Comment. Math. Helvetici 66 (1991) 362–367 0010-2571/91/030362-06$1.50 + 0.20/0

On the projective normality of the adjunction bundles

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Let X be a smooth projective variety of dimension n, polarized by a very ample line bundle L. Let K_X be the canonical line bundle of holomorphic n-forms and consider the line bundles given by: $\mathscr{L}_{(a,b)} = K_X{}^{\otimes a} \otimes L^{\otimes b}$ (adjoint bundles).

Let $\mathscr{A}$ be the following set of polarized varieties (X, L) : $\{(\mathbf{P}^n, O(1)), (\mathbf{P}^2, O(2)), (\mathbf{Q}^n, O_{\mathbf{Q}^n}(1))$, where $\mathbf{Q}^n$ is the smooth quadric, embedded in $\mathbf{P}^{n+1}$ in the standard way, X is a $\mathbf{P}^{n-1}$ bundle over a smooth curve, and the restriction of L to a fibre is $O_{\mathbf{P}^{n-1}}(1)\}$.

If (X, L) is not one of the pairs in $\mathscr{A}$ and $b \geq (n-1)a + 1$, then the map associated to $\Gamma(\mathscr{L}_{(a,b)})$ is an embedding; this is an easy consequence of [So] and [VdV] (see also [So-VdV]). In this paper we answer the natural question on the projective normality of this embedding; we will say that a pair $(X, \mathscr{L})$ is projectively normal if the natural maps $S^\rho H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^{\otimes \rho})$ are surjective for $\rho \geq 1$, where $S^\rho V$ is the ρ-symmetric product of a vector space. In particular we prove the following:

THEOREM. *Suppose that (X, L) is not one of the pairs in $\mathscr{A}$. If $a \geq 1$ and $b \geq (n-1)a + 1$, then $(X, \mathscr{L}_{(a,b)})$ is projectively normal.*

For $a = 1$ the bound on b is sharp; see [A-B] for some examples of polarized surfaces for which $\mathscr{L}_{(1,1)}$ is very ample but not projectively normal; in that paper it is shown that $\mathscr{L}_{(a,a)}$ is projectively normal for $a \geq 2$ (if it is very ample).

The appendix contains the proof of the lemma (2.6) which is necessary to prove the case $b = (n-1)a + 1$ of the theorem (i.e. the proof of the theorem for $b > (n-1)a + 1$ works without that lemma).

The question on the projective normality of $K_X \otimes L^{\otimes b}$, with $b \geq 2$ and $n = 2$, was posed by S. Mukai and M. L. Green at the conference on Algebraic Geometry in Trieste 1989, (Italy).

We would like to call attention to the preprint "A theorem on the syzygies of smooth projective varieties of arbitrary dimension" by L. Ein and R. Lazarsfeld which contains results similar to ours that they obtained independently.

The authors would like to thank the University of Notre Dame for making their collaboration possible. The first author would also like to thank the Ministero della Pubblica Istruzione (M.P.I. 40%). The second author would like to thank the National Science Foundation (D.M.S. 87-22330).

§1. Notation and preliminaries

(1.1) We consider in this paper a connected complex projective submanifold, X, of dimension n. We denote the structure sheaf of X by O_X. We denote by K_X the canonical sheaf (actually a line bundle) of holomorphic n-forms on X. If D is a divisor on X we will denote by $O_X(D)$ the invertible sheaf associated to D; for any coherent sheaf F on X, $h^i(F)$ denotes the complex dimension of $H^i(X, F)$.

We do not distinguish notationally between a divisor and its associated line bundle.

(1.2) Let $\mathscr{L}$ be a line bundle on a smooth projective variety X. $\mathscr{L}$ is said to be *numerically effective* (*nef*, for short) if $\mathscr{L} \cdot C \geq 0$ for all effective curves C on X, and in this case $\mathscr{L}$ is said to be *big* if $c_1(\mathscr{L})^n > 0$, where $c_1(\mathscr{L})$ is the first Chern class of $\mathscr{L}$.

Let $\mathscr{L}$ be *very ample*, i.e. $\mathscr{L}$ is spanned by global sections and the morphism associated to $\mathscr{L}$, denoted by $\varphi_{\mathscr{L}}: X \to \mathbf{P}^N$, is an embedding; then we give the following definition:

(1.3) DEFINITION. $\mathscr{L}$ is *projectively normal* if for every $\rho \geq 0$ the natural maps

$$(*)_\rho \quad H^0(P^N, O(\rho)) \cong S^\rho H^0(X, \mathscr{L}) \to H^0(X, \mathscr{L}^{\otimes \rho})$$

are surjective.

