$

When $B = B_1 \times B_2$, this yields the short exact sequence of (0.2).

§3. Naïve descent for cyclic homology

The purpose of this section is to analyze naïve descent for the functors $F = HC_n$. Because the cyclic homology groups $HC_n(A)$ are not A-modules in general, we cannot apply faithfully flat descent theory to the HC_n. Indeed, the following example shows that naïve descent fails for HC_n, even for faithfully flat étale extensions.

EXAMPLE 3.1. Let $A = k[x, y]/(y(y - x^2 + x))$ be the coordinate ring of a line and a parabola in the plane. The elements x and $t = x - 1$ are relatively prime; so

$B = A[1/x] \times A[1/t]$ is faithfully flat and ètale over A. Furthermore, $B \otimes_A B = B \times A[1/xt]$. Using the calculations of [GRW] we can show that if $n > 0$, then

$$HC_n(B) \cong HC_n(k \times k) \oplus HC_{n-1}(k \times k \times k \times k) \oplus V_n \oplus V_n$$

$$HC_n\left(A\left[\frac{1}{xt}\right]\right) \cong HC_n(k \times k) \oplus HC_{n-1}(k \times k \times k \times k),$$

where V_n is k if n is odd ($n > 0$) and 0 otherwise. The long exact sequence of [GRW, A.2] yields the (split) exact sequence for $n > 0$

$$0 \to HC_{n+1}(k) \to HC_n(A) \to HC_n(B) \to HC_n\left(A\left[\frac{1}{xt}\right]\right) \to HC_n(k) \to 0.$$

Since $HC_n(A)$ does not inject into $HC_n(B)$ for n odd, naïve descent fails for B/A. We remark that

$$HC_n(A) \cong HC_n(k) \oplus HC_{n+1}(k) \oplus V_n \oplus V_n.$$

Galois extensions provide one case in which the HC_n satisfy naïve descent. Recall ([KO, p. 121]) that if $A \subseteq B$ is a finite Galois extension and if F is any functor, then the Galois group G acts on $F(B)$, and the Amitsur cohomology $H^*(B/A; F)$ is the same as the group cohomology $H^*(G; F(B))$. Clearly naïve descent implies that $F(A) \cong F(B)^G$. If $F(B)$ is a $\mathbf{Q}$-module (or more generally if $|G|$ acts invertibly on $F(B)$, so that $H^i(G; F(B)) = 0$ for $i \neq 0$), then naïve descent is equivalent to the isomorphism $F(A) \cong F(B)^G$. Therefore the cyclic homology part of our Galois Descent Theorem (0.3) is equivalent to the following result.

PROPOSITION 3.2. *Every cyclic homology group HC_n satisfies naïve descent for finite Galois extensions $A \subseteq B$, assuming that $\mathbf{Q} \subseteq A$ or more generally that the order of the Galois group G is a unit in B. Moreover,*

$$HC_*(A) \cong HC_*(B)^G.$$

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
HC_{n-1}(A) & \to & HH_n(A) & \to & HC_n(A) & \to & HC_{n-2}(A) & \to & HH_{n-1}(A) \\
\downarrow & & \downarrow\cong & & \downarrow & & \downarrow & & \downarrow\cong \\
HC_{n-1}(B)^G & \to & HH_n(B)^G & \to & HC_n(B)^G & \to & HC_{n-2}(B)^G & \to & HH_{n-1}(B)^G
\end{array}$$

The top row is the SBI sequence, which is exact. If $|G|$ is a unit in B, then $|G|$ acts invertibly on $HH_*(B)$ and $HC_*(B)$, so the bottom row is exact. The theorem follows from induction on n and the 5-lemma. □

The following is another example in which naïve descent holds for a piece of cyclic homology. Let $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ be a graded k-algebra containing $\mathbf{Q}$, and write $F^+(A)$ for the kernel of $F(A) \to F(A_0)$ so that $F(A) \cong F(A_0) \oplus F^+(A)$. The usual SBI sequence of cyclic homology breaks up into short exact sequences ([G])

$$0 \to HC^+_{m-1}(A) \xrightarrow{B} HH^+_m(A) \xrightarrow{I} HC^+_m(A) \to 0.$$

