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Hermitian forms on link modules

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Simple knots of odd dimensions ≥ 5 comprise the class of knots admitting the most transparent algebraic classification. A knot $\Sigma^{2q-1} \subset S^{2q+1}$, Σ being a homotopy sphere, is *simple* if Σ bounds in S^{2q+1} a $(q-1)$-connected manifold. Such knots appear in algebraic geometry, where they describe isolated singularities of complex hypersurfaces [M1]. Simple knots were first studied by Kervaire [K]; different classification schemes for them have been constructed by J. Levine [L1], H. Trotter [T] and C. Kearton [K1]. Any simple knot determines naturally a covering $\tilde{X} \to X$, $X = S^{2q-1} - \Sigma$, with the infinite cyclic group acting on $\tilde{X}$ as the group of covering transformations. One associates with the knot the left $\Lambda = \mathbb{Z}[t, t^{-1}]$-module $H_q(\tilde{X})$, the Alexander module. This module supports a non-degenerate Hermitian form (the Blanchfield form)

$$H_q(\tilde{X}) \times H_q(\tilde{X}) \to \mathbb{Q}(t)/\Lambda,$$

where $\mathbb{Q}(t)$ is the field of rational functions of t with coefficients in $\mathbb{Q}$. The theorem of [T] and [K1] states that the Alexander module together with the Blanchfield form determine the knot uniquely. Thus, various geometric properties of simple knots can be read off from the Alexander module and the Blanchfield form.

A *simple link* is a natural generalization of a simple knot. A link $\Sigma^{2q-1} \subset S^{2q+1}$, where $\Sigma^{2q-1} = \Sigma_1 \cup \cdots \cup \Sigma_\mu$ is the ordered disjoint union of μ submanifolds of S^{2q+1}, each homeomorphic to S^{2q-1}, is called *simple* if each Σ_i bounds in S^{2q+1} a $(q-1)$-connected manifold V_i such that V_i is disjoint from V_j for $i \neq j$. (For a more invariant, but equivalent, definition see §6.) Any simple link determines naturally a covering $\tilde{X} \to X$, $X = S^{2q+1} - \Sigma$, with the free group F_μ on μ generators acting on $\tilde{X}$ as the group of covering transformations. One associates with the link the left

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$\Lambda = \mathbb{Z}[F_\mu]$-module $H_q(\tilde{X})$, a non-commutative generalization of the Alexander module. This module supports a non-degenerate Hermitian form

$$[\,,\,] : H_q(\tilde{X}) \times H_q(\tilde{X}) \to \Gamma/\Lambda,$$

where $\Gamma = \mathbb{Z}[[x_q, \ldots, x_\mu]]$ is the ring of formal power series in non-commuting variables $x_1, \ldots, x_\mu$ and Λ is embedded in Γ via the Magnus embedding. The main result of the present paper is that the module $H_q(\tilde{X})$ together with the form $[\,,\,]$ yield a complete system of invariants of simple $(2q-1)$-dimensional links, $q \geq 3$.

The method we apply to prove this result consists of two main ingredients: (1) the stable homotopy reduction of the problem which has been performed in [F2]; (2) a detailed algebraic analysis of the structure of link modules and of all possible Hermitian forms on them, which is done in the present paper.

Our plan is as follows. We first examine link modules assuming that the field $\mathbb{Q}$ of rational numbers is taken as the coefficient ring. We prove that in this case any link module contains a unique minimal lattice and this lattice determines the whole module. We show that any Hermitian form determines and is determined by a unique *scalar form* on the minimal lattice. In the case of knots ($\mu = 1$) the minimal lattice coincides with the whole module and the scalar form reduces to the Milnor form [M].

In the integral case the minimal lattice does not exist any more, but the scalar form is defined on a principal submodule (cf. §4), which does not form a lattice in general. We analyze the structure of self-dual lattices lying in the principal submodule, in similar fashion to [T]. In the last section we prove that each minimal Seifert manifold V determines a self-dual lattice which as an isometric structure is isomorphic to the isometric structure of the embedding $V \subset S^{2q+1}$.

A classification of simple links in terms of Seifert matrices has been obtained by Liang [Li] (up to umbient isotopy) and by Kobelskii [K2] (up to PL isotopy). A cobordism classification of simple links has been obtained by Cappell and Shaneson [CS] in terms of their general theory of homological surgery. A cobordism classification in terms of Seifert matrices has been established by Mio [Mi] and Ko [Ko]. The relationship between Γ-group classification of [CS] and the Seifert matrix approach was clarified by Ko in [Ko1]. Another cobordism classification was suggested by Duval [D]. Duval constructed a version of the Blanchfield form on the homology module of the free covering and proved that the Witt group of such forms is isomorphic to the group of link cobordisms.

Duval's version of the Blanchfield form takes values in Λ_Σ/Λ, where Λ_Σ is P. Cohn's localization of Λ with respect to a class of matrices. The version of the Blanchfield form $[\,,\,]$ we use here was constructed in [F2]; we found it more convenient for our purposes. One might guess that these two forms are in fact equivalent.

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§1. Lattices in link modules

Fix an integer $\mu > 0$ and a subring $k \subset \mathbb{Q}$. Let F_μ denote the free group on μ generators $t_1, \ldots, t_\mu$ and let $\Lambda = k[F_\mu]$ be the group ring.

1.1. A left Λ-module M has the *Sato property* if $\mathrm{Tor}_q^\Lambda(k, M) = 0$ for all q, where k is regarded as a right Λ-module with trivial action via the augmentation map. As was shown by Sato [S], this condition is equivalent to the following: the map

$$M^\mu = \underbrace{M \times \cdots \times M}_{\mu \text{ times}} \to M,$$

given by $(m_1, \ldots, m_\mu) \mapsto \Sigma_{i=1}^\mu (t_i - 1)m_i$, $m_i \in M$ is a bijection. In other words, each $m \in M$ has unique representation in the form

$$m = \sum_{i=1}^{\mu} (t_i - 1)m_i.$$

Let us define "*derivations*" $\partial_i : M \to M$, $i = 1, \ldots, \mu$, by

$$\partial_i(m) = m_i,$$

where $m_i \in M$ is the element appearing in the above decomposition. Thus,

$$m = \sum_{i=1}^{\mu} (t_i - 1)\,\partial_i(m), \qquad m \in M.$$

If $\lambda \in \Lambda$, then

$$\partial_i(\lambda m) = \partial_i(\lambda)m + \varepsilon(\lambda)\,\partial_i(m),$$

where $\partial_i(\lambda) \in \Lambda$ is the Fox derivative with respect to t_i [CF], and $\varepsilon(\lambda) \in k$ is the augmentation.

We can think of M as also having a left module structure over the ring $D = k[\partial_1, \ldots, \partial_\mu]$ of polynomials of non-commuting variables $\partial_1, \ldots, \partial_\mu$. Any Λ-homomorphism $f: M_1 \to M_2$ between modules with the Sato property is also a D-homomorphism. The converse is also true. Thus

$$\mathrm{Hom}_\Lambda (M_1, M_2) = \mathrm{Hom}_D (M_1 M_2).$$

1.2. The most important example of a module with the Sato property is the following.

Let $\Gamma = k[[x_1, \ldots, x_\mu]]$ be the ring of formal power series of non-commuting variables $x_1, \ldots, x_\mu$. The ring Λ may be embedded in Γ via the Magnus embedding $t_i \mapsto 1 + x_i$, $t_i^{-1} \mapsto 1 - x_i + x_i^2 - x_i^3 + \cdots$. Then Γ/Λ is a left Λ-module with the Sato property. The derivation $\partial_i : \Gamma/\Lambda \to \Gamma/\Lambda$ acts as cancellation of x_i from the left on monomials containing x_i on the leftmost position, and sends to zero all other monomials.

In fact, the above-mentioned rule defines an additive map $\partial_i : \Gamma \to \Gamma$ with the property

$$\gamma = \varepsilon(\gamma) + \sum_{i=1}^{\mu} x_i \, \partial_i(\gamma),$$

where $\varepsilon(\gamma) \in k$ is the augmentation. ∂_i maps Λ into itself and the restriction $\partial_i|_\Lambda$ coincides with the Fox derivative $\partial/\partial t_i$ [CF].

These remarks allow us to introduce a left D-module structure on Γ and Λ, which will be used later.

1.3. A *module of type L* is a left finitely generated Λ-module with the Sato property.

Γ/Λ is *not* a module of type L.

Modules of type L appear as homoloy of free coverings of boundary links [S], cf. also §6.