(1.4) *Remark.* 1) In the case $\rho = 0$ the map is surjective by the irreducibility of X; in the case $\rho = 1$ it is surjective since $\mathscr{L}$ is very ample.

2) If $\mathscr{I}_X$ is the ideal sheaf of X in $\mathbf{P}^N$, our assumption is equivalent to the condition: $H^1(\mathscr{I}_X(\rho)) = 0$ for all $\rho \geq 0$.

3) We say that $(X, \mathscr{L})$ is *arithmetically Cohen–Macaulay* if $\mathscr{L}$ is projectively normal and $H^q(X, \mathscr{L}^{\otimes \rho}) = 0$ for all $\rho \in \mathbf{Z}$ and $1 \leq q < \dim X$. In the case in which $\mathscr{L} = \mathscr{L}_{(a,b)}$, $b \geq (n-1)a + 1$ and (X, L) is not a pair in $\underline{\mathscr{A}}$, this last condition is true

for all $\rho \in \mathbf{Z} - 0$, by the Kawamata–Viehweg vanishing theorem; for $\rho = 0$ this is a topological condition on the vanishing of the betti numbers.

§2. Proof of the theorem

(2.1) Let X be a smooth projective variety of dimension n and L a very ample line bundle on X. We consider the projective $2n$-fold given by the product of X by itself, $P = X \times X$, and we denote by $\pi_i : P \to X$, for $i = 1, 2$, the projections. Moreover we will use the following notation: $H = \pi_1^* L \otimes \pi_2^* L$, $X \cong \Delta = \{(x, y) \in P : x = y\} \subset P$ is the diagonal and $K_P = \det(T_P^*) = \det(\pi_1^* T_X^* \otimes \pi_2^* T_X^*) = \pi_1^* K_X \otimes \pi_2^* K_X$ is the canonical bundle of P.

(2.2) GENERAL CONSTRUCTION. In the above notation, the projective normality of the line bundle $\mathscr{L}_{(a,b)} = K_X{}^{\otimes a} \otimes L^{\otimes b}$, when it is very ample, i.e. the surjectivity of the maps $(*)_\rho$ for $\mathscr{L} = \mathscr{L}_{(a,b)}$ and $\rho \geq 2$, is equivalent at the surjectivity of the maps

$$(*)'_\rho \quad H^0(P, K_P{}^{\otimes a} \otimes H^{\otimes b} \otimes \pi_1^*(K_X{}^{\otimes a} \otimes L^{\otimes b})^{\rho-2})$$
$$\to H^0(\Delta, [K_P{}^{\otimes a} \otimes H^{\otimes b} \otimes \pi_1^*(K_X{}^{\otimes a} \otimes L^{\otimes b})^{\rho-2}]_{|\Delta})$$

In fact the first factor, by the Künneth formula, is $H^0(X, (K_X{}^{\otimes a} \otimes L^{\otimes b})^{\rho-1}) \otimes H^0(X, K_X{}^{\otimes a} \otimes L^{\otimes b})$, while the second is $H^0(X, (K_X{}^{\otimes a} \otimes L^{\otimes b})^\rho)$.

Since by the Kawamata–Viehweg vanishing theorem we have

$$H^1(P, K_P{}^{\otimes a} \otimes H^{\otimes b} \otimes \pi_1^*(K_X{}^{\otimes a} \otimes L^{\otimes b})^{\rho-2}) = 0,$$

those surjectivities are in turn equivalent to

$$H^1(P, K_P{}^{\otimes a} \otimes H^{\otimes b} \otimes \mathscr{D} \otimes \mathscr{I}_\Delta) = 0,$$

for $\rho \geq 2$, where $\mathscr{I}_\Delta$ is the ideal sheaf of the diagonal and we denote $\pi_1^*(K_X{}^{\otimes a} \otimes L^{\otimes b})^{\otimes(\rho-2)} = \mathscr{D}$ for brevity.

We now blow up P along the diagonal Δ, $\pi : \bar{P} \to P$, and we will call $\bar{\Delta} = \pi^{-1}(\Delta)$, $E = [\bar{\Delta}]$ and $\bar{H} = \pi^* H$. Therefore $K_{\bar{P}} = (\pi^* K_P) \otimes E^{\otimes(n-1)}$.

It can be easily proved that $\pi_{(i)} \mathscr{I}_{\bar{\Delta}} = 0$ and $\pi_* \mathscr{I}_{\bar{\Delta}} = \mathscr{I}_\Delta$ (use for instance the theorem on formal functions, page 277 of [Ha]), where π_* and $\pi_{(i)}$ denote the direct image functor and the i-th derived functor of π_* respectively.