PROPOSITION 3.3. *If $A \subseteq B$ is an étale extension of graded k-algebras such that $\mathbf{Q} \subseteq A_0$ and $A_0 \subseteq B_0$ is also étale, then the relative cyclic homology functors HC^+_m satisfy naïve descent for $A \subseteq B$.*

Proof. The following is a short exact sequence of chain complexes. Note that $B \otimes_A \cdots \otimes_A B$ is graded with $B_0 \otimes_{A_0} \cdots \otimes_{A_0} B_0$ in degree zero.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & HC^+_{m-1}(A) & \to & HC^+_{m-1}(B) & \to & HC^+_{m-1}(B \otimes_A B) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & HH^+_m(A) & \to & HH^+_m(B) & \to & HH^+_m(B \otimes_A B) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
0 \to & HC^+_m(A) & \to & HC^+_m(B) & \to & HC^+_m(B \otimes_A B) & \to \cdots \\
 & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 &
\end{array}$$

Since the middle row is exact by (2.3) and $HC_m(A) = 0$ for $m < 0$, the result follows by induction on m and a diagram chase. □

APPLICATIONS 3.3.1. (i) If $A = A_0[t]$, then $F^+(A)$ is usually written as $NF(A_0)$; (ii) if $A = A_0[t]/(t^{p+1})$, then $F^+(A)$ is usually written as $C_pF(A_0)$. Proposition 3.3 state that the functors NHC_m and C_pHC_m satisfy naïve descent for faithfully flat étale extensions, assuming that $\mathbf{Q} \subseteq k$.

THEOREM 3.4 (ÉTALE DESCENT FOR CYCLIC HOMOLOGY). *Let A be a commutative, finite-dimensional noetherian k-algebra, and B an étale, faithfully flat* *extension of A. Then there is a fourth quadrant spectral sequence approaching $HC_*(A)$, beginning with Amitsur cohomology*:

$$E_2^{pq} = H^p(B/A; HC_{-q}) \Rightarrow HC_{p+q}(A).$$

It converges if any of the following conditions are met:

(a) $\mathbf{Q} \subseteq A$;

(b) $B = B_1 \times \cdots \times B_n$, *and each* B_i *is a localization of* A;

(c) B *is a Nisnevich cover of* A *([Nis]). That is, for every prime ideal* p *of* A, *there is a prime ideal* q *of* B *lying over* p *such that* $k(p) \cong k(q)$;

(d) $A = F[x_1, \ldots, x_m]/I$ *for some algebraically closed field* F;

(e) A *is finitely generated over one of the finite rings* $\mathbf{Z}[\frac{1}{2}]/n$ *or* $\mathbf{Z}[i]/n$.

Proof. These are all versions of the descent spectral sequence (A.5) obtained from the double complex $C^{pq} = HC_{-q}(B^{\otimes_A p+1})$. In case (a) we cite (4.9). In case (b) we cite (4.6.1) and (4.7). In case (c) we cite 4.8. In cases (d) and (e) we cite (4.9) and the remarks preceding it. □

Remark. If B/A is a finite Galois extension, the E_2^{pq} term is $H^p(G; HC_{-q}(B))$. This provides a more high-powered proof of Galois descent (3.2), for the spectral sequence will then collapse when $\mathbf{Q} \subseteq A$.

APPLICATION 3.5 (ČECH COHOMOLOGY). If $\mathcal{U} = \{U_1, \ldots, U_N\}$ is a finite covering of $X = \operatorname{Spec}(A)$ by affine opens, $U_i = \operatorname{Spec}(B_i)$, then $B = \Pi B_i$ is faithfully flat and étale over A. In this case the Amitsur cohomology $H^p(B/A; F)$ is the usual Čech cohomology $\check{H}^p(\mathcal{U}; F)$, which vanishes for $p > N$ ([TDTE], [M]). In this case we have a convergent spectral sequence

$$E^2_{pq} = \check{H}^{-p}(\mathcal{U}; HC_{-q}) \Rightarrow HC_{p+q}(A).$$

In the special case $N = 2$, $U_1 \cap U_2$ is $\operatorname{Spec}(B_1 \otimes_A B_2)$, and the spectral sequence degenerates into the long exact Mayer-Vietoris sequence (cf. [Bl], [Bry], [GRW])

$$\begin{aligned} \cdots \to HC_{n+1}(B_1 \otimes_A B_2) \to HC_n(A) \to HC_n(B_1) \oplus HC_n(B_2) \\ \to HC_n(B_1 \otimes_A B_2) \to \cdots. \end{aligned}$$

Example 3.1 shows that this sequence need not break up, as the Hochschild sequence did in (0.2). In fact, $H^1(B/A; HC_n) = HC_n(k)$ for this example.