We shall introduce now some more operations in modules M with the Sato property. If $m \in M$ then the equation

$$m = \sum_{i=1}^{\mu} (t_i - 1)\, \partial_i(m)$$

is equivalent to

$$m = \sum_{i=1}^{\mu} (t_i^{-1} - 1)\, \bar{\partial}_i(m),$$

where $\bar{\partial}_i : M \to M$, $i = 1, \ldots, \mu$ is defined by $\bar{\partial}_i(m) = -t_i \partial_i(m)$. Define

$$\pi_i(m) = -\partial_i(m) - \bar{\partial}_i(m) = (t_i - 1)\,\partial_i(m),$$

which will be called the i-th *component* of m. Then

$$m = \pi_1(m) + \cdots + \pi_\mu(m), \qquad m \in M,$$

$$\pi_i \circ \pi_i = \pi_i,$$

$$\pi_i \circ \pi_j = 0 \quad \text{for } i \neq j,$$

$$\partial_i = \partial_i \circ \pi_i,$$

$$\bar{\partial}_i = \bar{\partial}_i \circ \pi_i.$$

Let us also introduce an operator $z : M \to M$ by

$$z = -\partial_1 - \cdots - \partial_\mu.$$

One can express ∂_i and $\bar{\partial}_i$ in terms of z and π_i:

$$\partial_i = -z\pi_i,$$

$$\bar{\partial}_i = -\bar{z}\pi_i,$$

where

$$\bar{z} = 1 - z : M \to M.$$

Thus, the whole structure is given by a decomposition of unity $\{\pi_i\}_{i=\overline{1,\mu}}$, which gives a splitting of M into a direct sum (over k)

$$M \approx X_1 \oplus \cdots \oplus X_\mu,$$

and an endomorphism

$$z : M \to M.$$

1.4. Let M be a Λ-module of type L. A *lattice* in M is a k-submodule $A \subset M$ which:

(a) is invariant under ∂_i, $\bar{\partial}_i$, $i = 1, \ldots, \mu$;
(b) generates M over Λ;
(c) is finitely generated over k.

Condition (a) is equivalent to each of the following conditions (a′), (a″), (a‴):

(a′) A is invariant under z and π_i, $i = 1, \ldots, \mu$;
(a″) A is invariant under ∂_i and π_i, $i = 1, \ldots, \mu$;
(a‴) A is invariant under $\bar{\partial}_i$ and π_i, $i = 1, \ldots, \mu$.

1.5. LEMMA. (1) *Each Λ-module M of type L contains a lattice*; (2) *If A_1 and A_2 are two lattices in M then $A_1 + A_2$ and $A_1 \cap A_2$ are also lattices.*

Proof of (1). Let $m_1, \ldots, m_n \in M$ generate M over Λ. Then

$$\partial_i m_j = \sum_{k=1}^{n} \lambda_{ij}^k m_k, \qquad \lambda_{ij}^k \in \Lambda.$$

For sufficiently large N all the λ_{ij}^k are contained in the k-submodule S_N of Λ, generated by all monomials

$$t_{i_1}^{\varepsilon_1} t_{i_2}^{\varepsilon_2} \cdots t_{i_s}^{\varepsilon_s}$$

with

$$|\varepsilon_1| + |\varepsilon_2| + \cdots + |\varepsilon_s| \le N, \qquad i_1, \ldots, i_s \in \{1, \ldots, \mu\}.$$

S_N is closed under the Fox derivatives. Let A' be the set of all sums of the form

$$\sum_{i=1}^{n} \lambda_i m_i, \qquad \lambda_i \in S_N.$$

Then A' is invariant under ∂_i, generates M over Λ, and is finitely generated over k. Let $a_1, \ldots, a_q$ generate A' over k. Consider the k-submodule A of M generated by $\{\pi_i a_j\}$, where $i = 1, \ldots, \mu, j = 1, \ldots, q$. Then A is a lattice in M.

Proof of (2). Let $A_1, A_2 \subset M$ be two lattices. It is clear that $A_1 + A_2$ is a lattice and that $A_1 \cap A_2$ is invariant under ∂_i and $\bar{\partial}_i$ and is finitely generated over k. Hence we only have to prove that $A_1 \cap A_2$ generates M over Λ. This will follow from the following observation.

1.6. LEMMA. *Let M be a module of type L, $A \subset M$ be a lattice, and $B \subset M$ be a subset. Then*

$$A \cap (t_i - 1)B \subset \Lambda(A \cap B); \tag{1}$$

$$A \cap (t_i^{-1} - 1)B \subset \Lambda(A \cap B), \tag{2}$$

where the right-hand-side denotes the submodule generated by $A \cap B$ over Λ, $1 \leq i \leq \mu$.

Proof of Lemma 1.6. If $a \in A \cap (t_i - 1)B$ then $a = (t_i - 1)b$, $b \in B$ and $b = \partial_i a \in A \cap B$, i.e. $a \in \Lambda(A \cap B)$. The second inclusion may be proved similarly.

Proof of Lemma 1.5 (*continued*). Let A_1, A_2 be two lattices in M. From Lemma 1.6 we obtain by induction:

$$A_1 \cap (t_i - 1)(t_j - 1)A_2 \subset \Lambda(A_1 \cap (t_j - 1)A_2) \subset \Lambda(A_1 \cap A_2),$$

and similarly

$$A_1 \cap (t_{i_1}^{\varepsilon} - 1) \cdots (t_{i_s}^{\varepsilon_s} - 1)A_2 \subset \Lambda(A_1 \cap A_2),$$

for any $i_1, i_2, \ldots, i_s \in \{1, \ldots, \mu\}$ and $\varepsilon_i = \pm 1$, $i = 1, \ldots, s$. Since the union of all sets of the form

$$(t_{i_1}^{\varepsilon_1} - 1) \cdots (t_{i_s}^{\varepsilon_s} - 1)A_2$$

is equal to M, it follows that

$$A_1 \subset \Lambda(A_1 \cap A_2)$$

and so

$$\Lambda(A_1 \cap A_2) = M.$$

1.7. Let $A \subset B$ be two lattices. We will say that B is an *elementary extension* of A of the first kind (respectively, of the second kind) if $\pi_i B = \pi_i A$ for all but one index $i \in \{1, \ldots, \mu\}$ and $zB \subset A$ (resp. $\bar{z}B \subset A$).

If A is a lattice and $B = x_iA + A$, $C = \bar{x}_iA + A$, then $B \supset A$ is an elementary extension of the first kind and $C \supset A$ is an elementary extension of the second kind. Here

$$x_i = t_i - 1, \qquad \bar{x}_i = t_i^{-1} - 1.$$

1.8. PROPOSITION. *Let M be a module of type L and $A \subset M$ be a lattice. Then there exists a sequence of lattices.*

$$A = A_0 \subset A_1 \subset A_2 \subset \cdots$$

such that

$$\bigcup_{i=0}^{\infty} A_i = M$$

and A_i is an elementary extension of A_{i-1} for all $i = 1, 2, \ldots$.

Proof. Let us define A_i inductively. If $i \equiv j (\text{mod } 2\mu)$ with $0 \le j \le \mu$ put

$$A_i = x_jA_{i-1} + A_{i-1}.$$

If $i \equiv j (\text{mod } 2\mu)$ with $\mu + 1 \le j \le 2\mu$, define

$$A_i = \bar{x}_{j-\mu}A_{i-1} + A_{i-1}.$$

This gives the desired sequence $A_1 \subset A_2 \subset \cdots$.

1.9. COROLLARY. *If A and B are two lattices in M then there is a finite sequence of lattices $A = C_0, C_1, \ldots, C_N = B$ such that for each $i = 1, 2, \ldots, N$ either C_i is an elementary extension of C_{i-1} or else C_{i-1} is an elementary extension of C_i.*

Proof. First assume that $B \supset A$. Let $A_0 = A \subset A_1 \subset \cdots$ be a sequence of lattices as in Proposition 1.8. Take $C_i = B \cap A_i$. Then $C_i = B$ for sufficiently large i, $i \ge N$, say, and $C_1, \ldots, C_N$ is the desired sequence.

In the general case, we can construct a similar sequence of lattices joining A and $A + B$; in the same way we have a sequence of lattices joining B and $A + B$. Connecting the two sequences, we get the sequence of lattices joining A and B having the desired properties.

1.10. Assume that $k = \mathbb{Z}$ and M is a module of type L. We will say that M is *periodic* if there is an integer $N \in \mathbb{Z}$, $N \neq 0$ with $NM = 0$. As follows, from Lemma 1.5.(1), this is equivalent to $M = \mathrm{Tors}_{\mathbb{Z}} M$.

Any lattice of a periodic module of type L is finite and conversely, any module of type L admitting a finite lattice is periodic.

1.11. THEOREM. *Let M be a module of type L. Assume that either $k = \mathbb{Q}$ or $k = \mathbb{Z}$ and M is periodic. Then M contains a minimal lattice $A \subset M$, which is the intersection of all lattices in M. A lattice $A \subset M$ is the minimal lattice if and only if for any $k = 1, \ldots, \mu$*

$$\pi_k zA = \pi_k A \quad \text{and} \quad \pi_k \bar{z}A = \pi_k A$$

(where $\bar{z} = 1 - z$).

Proof. The existence of a minimal lattice follows from Lemma 1.5(2). If A is the minimal lattice and $\pi_k zA \neq \pi_k A$ for some k, define B, $B \subset A$, by

$$\pi_i B = \pi_i A \quad \text{for } i \neq k, \qquad i = 1, \ldots, \mu,$$

$$\pi_k B = \pi_k zA.$$

It is easy to check that B is a lattice with $B \subset A$, $B \neq A$ which gives a contradiction.

Similarly, if $\pi_k \bar{z}A \neq \pi_k A$ we can define B as

$$\pi_i B = \pi_i A,\ i \neq k,$$

$$\pi_k B = \pi_k \bar{z}A$$

and obtain a contradiction.

Assume now that A is a lattice with $\pi_k zA = \pi_k A$, $\pi_k \bar{z}A = \pi_k A$ for all $k = 1, \ldots, \mu$. If A contains another lattice $B \subset A$, $B \neq A$ then it follows from Proposition 1.8 that there exists a lattice $C \subset M$ such that $B \subset C \subset A$, $C \neq A$ and A is an elementary extension of C. This means that

$$\pi_i C = \pi_i A$$

for all $i = 1, \ldots, \mu$, $i \neq k$ and

$$zA \subset C, \quad \text{or} \quad \bar{z}A \subset C.$$

In the first case we have $\pi_k C \neq \pi_k A$ and

$$\pi_k zA \subset \pi_k C \underset{\neq}{\subset} \pi_k A$$

which is a contradiction. The second case can be treated similarly.