(2.3) Therefore, by the Leray spectral sequence applied at π, we have that

$$
\begin{aligned}
H^1(P, K_P{}^{\otimes a} \otimes H^{\otimes b} \otimes \mathscr{D} \otimes \mathscr{I}_\Delta) &= H^1(\bar{P}, \pi^*(K_P{}^{\otimes a} \otimes H^{\otimes b} \otimes \mathscr{D}) \otimes E^{\otimes -1}) \\
&= H^1(\bar{P}, K_{\bar{P}} \otimes \pi^*(K_P{}^{\otimes(a-1)} \otimes H^{\otimes(b-n)} \otimes \mathscr{D}) \\
&\quad \otimes \bar{H}^{\otimes n} \otimes E^{\otimes -n}) \\
&= H^1(\bar{P}, K_{\bar{P}} \otimes \mathscr{C}).
\end{aligned}
$$

(2.4) CONCLUSION. By the Kawamata–Viehweg vanishing theorem our theorem is proved for those value of a, b and n for which $\mathscr{C} := \pi^*(K_P{}^{\otimes(a-1)} \otimes H^{\otimes(b-n)} \otimes \mathscr{D}) \otimes \bar{H}^{\otimes n} \otimes E^{\otimes -n}$ is nef and big.

We prove first that $\bar{H} \otimes E^{\otimes -1}$ is nef, this is an immediate consequence of the following lemma.

(2.5) LEMMA. *In the above notations, if L is very ample then $H \otimes \mathscr{I}_\Delta$, and therefore $\bar{H} \otimes E^{\otimes -1}$, is spanned by global sections.*

Proof. Let first $z = (x, y)$ be a point of P not on Δ, i.e. $x \neq y$ and choose local trivializations in neighborhoods of x and y. Take an element f of $\Gamma(X, L)$ such that $f(x) = f(y) = 1$ and an element g of $\Gamma(X, L)$ such that $g(x) = 2$ and $g(y) = 1$. Therefore $[\pi_1^*(f) \otimes \pi_2^*(g) - \pi_1^*(g) \otimes \pi_2^*(f)]$ is a section of $H \otimes \mathscr{I}_\Delta$ non vanishing at z, therefore spanning $H \otimes \mathscr{I}_\Delta$ at z.

Let then $z = (x, x)$ be a point on the diagonal Δ. Choose a trivialization of L in a neighborhood of x in X. We will prove the spannedness of $H \otimes \mathscr{I}_\Delta$ at z by finding sections of H which together vanish to the first order along Δ in a neighborhood of z. To achieve this purpose, since L is very ample, we can choose sections $f_1, \ldots, f_n$ of L on X_1 (respectively $g_1, \ldots, g_n$ on X_2) whose differentials span the tangent space at x; therefore we can choose coordinates in a neighborhood $U \subset X_1$ of x, $\varphi = (z_1, \ldots, z_n)$ (respectively $\psi = (w_1, \ldots, w_n)$ on X_2) such that $f_i \circ \varphi(p) = z_i$ and $g_i \circ \psi(p) = w_i$ for all $p \in U$. Let f_0 and g_0 be the sections of L such that $f_0 \circ \varphi(p) = 1$, respectively $g_0 \circ \psi(p) = 1$.

Then $[\pi_1^*(f_0) \otimes \pi_2^*(g_i) - \pi_1^*(f_i) \otimes \pi_2^*(g_0)]$, for $i = 1, \ldots, n$, are the required sections.

(2.6) LEMMA. *Suppose that $(X, L) \neq (\mathbf{P}^n, O(1))$. Then $\bar{H} \otimes E^{\otimes -1}$ is big.*

Proof. The proof of this lemma is contained in the appendix.

Proof of the theorem. As pointed out in (2.4), we will search those value of a, b and n for which $\mathscr{C} := \pi^*(K_P{}^{\otimes(a-1)} \otimes H^{\otimes(b-n)} \otimes \mathscr{D}) \otimes \bar{H}^{\otimes n} \otimes E^{\otimes -n}$ is nef and big.

If $b \geq (n-1)a+2$, $a \geq 1$ and if (X, L) is not one of the pairs in the set $\mathscr{A}$, then the line bundle $K_X{}^{\otimes(a-1)} \otimes L^{\otimes(b-n)}$ is nef and big. Therefore the line bundles

$$\begin{aligned}\mathscr{F} &= \pi^*(K_P{}^{\otimes(a-1)} \otimes H^{\otimes(b-n)})\\ &= \pi^*(\pi_1^*(K_X{}^{\otimes(a-1)} \otimes L^{\otimes(b-n)})) \otimes \pi^*(\pi_1^*(K_X{}^{\otimes(a-1)} \otimes L^{\otimes(b-n)}))\end{aligned}$$

are also nef and big, and

$$\mathscr{D} = \pi_1^*((K_X{}^{\otimes a} \otimes L^{\otimes b})^{\otimes(\rho-2)}) \text{ is nef for } \rho \geq 2.$$

This and the lemma (2.5) imply that $\mathscr{C}$ is nef and big if $b \geq (n-1)a+2$.