§4. HH and HC for schemes

As motivation for the ideas of this section, let X be a scheme over k, and consider the (reindexed) cochain complex $\mathscr{H}\mathscr{H}^{\cdot}_{X/k}$ of Hochschild homology sheaves on X (see 0.4):

$$0 \to \mathscr{H}\mathscr{H}_0 \xrightarrow{d} \mathscr{H}\mathscr{H}_1 \xrightarrow{d} \mathscr{H}\mathscr{H}_2 \xrightarrow{d} \cdots.$$

The *de Rham cohomology* of X is the hypercohomology $H^*_{dR}(X) = H^*(X; \mathscr{H}\mathscr{H}^{\cdot}_{X/k})$. When X is smooth over k, this recovers the definition of [GdR] because in that case $\mathscr{H}\mathscr{H}_n = \Omega^n_{X/k}$. As in *op. cit.*, the hypercohomology spectral sequence

$$E_2^{pq} = H^p(X, \mathscr{H}\mathscr{H}_q) \Rightarrow H^{p+q}_{dR}(X)$$

converges for all X. Using Corollary 0.4, we see that if $X = \operatorname{Spec}(A)$ then the spectral sequence degenerates, showing that $H^*_{dR}(\operatorname{Spec}(A))$ is the cohomology of the complex

$$0 \to A \xrightarrow{d} HH_1(A) \xrightarrow{d} HH_2(A) \xrightarrow{d} \cdots,$$

which is $H^*_{dR}(A)$, as one might naïvely think.

All this works well because the cochain complex $\mathscr{H}\mathscr{H}^{\cdot}_{X/k}$ is bounded below. Now consider the problem of defining the Hochschild homology of any scheme X. Let $C_*(A)$ be a functorial chain complex whose homology is the Hochschild homology $HH_*(A)$ of a k-algebra A, and let $\tilde{C}_*$ be the chain complex of sheaves on X associated to the presheaf $C_*(U) = C_*(\Gamma(\mathscr{U}, \mathcal{O}_X))$. Note that the $\tilde{C}_*$ are not in general quasicoherent sheaves, and that the corresponding cochain complex $\tilde{C}^{-*}$ is not bounded below. We define the Hochschild homology of the k-scheme X to be the Zariski hypercohomology

$$HH_n(X) = \mathbf{H}^{-n}_{\mathrm{Zar}}(X, \tilde{C}_*) \cong \mathbf{H}^{-n}_{\mathrm{Zar}}(X, \tilde{C}^{-*}).$$

The definition of hypercohomology we have given in the appendix differs slightly from the usual hypercohomology of [EGA, 0_{III}]. If X were also noetherian of finite Krull dimension, $\mathbf{H}^*(X; \tilde{C}^{-*})$ would agree with the hypercohomology [EGA, 0_{III}] by the device of [M, pp. 311–312] or [Hart, I.5.38]. Although HH_n is a contravariant functor of X, we have indexed it with subscripts in order to have the following.

THEOREM 4.1. *If $X = \operatorname{Spec}(A)$ is an affine scheme over k, then $HH_n(X) \cong HH_n(A)$ for all n. In particular, $HH_n(X) = 0$ for $n < 0$.*

Proof. This is a consequence of (0.4) and the following more general result.

THEOREM 4.2. *There is a fourth quadrant (cohomology) spectral sequence with*

$$E_2^{pq} = H^p_{\mathrm{Zar}}(X; \mathscr{HH}_{-q}) \cong H^p_{\mathrm{Nis}}(X; \mathscr{HH}_{-q}) \cong H^p_{\mathrm{et}}(X; \mathscr{HH}_{-q})$$

which approaches $HH_{-p-q}(X)$, and converges if the spectral sequence is bounded, i.e., for each n there are only finitely many nonzero groups $H^p(X; \mathscr{HH}_q)$ with $-p+q=n$.