1.12. Our main interest is in the case $k = \mathbb{Z}$. A module of type L over $\mathbb{Z}$ does not have to have a minimal lattice in general. But the Proof of Theorem 1.11 shows that the following is true:

1.13. COROLLARY. *Let $k \subset \mathbb{Q}$ be an arbitrary subring. If A is a lattice in a module M of type L and*

$$\pi_k zA = \pi_k A, \qquad \pi_k \bar{z}A = \pi_k A$$

for any $k = 1, \ldots, \mu$ then A coincides with the intersection of all lattices in M.

1.14. REMARKS. 1. Consider minimal lattices in the case $\mu = 1$ (knots). *If $k = \mathbb{Q}$ then any module M of type L coincides with its minimal lattice.* It follows from the fact that M is finitely generated over $\mathbb{Q}$ (cf. [M]) and from Theorem 1.11: the conditions of the Theorem

$$zM = M, \qquad \bar{z}M = M$$

are satisfied because in the case $\mu = 1$ one has

$$z = (1-t)^{-1}, \qquad \bar{z} = (1-t^{-1})^{-1}$$

and both maps

$$1-t, \quad 1-t^{-1} : M \to M$$

are isomorphisms.

2. The same is true if $\mu = 1$, $k = \mathbb{Z}$ and M is periodic: in this case M also *coincides with its minimal lattice.* The proof is rather similar; instead of the result of Milnor [M] one may use a Lemma of Kervaire [K], ch. II, saying that under the above assumptions M is finite.

3. If $\mu > 1$ and $k = \mathbb{Q}$ then the minimal lattice $A \subset M$ *coincides with the whole module if and only if $M = 0$.*

Proof. $A = M$ implies M is finitely generated over $\mathbb{Q}$ and now the Sato property

$$\underbrace{M \times M \times \cdots \times M}_{\mu \text{ times}} \approx M$$

gives

$$\mu(\dim_{\mathbb{Q}} M) = \dim_{\mathbb{Q}} M.$$

Thus,

$$\dim_{\mathbb{Q}} M = 0, \qquad M = 0.$$

§2. The dual module

2.1. Since Γ/Λ is a $\Lambda - \Lambda$-bimodule, the group $\mathrm{Hom}_\Lambda(M; \Gamma/\Lambda)$ has a natural right Λ-module structure. We can transform it into a left Λ-module structure by using the standard involution of Λ, $t_i \mapsto t_i^{-1}$. In other words, we shall consider $\mathrm{Hom}_\Lambda(M, \Gamma/\Lambda)$ as a left Λ-module with

$$(t_i f)(m) = f(m) t_i^{-1}$$

for

$$f \in \mathrm{Hom}_\Lambda(M; \Gamma/\Lambda), \qquad m \in M, \qquad i = 1, \ldots, \mu.$$

2.2. PROPOSITION. *If M is a module of type L then* $\mathrm{Hom}_\Lambda(M, \Gamma/\Lambda)$ *is also of type L.*

For the proof we need a Lemma:

2.3. LEMMA. *Each $\gamma \in \Gamma$ can be uniquely represented in the form*

$$\gamma = \varepsilon(\gamma) + \sum_{i=1}^{\mu} \gamma_i \bar{x}_i, \qquad \gamma_i \in \Gamma,$$

where $\varepsilon(\gamma) = \gamma(0)$ is the augmentation and $\bar{x}_i$ denotes the following power series

$$\bar{x}_i = -x_i + x_i^2 - x_i^3 + \cdots \in \Gamma.$$

If γ belongs to Λ then $\gamma_i \in \Lambda$, $i = 1, \ldots, \mu$.

Proof. There is a continuous involution$^-: \Gamma \to \Gamma$ with $x_i \mapsto \bar{x}_i$, $i = 1, \ldots, \mu$. It follows that the decomposition

$$\gamma = \varepsilon(\gamma) + \sum_{i=1}^{\mu} \gamma_i \bar{x}_i, \qquad \gamma_i \in \Gamma,$$

holds if and only if $\gamma_i = \overline{\partial_i(\bar{\gamma})}$, cf. 1.2.

The involution$^-$ maps Λ into itself and $\partial_i(\Lambda) \subset \Lambda$ as well (∂_i coincides with the Fox derivative on Λ). Thus $\gamma_i \in \Lambda$ if $\gamma \in \Lambda$.

2.4. Let us define $\delta_i : \Gamma \to \Gamma$ by $\delta_i(\gamma = \overline{\partial_i(\bar{\gamma})}$, $\gamma \in \Gamma$. Then we have

$$\gamma = \varepsilon(\gamma) + \sum_{i=1}^{\mu} \delta_i(\gamma)\bar{x}_i$$

and

$$\delta_i(\gamma_1\gamma_2) = \delta_i(\gamma_1)\varepsilon(\gamma_2) + \gamma_1\delta_i(\gamma_2),$$

$$\delta_i(x_i) = -1 - x_i$$

$$\delta_i(x_j) = 0 \quad \text{if } i \neq j$$

for $\gamma, \gamma_1, \gamma_2 \in \Gamma$, $i = 1, \ldots, \mu$.

From Lemma 2.3 it follows that δ_i correctly defines a map $\delta_i : \Gamma/\Lambda \to \Gamma/\Lambda$. For $\lambda \in \Lambda, \gamma \in \Gamma$

$$\delta_i(\lambda\gamma) = \lambda\delta_i(\gamma)(\operatorname{mod} \Lambda);$$

that is, δ_i determines a left Λ-homomorphism $\Gamma/\Lambda \to \Gamma/\Lambda$.

2.5. *Proof of* 2.2. Let $f \in \operatorname{Hom}_\Lambda(M; \Gamma/\Lambda)$ be a Λ-homomorphism. The above remarks imply that $f_i = \delta_i \circ f$ belongs to $\operatorname{Hom}_\Lambda(M; \Gamma/\Lambda)$ and clearly

$$f = \sum_{i=1}^{\mu} (t_i - 1)f_i.$$

The uniqueness of this decomposition follows from 2.3, and this proves the Sato property for $\operatorname{Hom}_\Lambda(M; \Gamma/\Lambda)$. To complete the proof we have to show that

$\mathrm{Hom}_\Lambda(M; \Gamma/\Lambda)$ is finitely generated over Λ. This will follow from the next two lemmas (cf. 2.8).

2.6. LEMMA. *Let $A \subset M$ be a lattice in a module M of type L, and B be a Λ-module with the Sato property. Then any D-homomorphism $A \to B$ can be uniquely extended to a Λ-homomorphism $M \to B$. Thus,* $\mathrm{Hom}_\Lambda(M, B) = \mathrm{Hom}_D(A; B)$. *In particular, two modules of type L are isomorphic if and only if they admit lattices which are isomorphic as D-modules.*

Proof. Any Λ-homomorphism $f: M \to B$ is also a D-homomorphism (cf. 1.1) and so its restriction to A is a D-homomorphism $A \to B$. If $f|_A = 0$ then $f = 0$ (since A generates M over Λ). Thus the map

$$\mathrm{Hom}_\Lambda(M; B) \to \mathrm{Hom}_D(A; B)$$

is injective. To prove that it is onto, consider the sequence of lattices

$$A = A_0 \subset A_1 \subset A_2 \subset \cdots, \qquad \cup A_i = M,$$

constructed in Proposition 1.8. We want to show that any D-homomorphism $g_i: A_i \to B$ can be extended to a D-homomorphism $g_{i+1}: A_{i+1} \to B$; this would complete the proof of the Lemma.

Assume that $A_{i+1} \supset A_i$ is an elementary extension of the first kind. Define

$$g_{i+1}: A_{i+1} \to B$$

by

$$g_{i+1}(a) = \sum_{k=1}^{\mu} x_k g_i(\partial_k a), \qquad a \in A_{i+1}.$$

Because $\partial_k(A_{i+1}) \subset A_i$, this formula defines g_{i+1} correctly. If $a \in A_i$, then $g_{i+1}(a) = \Sigma_{k=1}^{\mu} x_k g_i(\partial_k a) = \Sigma_{k=1}^{\mu} x_k \partial_k g_i(a) = g_i(a)$; thus g_{i+1} is an extension of g_i. To show that g_{i+1} is a D-homomorphism, note that

$$g_{i+1}(\partial_l a) = g_i(\partial_l a)$$

(since $\partial_l a \in A_i$) and

$$g_i(\partial_l a) = \partial_l\left[\sum_{k=1}^{\mu} x_k g_i(\partial_k a)\right] = \partial_l g_{i+1}(a).$$

One may use similar arguments when $A_{i+1} \supset A_i$ is an elementary extension of the second kind.

2.7. LEMMA. *Let A be a lattice in a module M of type L. Then the image of*

$$\psi : \mathrm{Hom}_D(A; \Gamma) \to \mathrm{Hom}_D(A, \Gamma/\Lambda)$$

generates $\mathrm{Hom}_D(A, \Gamma/\Lambda)$ *over* Λ.

REMARK. Γ is assumed to be supplied with the D-module structure introduced in 1.2.