If $b = (n-1)a+1$, $a \geq 1$ and if (X, L) is not one of the pairs in the set $\mathscr{A}$ then again $\mathscr{F}$ and $\mathscr{D}$ are nef (the last for $\rho \geq 2$) but not necessarily big. We use both lemmas (2.5) and (2.6) to conclude that $\mathscr{C}$ is nef and big in this case too.

Appendix by M. ANDREATTA, E. BALLICO AND A. J. SOMMESE

The appendix is devoted to the proof of the lemma (2.6), which is needed to prove the case $b = (n-1)a+1$ of the theorem.

By the lemma (2.5) $\bar{H} \otimes E^{\otimes -1}$ is spanned by global sections; we call $h_{(X,L)}$, or just h_L, the map associated to the line bundle $\bar{H} \otimes E^{\otimes -1}$. It is sufficient to prove that for a general point the near fibers of this map are zero dimensional.

a) We first show that the fibers of $f := h_{(\mathbf{P}^n, O(1))}$ outside E are isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1 \backslash E$.

More precisely, fix $(u, v) \in \mathbf{P}^n \times \mathbf{P}^n \backslash \Delta$ and let m be the line $[u, v]$, we will show that on $P \backslash E\, f^{-1}(f(\pi^{-1}(u, v)))$ is $\pi^{-1}(m \times m)$.

Let $s : \mathbf{P}^n \times \mathbf{P}^n \to \mathbf{P} = \mathbf{P}^N$ be the Segre embedding (hence $O_{s(\mathbf{P}^n \times \mathbf{P}^n)}(1) \cong \pi_1^* L \otimes \pi_2^* L$); $s(\Delta)$ is the intersection of $s(\mathbf{P}^n \times \mathbf{P}^n)$ with a linear space V and f is induced by the set of hyperplanes containing V; thus, for every $x = s((u, v)) \in s(\mathbf{P}^n \times \mathbf{P}^n \backslash \Delta)$, the fiber of f through (u, v) and outside E is the intersection of $s(\mathbf{P}^n \times \mathbf{P}^n)$ with the linear space V_x spanned by V and x.

First assume $n = 1$, hence $N = 3$, $\dim V = 2$ and $V_x = \mathbf{P}^N$. Now assume $n > 1$; fix again $x = s((u, v))$, $u \neq v$, and let m be the line $[u, v]$; restricting to $s(m \times m)$ we see that the fiber contains the strict transform of $s(m \times m)$. We will show, using the group $\operatorname{Aut}(\mathbf{P}^n)$, that the fiber is not bigger. Observe first that if $x' = s((u', v'))$, $u' \neq v'$, m' the corresponding line and $V_x \supset x'$, then $V_x = V_{x'}$ and we have that the fiber contains the proper transform of $s(m' \times m')$; thus the problem for x' depends only on m'. Let $G = \{g \in \operatorname{Aut}(\mathbf{P}^n) : g(m) = m\}$. If $n = 2$ then G acts transitively on the set of lines $\neq m$; since f is not constant (because V is not an hyperplane of $\mathbf{P}^N$

if $n \geq 2$) we conclude in this case. Assume therefore $n > 2$. In this case we have two orbits for the lines different from $m: O = \{m' : m' \cap m = \emptyset\}$ and $O' = \{m' : m' \cap m \neq \emptyset\}$. If m' is in O then we conclude as in the case $n = 2$. Thus if the fiber of f is not what we want then it contains the proper transform of $s(m' \times m')$ for m' in O'. Fix m and m' with $m' \cap m \neq \emptyset$. Since $n > 2$ there is a line m'' with $m' \cap m'' \neq \emptyset$ and $m'' \cap m \neq \emptyset$; the fibers of f through m contains m' and hence m'', which is a contradiction to the case $n = 2$.

b) To prove the assertion at the beginning we take a pair $(X, L) \neq (\mathbf{P}^n, \mathcal{O}(1))$, $X \subset \mathbf{P}^r$. Fix $(u, v) \in X \times X \setminus \Delta$; the fibers of $h_{(X,L)}$ through $\pi^{-1}(u, v)$ is certainly contained in the corresponding fiber F of $h_{(\mathbf{P}^r, \mathcal{O}(1))}$. If the line $m = [u, v]$ is not contained in X then $F = s(m \times m)$ contains only finitely many pairs of points of $s(X \times X)$, as claimed.

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Received May 15, 1990