Proof. This is the hypercohomology spectral sequence (A.2), once we observe that the homology sheaf $H_n(\tilde{C}_*)$ is the sheaf $\mathscr{HH}_n$. This observation follows from the exactness of the sheafification functor ([M, p. 63]). The alternate characterizations of the E_2^{pq} term come from (0.4). □

COROLLARY 4.2.1. $HH_n(X) \cong \mathbf{H}^{-n}_{\mathrm{Nis}}(X; \tilde{C}_*) \cong \mathbf{H}^{-n}_{\mathrm{et}}(X; \tilde{C}_*)$.

Proof. This follows from 4.2 and the Comparison Theorem A.3. □

EXAMPLE 4.3. If X is noetherian of dimension d, then $HH_n(X) = 0$ for $n < -d$; $HH_{-d}(X) = H^d(X; \mathcal{O}_X)$, and there is a short exact sequence

$$H^{d-2}(X; \mathcal{O}_X) \to H^d(X; \Omega_{X/k}) \to HH_{1-d}(X) \to H^{d-1}(X, \mathcal{O}_X) \to 0.$$

In general, there will be infinitely many $n > 0$ with $HH_n(X) \neq 0$.

EXAMPLE 4.3.1. If X is a smooth projective curve of genus g over a field k, then $\mathscr{HH}_n = \Omega^n_{X/k}$, which is zero for $n \geq 2$. Hence $HH_{-1}(X) \cong H^1(X; \mathcal{O}_X) \cong k^g$, $HH_0(X) \cong k^2$ via an extension

$$0 \to H^1(X; \Omega_{X/k}) \to HH_0(X) \to H^0(X; \mathcal{O}_X) \to 0,$$

and $HH_1(X) \cong H^0(X, \Omega_{X/k}) \cong k^g$. $HH_n(X) = 0$ for $n \neq 0, 1, -1$.

DENNIS TRACE MAP (4.4). The Dennis trace map $D: K_n(A) \to HH_n(A)$ induces a map for every scheme X:

$$K_n(X) \xrightarrow{\eta} \mathbf{H}^{-n}(X; K) \to \mathbf{H}^{-n}(X; C_*) = HH_n(X).$$

Here η is the augmentation map of [AKTEC, 1.33]. The Dennis trace map induces a map of spectral sequences from the Brown-Gersten spectral sequence of algebraic K-theory to the spectral sequence of 4.2.

DEFINITION 4.5 (LODAY [L, 3.4]). Let $B_*(A)$ be any functorial chain complex whose homology is the cyclic homology, $HC_*(A)$, of a k-algebra A, and let $\tilde{B}_*$ be the chain complex of sheaves on X associated to the presheaf $B_*(U) = B_*(\Gamma(U, \mathcal{O}_X))$. The homology sheaf $H_n(\tilde{B}_*)$ is the sheaf $\mathscr{HC}_n$ associated to the presheaf $U \mapsto HC_n(\Gamma(U, \mathcal{O}_X))$. We define the cyclic homology of the k-scheme X to be the Zariski hypercohomology

$$HC_n(X) = \mathbf{H}^{-n}_{\mathrm{Zar}}(X, \tilde{B}_*) \cong \mathbf{H}^{-n}_{\mathrm{Zar}}(X, \tilde{B}^{-*}).$$

Our first job is to show that if X is affine then we recover the usual definition of cyclic homology. A proof of the formula $HC_*(\mathrm{Spec}\,(A)) = HC_*(A)$ for arbitrary k-algebras A is beyond the scope of this paper. In this paper we shall only prove this if A is noetherian and finite-dimensional. For this we need:

SBI SEQUENCE (4.5.1). *There is a long exact sequence*

$$\cdots \to HC_{n+1}(X) \xrightarrow{S} HC_{n-1}(X) \xrightarrow{B} HH_n(X) \xrightarrow{I} HC_n(X) \xrightarrow{S} \cdots.$$

Proof. This is the hypercohomology exact sequence (see A.4) of the exact sequence $0 \to C_* \xrightarrow{I} B_* \xrightarrow{S} B_*[-2] \to 0$ of chain complexes in [LQ, 1.6]. □

LEMMA 4.6. *If X is a noetherian k-scheme of dimension d, then $HC_n(X) = 0$ for $n < -d$ and $HC_{-d}(X) \cong HH_{-d}(X) \cong H^d(X; \mathcal{O}_X)$.*