Proof. Let $f : A \to \Gamma/\Lambda$ be a D-homomorphism. It is clear that f admits a k-lifting $g : A \to \Gamma$

$$\begin{tikzcd} & \Gamma \arrow[d] \\ A \arrow[ur, "g"] \arrow[r, "f"'] & \Gamma/\Lambda \end{tikzcd}$$

For $i = 1, \dots, \mu$ consider the map $g_i : A \to \Lambda$, $g_i(a) = \partial_i g(a) - g(\partial_i a)$, $a \in A$. Let us write

$$g_i(a) = \sum_{\pi \in F_\mu} \pi \cdot f^i_\pi(a),$$

where $f^i_\pi : A \to k$ is a k-homomorphism and f^i_π is nonzero only for a finite number of pairs (i, π), $i = 1, \dots, \mu$, $\pi \in F_\mu$. Let $\hat{f}^i_\pi : A \to \Gamma$ be the D-homomorphism

$$\hat{f}^i_\pi(a) = \sum_\alpha x^\alpha f^i_\pi(\partial^\alpha a),$$

where α runs over all multi-indices $\alpha = (i_1, \dots, i_s)$, $i_j \in \{1, \dots, \mu\}$ and $x^\alpha = x_{i_1} x_{i_2}, \dots, x_{i_s}$, $\partial^\alpha = \partial_{i_s} \partial_{i_{s-1}}, \dots, \partial_{i_1}$. Define $\hat{g} : A \to \Gamma$ by

$$\hat{g}(a) = \sum_{i,\pi} \hat{f}^i_\pi(a) x_i \pi,$$

where i runs over $1, 2, \ldots, \mu$ and $\pi \in F_\mu$, the sum is in fact finite. $\hat{g}$ is a k-homomorphism. Let us show that $h = \hat{g} - g : A \to \Gamma$ is a D-homomorphism:

$$\partial_j \hat{g}(a) - \hat{g}(\partial_j a) = \sum_{i,\pi,\alpha} x^\alpha f^i_\pi(\partial^\alpha \partial_j a) x_i \pi + \sum_\pi f^j_\pi(a) \cdot \pi - \sum_{i,\pi,\alpha} x^\alpha f^i_\pi(\partial^\alpha \partial_j a) x_i \pi$$

$$= \sum_\pi f^j_\pi(a)\pi = g_j(a) = \partial_j g(a) - g(\partial_j a).$$

The map $\hat{g} : A \to \Gamma$ is not a D-homomorphism in general. But its reduction modulo Λ, that is $\hat{f} : A \to \Gamma/\Lambda$, $\hat{f}(a) = \hat{g}(a) \bmod \Lambda$, is a D-homomorphism. From the formula defining $\hat{g}$ it is clear that $\hat{f}$ belongs to $\Lambda(\mathrm{im}\,(\psi))$, the Λ-submodule, generated by $\mathrm{im}\,(\psi)$. On the other hand, $f = \hat{f} - \hat{h}$, where $\hat{h} : A \to \Gamma/\Lambda$ is the reduction of h modulo Λ. The above remark shows that $\hat{h} \in \mathrm{im}\,(\psi)$ and thus

$$f \in \Lambda(\mathrm{im}\,(\psi)),$$

which proves the lemma.

2.8. *Proof of* 2.2 (*continued*). We have to show that $\mathrm{Hom}_\Lambda(M, \Gamma/\Lambda)$ is finitely generated over Λ. By Lemma 2.6 we can identify $\mathrm{Hom}_\Lambda(M, \Gamma/\Lambda)$ with $\mathrm{Hom}_D(A; \Gamma/\Lambda)$, where A is a lattice in M, and by virtue of Lemma 2.7 it is enough to show that $\mathrm{Hom}_D(A; \Gamma)$ is finitely generated over k. We will do this by showing that

$$\mathrm{Hom}_D(A; \Gamma) \approx A^* = \mathrm{Hom}_k(A, k)$$

as modules over k. Let $F : A \to \Gamma$ be a D-homomorphism. For $a \in A$, $F(a) = \Sigma\, x^\alpha f_\alpha(a)$, where α runs over all tuples $(i_1, \ldots, i_s)$ with $i_1, \ldots, i_s \in \{1, \ldots, \mu\}$ and x^α denotes the monomial

$$x_{i_1} x_{i_2} \cdots x_{i_s}$$

with the convention $x^\phi = 1$.

For each multi-index α, $f_\alpha : A \to k$ is a k-linear map. Since F is a D-homomorphism, $\partial_i F(a) = F(\partial_i a)$ and so

$$f_\alpha(\partial_i a) = f_{i\alpha}(a).$$

This formula shows that all k-homomorphisms f_α can be expressed in terms of $f_\phi : A \to k$:

$$f_\alpha(a) = f_\phi(\partial_{i_s} \cdots \partial_{i_2} \partial_{i_1} a)$$

for

$$\alpha = (i_1, \ldots, i_s).$$

Conversely, given a k-homomorphism $f_\phi : A \to k$ we can define f_α by the previous formula and then construct $F : A \to \Gamma$ as a D-homomorphism

$$F(a) = \sum x^\alpha f_\alpha(a).$$

Thus, the map $F \mapsto f_\phi$ is a k-isomorphism

$$\mathrm{Hom}_D(A, \Gamma) \overset{\to}{\approx} \mathrm{Hom}_k(A; k) = A^*.$$

This proves Proposition 2.2.

2.9. THEOREM. *Assume that $k = \mathbb{Q}$. Let M be a module of type L and let $A \subset M$ be its minimal lattice. Consider the following homomorphism*

$$\varphi : A^* = \mathrm{Hom}_{\mathbb{Q}}(A; \mathbb{Q}) \to \mathrm{Hom}_D(A; \Gamma/\Lambda) = \mathrm{Hom}_\Lambda(M; \Gamma/\Lambda),$$

$$\varphi(f)(a) = \sum_{i=1}^{\mu} \sum_\alpha x^\alpha f(\pi_i \partial^\alpha a) x_i \pmod{\Lambda},$$

where $f \in A^*$, $a \in A$, and α runs over all multi-indices. Then φ is a monomorphism and its image coincides with the minimal lattice of the dual module $\mathrm{Hom}_\Lambda(M; \Gamma/\Lambda)$.

For the proof we need two lemmas.

2.10. LEMMA. *Let $k = \mathbb{Q}$ and let X be a D-module which is finitely generated over k. Then* $\mathrm{Hom}_D(X; \Lambda) = 0$ *if and only if*

$$\sum_{i=1}^{\mu} \partial_i(X) = X$$

and for any $k = 1, \ldots, \mu$

$$\sum_{\substack{i=1\\ i \neq k}}^{\mu} \partial_i(X) + (1 + \partial_k)(X) = X.$$

Proof. Let A_0 be a D-module with $A_0 = k$ and all $\partial_i : A_0 \to A_0$ equal to zero. For each $i \in \{1, \ldots, \mu\}$ denote by A_i a D-module with $A_i = k, \partial_j = 0$ for $j \neq i$ and $\partial_i = -1$.

The modules A_i, $i = 0, 1, \ldots, \mu$ are isomorphic to D-submodules of Λ (A_0 is isomorphic to the submodule generated by $1 \in \Lambda$ and A_i for $i \in \{1, \ldots, \mu\}$ is isomorphic to the submodule generated by t_i^{-1}). Thus $\mathrm{Hom}_D(X; \Lambda) = 0$ implies $\mathrm{Hom}_D(X; A_i) = 0$ for all $i \in \{0, 1, \ldots, \mu\}$.

The condition $\mathrm{Hom}_D(X; A_0) = 0$ means that each k-linear map $f : X \to k$ with $f(\partial_j x) = 0$ for all $x \in X$ and $j \in \{1, \ldots, \mu\}$, is zero. This is equivalent to $\Sigma_{j=1}^{\mu} \partial_j(X) = X$.

The condition $\mathrm{Hom}_D(X, A_i) = 0$ for $i = 1, \ldots, \mu$ implies that each k-linear map $f : X \to k$ satisfying $f(\partial_j x) = 0$ for all $x \in X$ and $j \in \{1, \ldots, \mu\}, j \neq i$ and $f(\partial_i x) = -x$ (which is equivalent to $f((1 + \partial_i)x) = 0$) is the zero map, $f = 0$. This is equivalent to

$$\sum_{j \neq i} \partial_j X + \mathrm{im}\,[(1 + \partial_i) : X \to X] = X.$$

Thus, $\mathrm{Hom}_D(X; \Lambda) = 0$ implies the above-mentioned conditions.

Suppose now that the above conditions are satisfied, that is $\mathrm{Hom}_D(X; A_i) = 0$ for $i = 0, 1, \ldots, \mu$. Consider the filtration $0 = L_{-1} \subset L_0 \subset L_1 \subset \cdots$ of $\Lambda = \cup_{i=0}^{\infty} L_i$, where $L_0 = \mathbb{Q} \subset \Lambda$ and

$$L_i = L_{i-1} + x_j L_{i-1} \qquad \text{for } i \equiv j (\mathrm{mod}\ 2\mu), \quad i \leq j \leq \mu$$

$$L_i = L_{i-1} + \bar{x}_{j-\mu} L_{i-1} \qquad \text{for } i \equiv j (\mathrm{mod}\ 2\mu), \quad \mu + 1 \leq j \leq 2\mu.$$

Then each L_i is a D-submodule of Λ and L_i / L_{i-1} is isomorphic to a finite direct sum of several copies of some of $A_0, \ldots, A_\mu$. If $f : X \to \Lambda$ is a D-homomorphism then $f(X) \subset L_N$ for some N and $\mathrm{Hom}\,(X; A_i) = 0$ implies $f(X) \subset L_{N-1}$. Thus by induction we get $f(X) \subset L_{-1} = 0$.