Proof. The hypercohomology spectral sequence of (A.2),

$$E_2^{pq} = H^p_{\mathrm{Zar}}(X; \mathscr{HC}_{-q}) \Rightarrow HC_{-p-q}(X),$$

converges, showing that $HC_n(X) = 0$ for $n < -d$. □

COROLLARY 4.6.1. *If $X = \mathrm{Spec}\,(A)$ is affine, noetherian and finite dimensional, then $HC_n(X) \cong HC_n(A)$ for all n. In particular, $HC_n(X) = 0$ for $n < 0$.*

Proof. Using Theorem 4.1, the SBI sequence (4.5.1), and induction on n, we see that it suffices to show for some $d \geq 0$ that $HC_n(X) = 0$ for $n < -d$. □

QUESTION 4.6.2. If $X = \mathrm{Spec}\,(A)$, is $H^p(X; \mathscr{HC}_q) = 0$ for $p \geq q$ $(p \neq 0)$? The proof of (4.6) and (A.2.1) show that an affirmative answer to this question would yield the formula $HC_*(\mathrm{Spec}\,(A)) = HC_*(A)$ for arbitrary k-algebras A.[1]

Remark 4.6.3. If A is smooth over a field k of characteristic zero, we know by [LQ] that $HC_n(A) = \Omega^n_{A/k}/d\Omega^{n-1}_{A/k} \oplus H^{n-2}_{dR} \oplus H^{n-4}_{dR} \oplus \cdots$. Let $X = \mathrm{Spec}\,(A)$. By Bloch-Ogus ([BO, 2.2]), we know that $H^p(X; \mathscr{H}^q_{dR}) = 0$ for $p > q$. It follows that $H^p(X; \mathscr{H}\mathscr{C}_q) = 0$ for $p \geq q$, except for $H^0(X; \mathscr{H}\mathscr{C}_0) = A$, so the question has an affirmative answer in this case.

PROPOSITION 4.7. *Let X be a finite dimensional noetherian k-scheme and $\mathscr{U}$ a Zariski cover of X by open subsets. Then the descent spectral sequence for Čech cohomology converges*:

$$E_2^{pq} = \check{H}^p(\mathscr{U}; HC_{-q}) \Rightarrow HC_{-p-q}(X).$$

Proof. See (A.5) or [AKTEC, 1.47]. □

There are other topologies to use other than the Zariski topology, such as the étale topology ([M]) or the Nisnevich topology ([Nis]). If X is a noetherian scheme of dimension d, then $H^n_{\mathrm{Nis}}(X; -) = 0$ for $n > d$, so the argument in (4.5)–(4.7) goes through verbatim for the Nisnevich site. There is a map from the Zariski SBI sequence to the Nisnevich SBI sequence. As in 4.6, $\mathbf{H}^n_{\mathrm{Zar}}(X; \tilde{B}_*) = \mathbf{H}^n_{\mathrm{Nis}}(X; \tilde{B}_*) = 0$ for $n > \dim(X)$. Since $\mathbf{H}^n_{\mathrm{Zar}}(X; \tilde{C}_*) = \mathbf{H}^n_{\mathrm{Nis}}(X; \tilde{C}_*)$ by (4.2.1), we may apply induction on n and the 5-lemma to prove the following result.

PROPOSITION 4.8. *Let X be a finite dimensional noetherian k-scheme. Then $HC_n(X) \cong H^{-n}_{\mathrm{Nis}}(X; B_*)$. Moreover, if $\mathscr{U}$ is a Nisnevich cover of X, then the descent spectral sequence for Čech cohomology converges*:

$$E_2^{pq} = \check{H}(\mathscr{U}; HC_{-q}) \Rightarrow HC_{-p-q}(X).$$

We now compare our Zariski hypercohomology to an étale hypercohomology construction. For simplicity, we shall assume that $\mathbf{Q} \subseteq k$, noting that, as in[AKTEC, 1.48], our methods would also apply to

(a) schemes of finite type over an algebraically closed field, and
(b) schemes of finite type over $\mathbf{Z}[\frac{1}{2}]/n$ or $\mathbf{Z}[i]/n$.