2.11. COROLLARY. *Let $k = \mathbb{Q}$ and A be the minimal lattice in a module M of type L. Then any D-homomorphism $f : A \to \Lambda$ vanishes on $zA \subset A$ and takes values in $\mathbb{Q} \subset \Lambda$. More precisely,*

$$\mathrm{Hom}_D(A; \Lambda) = \mathrm{Hom}_{\mathbb{Q}}(A / zA; \mathbb{Q}) = (A / zA)^*.$$

Proof. Denote by A' a D-module with underlying set $A' = A$ having the same k-module structure and with operators $\partial_i : A' \to A'$, defined by

$$\partial_i(a) = -\pi_i z a, \qquad a \in A', \quad i = 1, \dots, \mu.$$

(Recall, that $\partial_i a = -z\pi_i a$ in A). The map

$$z : A' \to A, \qquad a \mapsto z\dot{a},$$

is a D-homomorphism. Note that the module A' satisfies the properties of Lemma 2.10. Thus, if $f : A \to \Lambda$ is a D-homomorphism, then $f(zA) = 0$. It follows that for $a \in A$ and $i = 1, \dots, \mu$.

$$\partial_i f(a) = f(\partial_i a) = f(-z\pi_i a) = 0$$

and so $f(a) = \text{const} \in \mathbb{Q} \subset \Lambda$.

Conversely, given a $\mathbb{Q}$-homomorphism $f : A \to \mathbb{Q}$ which is zero on zA, one can consider the composite

$$A \xrightarrow{f} \mathbb{Q} \xrightarrow{\text{incl}} \Lambda$$

as representing a D-homomorphism $A \to \Lambda$.

2.12. *Proof of Theorem* 2.9. Let us first show that φ is a monomorphism. If $f \in A^*$ belongs to $\ker(\varphi)$ then

$$\varphi(f)(a) = \sum_{i=1}^{\mu} \sum_{\alpha} x^{\alpha} f(\pi_i \partial^{\alpha} a) x_i$$

belongs to Λ for any $a \in A$. Thus the map $\hat{\varphi}_i(f) : A \to \Gamma$ defined by

$$\hat{\varphi}_i(f)(a) = \sum_{\alpha} x^{\alpha} f(\pi_i \partial^{\alpha} a), \qquad a \in A$$

is a D-homomorphism (obviously) taking values in $\Lambda \subset \Gamma$. From 2.11 it follows that $f(\pi_i zA) = 0$, that is $f(\pi_i A) = 0$ (since $\pi_i zA = \pi_i A$). This is true for all $i = 1, \dots, \mu$ and so $f = 0$.

Our next step will be to show that $\text{im}(\varphi)$ is invariant under $\bar{z}$ and $\pi_1, \dots, \pi_\mu$. To do this we will introduce operations

$$\bar{z}, \pi_1, \dots, \pi_\mu : A^* \to A^*$$

by

$$(\bar{z}f)(a) = f(za)$$
$$(\pi_i f)(a) = f(\pi_i a)$$

for

$$f \in A^*, \qquad a \in A, \quad i = 1, \ldots, \mu,$$

and will show that φ commutes with $\bar{z}$ and $\pi_1, \ldots, \pi_\mu$.

Compute

$$\varphi(\bar{z}f)(a) = \sum_{i=1}^{\mu} \sum_{\alpha} x^\alpha f(z\pi_i \partial^\alpha a)x_i = -\sum_{i=1}^{\mu} \sum_{\alpha} x^\alpha f(\partial_i \partial^\alpha a)x_i = (\bar{z}\varphi(f))(a)$$

and also

$$\varphi(\pi_j f)(a) = \sum_{i=1}^{\mu} \sum_{\alpha} x^\alpha f(\pi_j \pi_i \partial^\alpha a)x_i = \sum_{\alpha} x^\alpha f(\pi_j \partial^\alpha a)x_j = (\pi_j \varphi(f))(a)$$

To prove that $\operatorname{im}(\varphi)$ is a lattice we have left to show that $\operatorname{im}(\varphi)$ generates $\operatorname{Hom}_D(A; \Gamma/\Lambda)$ over Λ. By Lemma 2.7 it is enough to show that $\operatorname{im}(\varphi)$ contains $\operatorname{im}(\psi)$, where ψ is the homomorphism of Lemma 2.7. Let $F: A \to \Gamma/\Lambda$ belong to $\operatorname{im}(\psi)$. Then by the arguments of the Proof of Theorem 2.2 (cf. 2.8) there exists a $\mathbb{Q}$-homomorphism $g: A \to \mathbb{Q}$ such that

$$F(a) = \sum x^\alpha g(\partial^\alpha a) \pmod{\Lambda}$$

for all $a \in A$. It follows that

$$F = \varphi(f),$$

where $f \in A^*$, $f(a) = -g(za)$ for $a \in A$.

In order to show that $\operatorname{im}(\varphi)$ is the *minimal* lattice we can check the conditions of Theorem 1.11 for A^*. Since

$$(\pi_k z f)(a) = f(\bar{z}\pi_k a)$$

for $f \in A^*$, $a \in A$, the identity

$$\pi_k z A^* = \pi_k A^*$$

is equivalent to the following statement: for each $\mathbb{Q}$-homomorphism $g : \pi_k A \to \mathbb{Q}$ there exists a $\mathbb{Q}$-homomorphism $h : A \to \mathbb{Q}$ such that the diagram

$$\begin{tikzcd} \pi_k A \arrow[rr, "\bar{z}"] \arrow[dr, "g"'] & & A \arrow[dl, "h"] \\ & \mathbb{Q} & \end{tikzcd}$$

commutes. The last statement is equivalent to the fact that $\bar{z}|_{\pi_k A}$ is a monomorphism, which is in fact true: if $a \in \pi_k A$ and $\bar{z}a = 0$ then $\bar{\partial}_i a = 0$ for all $i = 1, \ldots, \mu$ and $a = \Sigma_{i=1}^{\mu} \bar{x}_i \bar{\partial}_i(a) = 0$. The identity

$$\pi_k \bar{z} A^* = \pi_k A^*$$

follows similarly.

This proves the Theorem.

§3. Hermitian forms

In this section we consider modules of type L supplied with Hermitian forms with values in Γ/Λ. Assuming that the ground ring k is $\mathbb{Q}$, we show that such a form defines (and can be expressed in terms of) a unique scalar form defined on the minimal lattice.

In the case of knots ($\mu = 1$) the minimal lattice coincides with the whole module and the scalar form constructed here reduces to the Milnor form [M].

3.1. Let M_1, M_2 be two modules of type L. Consider a $\mathbb{Q}$-bilinear pairing

$$[\,,\,] : M_1 \times M_2 \to \Gamma/\Lambda, \qquad (a, b) \mapsto [a, b] \in \Gamma/\Lambda$$

with the properties:

(a) $[\lambda a, b] = \lambda[a, b]$ for $\lambda \in \Lambda$, $a \in M_1$, $b \in M_2$;
(b) $[a, \lambda b] = [a, b]\bar{\lambda}$;

(c) $[\,,\,]$ is nondegenerate in the following sense: for $b \in M_2$ let $\varphi_b : M_1 \to \Gamma/\Lambda$ be the Λ-homomorphism defined by $\varphi_b(a) = [a, b]$; then the map

$$M_2 \to \operatorname{Hom}_\Lambda(M_1; \Gamma/\Lambda), \qquad b \mapsto \varphi_b,$$

is an isomorphism. (Note, that (b) means that this map is a Λ-homomorphism).

In the case $M_1 = M_2$ we will consider an additional property:

(d) $[a, b] = \varepsilon[b, a]$ for $a, b \in M$, $\varepsilon = \pm 1$.

3.2. THEOREM. *Let $k = \mathbb{Q}$ and M_1, M_2 be two modules of type L supplied with a pairing*

$$[\,,\,] : M_1 \times M_2 \to \Gamma/\Lambda$$

satisfying (a), (b), (c) *of subsection* 3.1. *Then there exists a unique $\mathbb{Q}$-bilinear map*

$$\langle\,,\,\rangle : A_1 \times A_2 \to \mathbb{Q}$$

(*the scalar form*), *defined on the minimal lattices $A_1 \subset M_1$, $A_2 \subset M_2$ such that*

(1) *for $a \in A_1, b \in A_2$*

$$[a, b] = \sum_{i=1}^{\mu} \sum_{\alpha} x^\alpha \langle \partial^\alpha a, \pi_i b\rangle x_i \pmod{\Lambda}$$

where α runs over all multi-indices $\alpha = (i_1, \ldots, i_s)$ with $i_j \in \{1, \ldots, \mu\}$ and $x^\alpha = x_{i_1} x_{i_2} \cdots x_{i_s}$, $\partial^\alpha = \partial_{i_s} \partial_{i_{s-1}} \cdots \partial_{i_1}$;

(2) *$\langle \pi_i a, b\rangle = \langle a, \pi_i b\rangle$ for all $a \in A_1, b \in A_2$, $i = 1, \ldots, \mu$;*

(3) *$\langle za, b\rangle = \langle a, \bar{z} b\rangle$, where $\bar{z} = 1 - z$;*

(4) *$\langle\,,\,\rangle$ is non-degenerate; i.e. the associated map $A_2 \to A_1^* = \operatorname{Hom}_{\mathbb{Q}}(A_1; \mathbb{Q})$ is an isomorphism.*

Conversely, given a scalar form with the above properties, the formula in (1) *defines a pairing $A_1 \times A_2 \to \Gamma/\Lambda$ which can be uniquely extended to a pairing $M_1 \times M_2 \to \Gamma/\Lambda$ satisfying* (a), (b), (c) *of* 3.1.