---

[1] **Note added in proof.** L. Barbiéri-Viale has shown that this question has a negative answer for $q = 1$, because $H^p(X; \mathscr{H}\mathscr{C}_1)$ is sometimes isomorphic to $H^{p+2}(X; k)$, which can be nonzero if X is not irreducible. If A is of finite type over a field k then Remark 4.6.3 shows that $H^p(X, \mathscr{H}\mathscr{C}_q) = 0$ if $p \geq \max(q, 2 + \dim(\mathrm{Sing}\, X))$.

PROPOSITION 4.9. *Let X be a finite dimensional noetherian k-scheme, and assume that $\mathbf{Q} \subseteq k$. Then the Zariski and étale hypercohomologies agree*:

$$HC_n(X) \cong \mathbf{H}_{\mathrm{Zar}}^{-n}(X; \tilde{B}_*) \cong \mathbf{H}_{\mathrm{\acute{e}t}}^{-n}(X; \tilde{B}_*).$$

Moreover, if $\mathscr{U}$ is an étale cover of X, then the descent spectral sequence for Čech cohomology converges:

$$E_2^{pq} = \check{H}(\mathscr{U}; HC_{-q}) \Rightarrow HC_{-p-q}(X).$$

Proof. We replay the above tape for the étale site. There is an étale hypercohomology SBI sequence as in (4.5.1) and a map from the Zariski SBI sequence to the étale SBI sequence. By (4.2.1), $\mathbf{H}_{\mathrm{Zar}}^*(X; \tilde{C}_*)$ and $\mathbf{H}_{\mathrm{\acute{e}t}}^*(X; \tilde{C}_*)$ agree with $HH_*(X)$. As in (4.6), $\mathbf{H}_{\mathrm{\acute{e}t}}^n(X; \tilde{B}_*) = 0$ for $n > \dim(X)$. We complete the proof by using induction and the 5-lemma. □

§5. *K*-theory of seminormal curves

In the proof of Theorem 0.5 we need the following special case of the "KABI conjecture" of [GRW]:

PROPOSITION 5.1. *Let $l \subset l_1$ be fields of characteristic zero, $A = l \oplus tl_1[t]$, and $I = tl_1[t]$. Then the map*

$$v : K_n(A, B, I) \to HC_{n-1}^{\mathbf{Q}}(A, B, I)$$

is an isomorphism for all n.

Proof. Choose a Galois extension L of l containing l_1, and note that, as on p. 74 of [GRW],

$$A \otimes_l L \cong L[x_0, \ldots, x_n]/(x_i x_j = 0, i \neq j)$$

and

$$B \otimes_l L \cong \Pi L[x_i]$$

where $n = [l_1 : l]$. By Theorem A.2 of [WA], the natural map

$$v_L : K_n(A \otimes_l L, B \otimes_l L, I \otimes_l L) \to HC_{n-1}^{\mathbf{Q}}(A \otimes_l L, B \otimes_l L, I \otimes_l L)$$

is an isomorphism. By naturality, v_L is compatible with the action of the Galois group G of L/l; so the G-invariant subgroups are also isomorphic. By [vdK, 1.10] it follows that

$$K_n(A, B, I) \cong K_n(A \otimes_l L, B \otimes_l L, I \otimes_l L)^G;$$

while by Galois descent for cyclic homology (0.3), it follows that

$$HC_m(A, B, I) \cong HC_m(A \otimes_l L, B \otimes_l L, I \otimes_l L)^G.$$

The result is now evident. □

Proof of Theorem 0.5. Choose a Galois extension L of l with group G and containing all of the l_i. Then $B = A \otimes_l L$ is a Galois extension of A with Galois group G, and $B \cong L[x_0, \ldots, x_b]/\{x_i x_j = 0 \text{ for } i \neq j\}$. The calculation of $HC_*(B)$ in [GRW, 3.12] and (0.3) gives the cyclic homology of A. We may now copy the proof of [GRW, 8.4]. □

Appendix: Hypercohomology

If $K^\cdot$ is a cochain complex of sheaves on a site X, the hypercohomology of $K^\cdot$ is well-known as long as $K^\cdot$ is bounded below or if X has finite cohomological dimension ([Hart], [EGA, 0_{III}], [M, Appendix C]). Unfortunately, we need the hypercohomology when $K^\cdot$ is *bounded above*, as is the case with the Hochschild complex, and we could find nothing in the literature more explicit than the sheaves of spectra approach of Thomason [AKTEC]. This appendix is largely a translation of [AKTEC] into the language of homological algebra, and is included for the convenience of the reader.