Proof. To define a pairing $[\,,\,] : M_1 \times M_2 \to \Gamma/\Lambda$ satisfying (a), (b), (c) of 3.1 is equivalent to specifying a Λ-isomorphism

$$M_2 \to \operatorname{Hom}_\Lambda(M_1; \Gamma/\Lambda),$$

and by Lemma 2.9 and Theorem 2.9 this is equivalent to specifying a D-isomorphism

$$A_2 \to A_1^*$$

which is the restriction of the above homomorphism to the minimal lattices and represents the scalar form.

3.3. THEOREM. *Let $k = \mathbb{Q}$ and $[\,,\,] : M \times M \to \Gamma/A$ be a pairing satisfying* (a), (b), (c) *of* 3.1. *The pairing* $[\,,\,]$ *satisfies* (d) *of* 3.3 *if and only if the scalar form* $\langle\,,\,\rangle$ *is* $(-\varepsilon)$*-symmetric*:

$$\langle a, b\rangle = -\varepsilon\langle b, a\rangle$$

for $a, b \in A$.

Proof. Consider the pairing

$$\{\,,\,\} : A \times A \to \Gamma,$$

$$\{a, b\} = \sum_{i=1}^{\mu} \sum_{\alpha} x^{\alpha}\langle \partial^{\alpha} a, \pi_i b\rangle x_i, \qquad a, b \in A.$$

Let $\delta_i : \Gamma \to \Gamma$ denote the map of 2.4. Then

$$\begin{aligned}
\delta_i\{a, b\} &= -\sum_{\alpha} x^{\alpha}\langle \partial^{\alpha} a, \pi_i b\rangle x_i - \sum_{\alpha} x^{\alpha}\langle \partial^{\alpha} a, \pi_i b\rangle \\
&= -\sum_{\alpha} x^{\alpha}\langle \partial^{\alpha} a, \pi_i b\rangle x_i - \langle a, \pi_i b\rangle \\
&\quad -\sum_{\alpha} x^{\alpha}\langle \partial_i \partial^{\alpha} a, \pi_i b\rangle x_i - \sum_{\substack{k=1\\ k\neq i}}^{\mu} \sum_{\alpha} x^{\alpha}\langle \partial_k \partial_a^{\alpha}, \pi_i b\rangle x_k \\
&= \sum_{\alpha} x^{\alpha}\langle \partial^{\alpha} a, \pi_i \partial_i b\rangle x_i \quad \text{(from the first and the third term)} \\
&\quad + \sum_{\substack{k=1\\ k\neq i}}^{\mu} \sum_{\alpha} x^{\alpha}\langle \partial^{\alpha} a, \pi_k \partial_i b\rangle x_k \quad \text{(from the fourth term)} - \langle a, \pi_i b\rangle \\
&= \{a, \partial_i b\} - \langle a, \pi_i b\rangle.
\end{aligned}$$

From this one obtains by induction

$$\delta_{i_1}\delta_{i_2}\cdots\delta_{i_s}\{a, b\} = \{a, \partial_{i_1}\partial_{i_2}\cdots\partial_{i_s}b\} - \langle a, \pi_{i_1}\partial_{i_2}\cdots\partial_{i_s}b\rangle$$

and

$$\varepsilon(\delta_{i_1}\delta_{i_2}\cdots\delta_{i_s}\{a, b\}) = -\langle a, \pi_{i_1}\partial_{i_2}\cdots\partial_{i_s}b\rangle,$$

where ε denotes the augmentation $\Gamma \to \mathbb{Q}$. Thus

$$\{a, b\} = -\sum_{(i_1,\ldots,i_s)} \bar{x}_{i_1}\bar{x}_{i_2}\cdots\bar{x}_{i_s}\langle a, \pi_{i_1}\partial_{i_2}\cdots\partial_{i_s}b\rangle$$

and

$$\overline{\{a, b\}} = -\sum_{i=1}^{\mu}\sum_{\alpha} x^{\alpha}\langle\pi_i a, \partial^{\alpha}b\rangle x_i.$$

Reducing the last formula modulo Λ, we see that the scalar form $\langle\,,\rangle_1 : A \times A \to \mathbb{Q}$ corresponding to

$$[\,,]_1 : M \times M \to \Gamma/\Lambda, \qquad [a, b]_1 = \overline{[b, a]}$$

is

$$\langle a, b\rangle_1 = -\langle b, a\rangle$$

and so our statement follows from Theorem 3.2.

§4. Quasi-minimal lattices

In this section we come to our main goal and start the study of the structure of modules of type L assuming that the ground ring k is $\mathbb{Z}$.

We have to change our notation slightly. The symbols Λ, Γ, D will denote the rings introduced in 1.1, 1.2, 1.3, with $k = \mathbb{Z}$. For example $\Lambda = \mathbb{Z}[F_\mu]$ and so on. The symbols $\mathbb{Q}\Lambda, \mathbb{Q}\Gamma, \mathbb{Q}D$ will denote similar rings where $k = \mathbb{Q}$.

A module of type L is now a left Λ-module with the Sato property.

4.1. Let M be a module of type L. Then $\mathbb{Q}M = \mathbb{Q} \otimes_{\mathbb{Z}} M$ is a module of type L over $\mathbb{Q}\Lambda$ and by Theorem 1.11 there exists a unique minimal lattice $\mathbb{Q}N \subset \mathbb{Q}M$. Consider the canonical map

$$\pi : M \to \mathbb{Q}M$$

and define

$$N = \pi^{-1}(\mathbb{Q}N).$$

We will call N the *principal submodule* of M. It has the following properties:

(a) if $m \in M$ and $nm \in N$ for some $n \in \mathbb{Z}, n \neq 0$, then $m \in N$;
(b) N is invariant under $\partial_i, \pi_i : M \to M, i = 1, \ldots, \mu$;
(c) N generates M over Λ.

A lattice A in M will be called *quasi-minimal* if it has the following property: the groups

$$\pi_k A/\pi_k zA, \qquad \pi_k A/\pi_k \bar{z}A,$$

are finite for all $k = 1, \ldots, \mu$. From Theorem 1.11 it follows that any quasi-minimal lattice A lies in N and the group N/A is $\mathbb{Z}$-torsion. Conversely, any lattice A which lies in N is quasi-minimal. The arguments at the beginning of the proof of Theorem 1.11 show that N always contains a quasi-minimal lattice.

If A and B are two quasi-minimal lattices with $A \subset B$ then B/A is finite.

The following statements are designed to enable us to compute the principal submodule and quasi-minimal lattices of the dual module $\mathrm{Hom}_\Lambda(M; \Gamma/\Lambda)$.

4.2. PROPOSITION. *Let M be a module of type L with no $\mathbb{Z}$-torsion and let N be its principal submodule. Any $\mathbb{Z}$-homomorphism $f: N \to \mathbb{Q}$ defines a D-homomorphism*

$$\varphi(f) : N \to \mathbb{Q}\Gamma/\mathbb{Q}\Lambda$$

(*where*

$$\varphi(f)(a) = \sum_{i=1}^{\mu} \sum_{\alpha} x^\alpha f(\pi_i \partial^\alpha a) x_i \pmod{\mathbb{Q}\Lambda},$$

for $a \in N$, see th. 2.9) *which has a unique extension* (*by Lemma* 2.6)

$$\bar{\varphi}(f) : M \to \mathbb{Q}\Gamma/\mathbb{Q}\Lambda.$$

The homomorphism $\bar{\varphi}(f)$ takes values in $\Gamma/\Lambda \subset \mathbb{Q}\Gamma/\mathbb{Q}\Lambda$ if and only if there exists a quasi-minimal lattice $A \subset N$ such that $f(A) \subset \mathbb{Z}$.

Proof. Suppose that there exists a lattice $A \subset N$ with $f(A) \subset \mathbb{Z}$. Then $\varphi(f)|_A$ takes values in Γ/Λ and by Lemma 2.6 $\varphi(f)|_A$ has a unique extension $M \to \Gamma/\Lambda$; thus $\bar{\varphi}(f)(M) \subset \Gamma/\Lambda$.

Conversely, suppose that $\bar{\varphi}(f)$ takes values in $\Gamma/\Lambda \subset \mathbb{Q}\Gamma/\mathbb{Q}\Lambda$. Let $B \subset N$ be a quasi-minimal lattice. For $i = 1, \dots, \mu$ and $b \in B$ define

$$\varphi_i(b) = \sum_{\alpha} x^{\alpha} f(\pi_i \partial^{\alpha} b).$$

Then φ_i is a D-homomorphism $B \to \Gamma + \mathbb{Q}\Lambda$. Consider the composite map

$$\psi_i : B \xrightarrow{\varphi_i} \Gamma + \mathbb{Q}\Lambda \to \Gamma + \mathbb{Q}\Lambda/\Gamma = \mathbb{Q}\Lambda/\Lambda.$$

If $b \in B$, the condition $b \in \ker(\psi_i)$ is equivalent to

$$f(\pi_i \partial^{\alpha} b) \in \mathbb{Z}$$

for any multi-index α (including $\alpha = \varnothing$). It is clear that $b \in \ker(\psi_i)$ implies $\partial_k b \in \ker(\psi_i)$ and $\pi_k b \in \ker(\psi_i)$ for any $k = 1, \dots, \mu$. Thus the subgroup

$$A = \bigcap_{i=1}^{\mu} \ker(\psi_i), \qquad A \subset B$$

is invariant under ∂_k and π_k. Lemma 4.4 below applied to the map

$$g : B \to \underbrace{\mathbb{Q}\Lambda/\Lambda \oplus \mathbb{Q}\Lambda/\Lambda \oplus \cdots \oplus \mathbb{Q}\Lambda/\Lambda}_{\mu \text{ times}},$$

where $g(b) = \psi_1(b) \oplus \psi_2(b) \oplus \cdots \oplus \psi_\mu(b)$, $b \in B$, shows that A generates M over Λ. Thus A is a lattice with $f(A) \subset \mathbb{Z}$.