Because there are enough injective sheaves, we can form an injective Cartan-Eilenberg resolution $I^{\cdot\cdot}$ of $K^\cdot$, i.e., a right half-plane cochain double complex of injective sheaves I^{pq} together with an augmentation $K^\cdot \to I^{\cdot q}$ so that each $K^\cdot \to I^{\cdot q}$ is an injective resolution and the conditions of [CE, p. 363], [Hart, p. 76] or [EGA, 0_{III}, 11.4.2] are met. If $L^\cdot$ is another complex, with injective Cartan-Eilenberg resolution $J^{\cdot\cdot}$, every morphism $K^\cdot \to L^\cdot$ lifts to a map $I^{\cdot\cdot} \to J^{\cdot\cdot}$ of double complexes, which is unique up to chain homotopy [CE, XVII.1.2].

The total complex $\mathrm{Tot}(I^{\cdot\cdot})$, which is the product $\Pi_{p+q=n} I^{pq}$ in degree n, is a cochain complex of sheaves. Since Tot converts chain homotopy equivalent maps of double complexes to chain homotopy equivalent maps of complexes, it follows that $\mathrm{Tot}(I^{\cdot\cdot})$ is well-defined up to chain homotopy equivalence, and that the lift

$\mathrm{Tot}(I^{\cdot\cdot}) \to \mathrm{Tot}(J^{\cdot\cdot})$ of a map $K^\cdot \to L^\cdot$ is unique up to chain homotopy. Taking global sections yields hypercohomology.

DEFINITION A.1. The *hypercohomology group* $\mathbf{H}^n(X; K^\cdot)$ is the n^{th} cohomology group of the complex $\mathrm{Tot}(H^0(X; I^{\cdot\cdot})) \cong H^0(X; \mathrm{Tot}(I^{\cdot\cdot}))$ of global sections of $\mathrm{Tot}(I^{\cdot\cdot})$. The above remarks show that, up to isomorphism, $\mathbf{H}^n(X; K^\cdot)$ is independent of the choice of $I^{\cdot\cdot}$. Moreover, every map $K^\cdot \to L^\cdot$ gives rise to a unique map $\mathbf{H}^n(X; K^\cdot) \to \mathbf{H}^n(X; L^\cdot)$. If $K^\cdot$ is bounded below, this is the usual definition of [EGA, 0_{III}] or [Hart].

VARIANT. If $C_\bullet$ is a chain complex of sheaves on X, we form the usual cochain complex $K^\cdot$ with $K^n = C_{-n}$ and set $\mathbf{H}^n(X; C_\bullet) = \mathbf{H}^n(X; K^\cdot)$.

Remark. The Godement double complex T^pK^q of [AKTEC, 1.31] is not a Cartan–Eilenberg resolution, but there is a map $T^\cdot K^\cdot \to I^{\cdot\cdot}$, unique up to chain homotopy, which induces a quasi-isomorphism from $\mathrm{Tot}(T^pK^q)$ to $\mathrm{Tot}(I^{\cdot\cdot})$ and isomorphisms from Thomason's $\pi_{-n}\mathbf{H}^\cdot(X; K^\cdot)$ to our $\mathbf{H}^n(X; K^\cdot)$. This may be seen using the Eilenberg–Moore Comparison Theorem (see [EM] or A.3 below).

We could also define the hypercohomology complex $\mathbf{H}^\cdot(X; K^\cdot)$ to be the cochain complex $H^0(X; \mathrm{Tot}(I^{\cdot\cdot}))$, considered as an object in the derived category of cochain complexes of abelian groups. The map from Thomason's $\mathbf{H}^\cdot(X; K^\cdot)$ to ours is a quasi-isomorphism but probably not an isomorphism in general. However, we will have no use for this notion.

By choosing a fixed Cartan-Eilenberg resolution for every $K^\cdot$ in whichever small category we have under consideration, we see that the $\mathbf{H}^n(X; -)$ may be made functorial. Of course, Thomason's choices are already functorial ([AKTEC, 1.33]).

HYPERCOHOMOLOGY SPECTRAL SEQUENCE (A.2). *There is a right half-plane hypercohomology spectral sequence*

$$E_2^{pq} = H^p(X; H^q(K^\cdot)) \Rightarrow \mathbf{H}^{p+q}(X; K^\cdot)$$

which converges strongly under any of the following conditions:

(a) *$K^\cdot$ is cohomologically bounded below*;

(b) *X has finite cohomological dimension*;

(c) *The spectral sequence is bounded, i.e., each diagonal $p+q=n$ of E_2^{**} contains only finitely many nonzero terms.*

(d) *For each p and q,* $\lim^1 E_r^{pq} = 0$.

Proof. This is the usual hypercohomology spectral sequence of [EGA, 0_{III}, 12.4] obtained by filtering the double complex $H^0(X; \mathrm{Tot}\,(I^{\cdot\cdot}))$ by columns. Boardman ([B, Theorem 8.1]) proved that the spectral sequence converges strongly if and only if (d) holds. Since (a), (b), and (c) all imply (d), we are done. □

ADDENDUM A.2.1. More precisely, for each n, the spectral sequence converges completely to $\mathbf{H}^n(X; K^\cdot)$ if $\lim^1 E_r^{pq} = 0$ for all p and q with $p + q = n$ or $p + q = n - 1$.

COMPARISON THEOREM (A.3). *If $K^\cdot \to L^\cdot$ is a morphism of chain complexes such that $H^p(X; H^q(K^\cdot)) \to H^p(X; H^q(L^\cdot))$ is an isomorphism for every p and q, then for all n:*

$$\mathbf{H}^n(X; K^\cdot) \cong \mathbf{H}^n(X; L^\cdot).$$

Proof. [EM, Theorem 7.4]. □

We conclude with two results of Thomason, phrased in the language of homological algebra for ease of reference.

HYPERCOHOMOLOGY EXACT SEQUENCE (A.4). *Hypercohomology is a cohomological δ-functor. That is, if $0 \to K^\cdot \to L^\cdot \to M^\cdot \to 0$ is a short exact sequence of complexes of sheaves on X, then there is a long exact sequence*

$$\cdots \to \mathbf{H}^{n+1}(X; M^\cdot) \xrightarrow{\delta} \mathbf{H}^n(X; K^\cdot) \to \mathbf{H}^n(X; L^\cdot) \to \mathbf{H}^n(X; M^\cdot) \xrightarrow{\delta} \cdots.$$

Proof. [AKTEC, 1.35]. Note that the usual result from [EGA, 0_{III}.11.5.2] does not apply unless the complexes are bounded below, which is emphatically not our case. □

CARTAN-LERAY DESCENT SPECTRAL SEQUENCE (A.5). *Let $K^\cdot$ be a presheaf of cochain complexes on X, and write $\mathbf{H}^q$ for the presheaf $U \mapsto \mathbf{H}^q(U; K^\cdot)$. Then for every cover $\mathscr{U}$ of X, there is a spectral sequence*

$$E_2^{pq} = \check{H}^p(\mathscr{U}; \mathbf{H}^q) \Rightarrow \mathbf{H}^{p+q}(X; \tilde{K}^\cdot)$$

which converges if there is a bound d such that if $p > d$ then for all q, $H^p(X; \mathscr{H}^q) = 0$ and $H^p(U; \mathscr{H}^q) = 0$ for all $U \in \mathscr{U}$.

Proof. [AKTEC, 1.46]. The following is a sketch of the proof. If $K^\cdot$ is bounded below, we cite the classical result, say from [EGA, 0_{III}, 12.4.6]. Now let $K^\cdot\langle n\rangle$

denote the "good" bounded below truncations of K and observe that for $q < n - d$, the sheaves $\mathscr{H}^q(-; K^{\cdot})$ and $\mathscr{H}^q(-; K^{\cdot}\langle n \rangle)$ agree, and $\mathbf{H}^q(X; K^{\cdot}) \cong \mathscr{H}^q(X; K^{\cdot}\langle n \rangle)$ as well. A use of the Comparison Theorem (A.3) completes the proof. $\square$

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Department of Mathematics
Rutgers University
New Brunswick, New Jersey 08903, USA

and

Department of Mathematics
Texas A&M University
College Station, Texas 77843, USA

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