4.3. Let C be a D-module. We will say that C is a *D-module of type* 0 if $\partial_k C = 0$ for all $k = 1, \ldots, \mu$. We will say that C is a *D-module of type* i (where $i \in \{1, 2, \ldots, \mu\}$) if $\partial_k C = 0$ for $k \neq i$, $k \in \{1, \ldots, \mu\}$ and $(1 + \partial_i)C = 0$.

4.4. LEMMA. *Let M be a module of type L. Let $X \subset M$ be a D-submodule which is finitely generated over $\mathbb{Z}$ and generates M over Λ. Let Y be a D-module having a D-filtration*

$$0 = Y_0 \subset Y_1 \subset Y_2 \subset \cdots, \qquad \cup Y_i = Y,$$

with the property that for each $j = 1, 2, \ldots$ there exists a number $i = i(j) \in \{0, 1, \ldots, \mu\}$ such that Y_j / Y_{j-1} is a D-module of type i. Then the kernel of any D-homomorphism

$$g : X \to Y$$

generates M over Λ.

Proof. It is sufficient to prove the Lemma under the assumption that Y is a D-module of type i, for $i \in \{0, 1, \ldots, \mu\}$.

Let $i = 0$. Then for any $x \in X$ all $\partial_k x$, $k = 1, 2, \ldots, \mu$, belong to $\ker(g)$, and

$$x = \sum_{k=1}^{\mu} (t_k - 1)\, \partial_k x.$$

Thus, x belongs to $\Lambda(\ker(g))$, the Λ-submodule of M, generated by $\ker(g)$.

Suppose that $i \in \{1, \ldots, \mu\}$ and Y is a D-module of type i. For $x \in X$, let $\sigma_k = \partial_k x$ for $k \neq i$, $k \in \{1, \ldots, \mu\}$ and $\sigma_i = x + \partial_i x$. Then $\sigma_j \in \ker(g)$ for all j and

$$x = t_i^{-1} \sum_{k=1}^{\mu} (t_k - 1)\sigma_k,$$

which proves that $x \in \Lambda(\ker(g))$.

Thus,

$$\Lambda(\ker(g)) = M$$

in both cases.

From Proposition 4.2 and Theorem 2.9 we get:

4.5. COROLLARY. *Let M be a module of type L and let $N \subset M$ be its principal submodule. Then the homomorphism φ of Theorem 2.9 gives an identification of the principal submodule of the dual module with the set $D(N)$ of all $\mathbb{Z}$-homomorphisms $f: N \to \mathbb{Q}$ having the property that $f(A) \subset \mathbb{Z}$ for some lattice $A \subset N$.*

§5. Self-dual lattices

5.1. In this section we will consider a module M of type L (with the ground ring $k = \mathbb{Z}$) supplied with a Hermitian pairing

$$[\,,\,]: M \times M \to \Gamma/\Lambda$$

satisfying (a), (b), (c), (d) of 3.1.

Any such module M has no $\mathbb{Z}$-torsion and we can consider it as embedded in $\mathbb{Q}M$ via the canonical map $M \to \mathbb{Q}M$. From Theorem 3.2 we know that there exists a unique scalar form

$$\langle\,,\,\rangle: N \times N \to \mathbb{Q},$$

where N is the principal submodule of M, such that

$$[a, b] = \sum_{i=1}^{\mu} \sum_{\alpha} x^{\alpha}\langle \partial^{\alpha} a, \pi_i b\rangle x_i \pmod{\Lambda}$$

for $a, b \in N$.

The pairing $[\,,\,]: M \times M \to \Gamma/\Lambda$ defines a map

$$\lambda: M \to \operatorname{Hom}_{\Lambda}(M; \Gamma/\Lambda),$$

where $\lambda(a)(b) = [b, a]$, for $a, b \in M$.

Since λ is an isomorphism, it maps the principal submodule N of M isomorphically onto the principal submodule of $\operatorname{Hom}_{\Lambda}(M; \Gamma/\Lambda)$ which can be identified with $D(N)$ (cf. Corollary 4.5). This means that the scalar form $\langle\,,\,\rangle$ has the following property (additional to that of Theorems 3.2 and 3.3):

5.2. COROLLARY. *Let $\langle\,,\,\rangle: N \times N \to \mathbb{Q}$ be the scalar form, corresponding to the Hermitian form $[\,,\,]$. Then* (1) *for each $a \in N$ there exists a lattice $A \subset N$ such that*

$\langle b, a\rangle \in \mathbb{Z}$ for all $b \in A$; (2) for any $\mathbb{Z}$-homomorphism $f: N \to \mathbb{Q}$ having the property that $f(A) \subset \mathbb{Z}$ where A is a quasi-minimal lattice, there exists unique $a \in N$ such that

$$f(b) = \langle b, a\rangle$$

for all $b \in N$.

5.3. PROPOSITION. *Let $A \subset N$ be a subset. Define*

$$A^{\perp} = \{b \in N; \langle b, a\rangle \in \mathbb{Z}, \forall a \in A\}.$$

If A is a lattice then $A^{\perp}$ is also a lattice.

Proof. The formulas

$$\langle zb, a\rangle = \langle b, \bar{z}a\rangle,$$

$$\langle \pi_i b, a\rangle = \langle b, \pi_i a\rangle$$

show that $A^{\perp}$ is invariant under z and π_i, $i = 1, \dots, \mu$ and we have only to prove that $A^{\perp}$ generates M over Λ. Let $e_1, \dots, e_n$ be a basis of A. By Corollary 5.2 for each $i = 1, \dots, n$ there exists a lattice $B_i \subset N$ with $\langle e_i, B_i\rangle \subset \mathbb{Z}$, $i = 1, \dots, n$. Then the intersection

$$B = \bigcap_{i=1}^{n} B_i$$

is contained in $A^{\perp}$ and by Lemma 1.5, B generated M over Λ.

This proves the statement.

5.4. It is clear that $A \subset A^{\perp\perp}$. The reverse inclusion is also true. For if $b \notin A$ then there exists a functional $f: N \to \mathbb{Q}$ with $f(A) \subset \mathbb{Z}$ and $f(b) \notin \mathbb{Z}$. By 5.2 there exists $c \in N$ with $f(x) = \langle x, c\rangle$ for all $x \in N$. Thus $c \in A^{\perp}$ and $\langle c, b\rangle \notin \mathbb{Z}$ which means $b \notin A^{\perp\perp}$. So

$$A^{\perp\perp} = A.$$

5.5. A quasi-minimal lattice $A \subset N$ will be called *self-dual* if $A = A^{\perp}$. Two self-dual lattices A and B will be called *adjacent* if $zA \subset B$ or $zB \subset A$.

5.6. THEOREM. *Let M be a module of type L over the ring $\Lambda = \mathbb{Z}[F_\mu]$. Assume that M is supplied with a non-degenerate Hermitian form*

$$[\,,\,] : M \times M \to \Gamma/\Lambda$$

satisfying (a), (b), (c), (d) *of* 3.1. *Then*

(1) *there exists a self-dual lattice $A \subset M$;*

(2) *for any two self-dual lattices $A, B \subset M$ there exists a finite sequence of self-dual lattices*

$$C_1, C_2, \ldots, C_s$$

such that

$$C_1 = A, \qquad C_s = B$$

and lattices C_i and C_{i+1} are adjacent for each $i = 1, \ldots, s-1$.

Proof of (1). Let $X \subset N$ be an arbitrary quasi-minimal lattice. Then $Y = X \cap X^\perp$ is also a lattice (by Proposition 5.3 and Lemma 1.5) and $Y^\perp \supset X + X^\perp \supset Y$, that is $Y^\perp \supset Y$. The group $Y^\perp/Y$ is finite and so we will prove statement (1) of Theorem 5.6 by showing that if $Y^\perp \neq Y$ then there exists a lattice $Z \subset N$ with $Z^\perp \supset Z$ and the order of $Z^\perp/Z$ is smaller than the order of $Y^\perp/Y$. Assume that $Y^\perp \neq Y$. It follows from Proposition 1.8 that there is an element $a \in Y^\perp$, $a \notin Y$ with $a = \pi_k a$ for some $k \in \{1, \ldots, \mu\}$ and $za \in Y$ or $\bar{z}a \in Y$. In both cases let Z be the subgroup generated by Y and a. It is clear that Z is a lattice. The identities

$$\langle a, a \rangle = \langle za, a \rangle + \langle a, za \rangle$$

$$\langle a, b \rangle = \langle \bar{z}a, a \rangle + \langle a, \bar{z}a \rangle$$

show that $Z \subset Z^\perp$. Thus we have

$$Y \subset Z \subset Z^\perp \subset Y^\perp$$

and

$$\operatorname{ord}(Z^\perp/Z) < \operatorname{ord}(Y^\perp/Y).$$

Part (2) of Theorem 5.6 will follow from Lemma 5.10 below.

5.7. LEMMA. *Let $X \subset N$ be a quasi-minimal lattice with $X \subset X^\perp$, $\pi_i X = \pi_i X^\perp$ for $i \neq k$, $i \in \{1, \dots, \mu\}$ and*

$$z\bar{z}\pi_k X^\perp \subset X.$$

Let $A_+ \subset X^\perp$ be the set of all elements $y \in X^\perp$ with $zy \in X$. Then A_+ is a self-dual lattice and any other self-dual lattice B with $X \subset B \subset X^\perp$ is adjacent to A_+. Similarly, define $A_- = \{y \in X^\perp; \bar{z}y \in X\}$. Then A_- is a self-dual lattice adjacent to any lattice B with $X \subset B \subset X^\perp$.

Proof. It is clear that A_+ is a lattice. Let us show that A_+ is self-dual. For $a_1, a_2 \in A_+$ we have

$$\langle a_1, a_2 \rangle = \langle za_1, a_2 \rangle + \langle a_1, za_2 \rangle \in \mathbb{Z}.$$

Thus $A_+ \subset A_+^\perp$. Suppose $a \in A_+^\perp$. For any $y \in X^\perp$ the element $\bar{z}y$ belongs to A_+ and so

$$\langle za, y \rangle = \langle a, \bar{z}y \rangle \in \mathbb{Z}$$

which implies that $za \in X$ and $a \in A_+$.

The inclusions

$$zA_+ \subset X \subset B$$

show that A_+ and B are adjacent, where B is any self-dual lattice with $X \subset B \subset X^\perp$. The second statement concerning A_- can be proved similarly.

5.8. LEMMA. *Let X be a quasi-minimal lattice, $X \subset N$, with $X \subset X^\perp$. Then for any two self-dual lattices $A \supset X$ and $B \supset X$, the order of A/X is equal to the order of B/X. More precisely,*

$$\operatorname{ord}(A/X)^2 = \operatorname{ord}(B/X)^2 = \operatorname{ord}(X^\perp/X).$$

Proof. The scalar form $\langle\ ,\ \rangle$ defines a non-degenerate form

$$l : X^\perp/X \times X^\perp/X \to \mathbb{Q}/\mathbb{Z}$$

and for any self-dual lattice $A \supset X$ the subgroup $A/X \subset X^\perp/X$ coincides with its own annihilator with respect to l. Now

$$\operatorname{ord}(A/X)^2 = \operatorname{ord}(X^\perp/X)$$

follows.

5.9. Let us define the *distance* between two self-dual lattices A, B as

$$d(A, B) = \text{ord}\,(A/A \cap B) = \text{ord}\,(B/A \cap B).$$

We shall say that A and B are *equivalent* iff there exists a sequence of self-dual lattices $C_1, \ldots, C_s$ such that

$$C_1 = A, \qquad C_s = B$$

and lattices C_i and C_{i+1} are adjacent for each $i = 1, \ldots, s-1$. Statement (2) of Theorem 5.6 (which are are going to prove) says that any two self-dual lattices are equivalent.

5.10. LEMMA. *Let A and B be two self-dual lattices $A \neq B$. Then there exists a self-dual lattice A_1 which is equivalent to A and*

$$d(A_1, B) < d(A, B).$$

Proof. From Proposition 1.8 it follows that there is an element $b \in B$ with $b = \pi_k b$ for some $k \in \{1, \ldots, \mu\}$, $b \notin A \cap B$ and

$$zb \in A \cap B \quad \text{or} \quad \bar{z}b \in A \cap B.$$

Let us consider the first possibility; the arguments in the case $\bar{z}b \in A \cap B$ are quite similar. Denote by X the subgroup of N generated by A and b. It is clear that X is a lattice and

$$A \cap B \subset X^{\perp} \subset A \subset X,$$

$$\pi_i X^{\perp} = \pi_i X \qquad \text{for} \quad i \neq k,$$

$$z\bar{z}\pi_k X \subset X^{\perp}.$$

Denote by A_1 the set of all $y \in X$ with $zy \in X^{\perp}$. By Lemma 5.7, A_1 is a self-dual lattice equivalent to A. Furthermore, $A_1 \cap B$ contains $A \cap B$ and b and thus

$$d(A_1, B) = \text{ord}\,(B/A_1 \cap B) < \text{ord}\,(B/A \cap B) = d(A, B).$$

5.11. Lemma 5.10 obviously implies part (2) of Theorem 5.6.

§6. The classification of simple links

In this section we apply the results of the previous algebraic study of modules of type L to the geometric problem of describing simple links up to ambient isotopy. We will find a connection between minimal Seifert manifolds of a link and self-dual lattices in the homology module of the corresponding free covering.

6.1. An *n-dimensional μ-component link* is an oriented smooth submanifold Σ^n of S^{n+2}, where $\Sigma^n = \Sigma_1^n \cup \cdots \cup \Sigma_\mu^n$ is the ordered disjoint union of μ submanifolds of S^{n+2}, each homeomorphic to S^n. Σ is a *boundary link* if there is an oriented smooth submanifold V^{n+1} of S^{n+2}, $V^{n+1} = V_1^{n+1} \cup \cdots \cup V_\mu^{n+1}$ the disjoint union of the submanifolds V_i^{n+1}, such that $\partial V_i = \Sigma_i$ $(i = 1, \ldots, \mu)$. If each V_i is connected, we say that V is a *Seifert manifold* for Σ.

6.2. Let Σ^n be a μ-component link in S^{n+2}, and let $X = S^{n+2} - T(\Sigma)$ be the complement of a tabular neighbourhood $T(\Sigma)$ of Σ in S^{n+2}. Fix a base point $* \in X$; for each $i = 1, \ldots, \mu$ the *meridian* $m_i \in \pi_1(X, *)$ (an element represented by a small loop around Σ_i joined by a path to the base point) is defined up to conjugation.

A *splitting* [CS] is a homomorphism (which is defined up to conjugation) $S : \pi_1(X, *) \to F_\mu$ onto the free group with μ generators $t_1, \ldots, t_\mu$ and has the following property: the image of the conjugacy class of the i-th meridian m_i coincides with the conjugacy class $[t_i]$ of $t_i \in F_\mu$.

This notion does not depend on the choice of the base point.

If Σ is a boundary link then each Seifert manifold V defines an obvious splitting S_V: if α is a loop in X which is in general position with respect to V, then $\mathscr{S}_V([\alpha])$ is a word in $t_1, \ldots, t_\mu$, obtained by writing down $t_i^{\varepsilon_i}$ $(\varepsilon_i = \pm 1)$ for each intersection p of α and V (where i is the number with $p \in V_i \cap \alpha$ and ε_i is the local intersection number of α and V_i at p) and then multiplying these words in order of their appearance in α.

A theorem of Gutiérrez [G] states that any link admitting a splitting, is a boundary link; cf also [Sm].

6.3. An *$\mathscr{F}$-link* [CS] (of dimension n multiplicity μ) is a pair $(\Sigma, \mathscr{S})$, where Σ is a link (of dimension n multiplicity μ) and $\mathscr{S}$ is a splitting for Σ. Two $\mathscr{F}$-links $(\Sigma_1, \mathscr{S}_1)$ and $(\Sigma_2, \mathscr{S}_2)$ are *equivalent* if there exists a diffeomorphism $h : S^{n+2} \to S^{n+2}$, taking Σ_1 onto Σ_2, preserving orientations of S^{n+2} and Σ_ν, $\nu = 1, 2$, and mapping $\mathscr{S}_2$ onto $\mathscr{S}_1$.

6.4. An $\mathscr{F}$-link $(\Sigma, \mathscr{S})$ is called *r-simple* (where r is an integer, $r \geq 1$) if (a) $\mathscr{S}$ is an isomorphism $\pi_1(X, *) \to F_\mu$; (b) $\pi_i(X, *) = 0$ for all $1 < i \leq r$. We will consider every $\mathscr{F}$-link as being *0-simple*.

Another theorem of Gutiérrez [G] states that any r-simple n-dimensional $\mathcal{F}$-link $(\Sigma, \mathcal{S})$ admits a Seifert manifold V with each component r-connected and $\mathcal{S} = \mathcal{S}_V$, provided $n \geq 4$.

6.5. Let $(\Sigma, \mathcal{S})$ be an $\mathcal{F}$-link. Fix a particular epimorphism $\mathcal{S}_0 : \pi_1(X, *) \to F_\mu$ onto F_μ (the free group in $t_1, \ldots, t_\mu$) conjugate to $\mathcal{S}$. Consider the covering

$$\tilde{X} \to X$$

corresponding to the kernel of $\mathcal{S}_0$; it has the group F_μ acting on $\tilde{X}$ as the group of covering translations. The diffeomorphism type of $\tilde{X}$ (considered as manifold together with F_μ-action) does not depend on the choice of $\mathcal{S}_0$ in $\mathcal{S}$. Thus, the homology

$$H_k(\tilde{X}; \mathbb{Z}), \qquad k = 1, 2, \ldots, n$$

considered as left $\Lambda = \mathbb{Z}[F_\mu]$-modules, are invariants of $(\Sigma, \mathcal{S})$.

Sato has shown that these homology modules of the free covering are Λ-modules of type L (cf. [S]).

A multiplicative structure on $H_*(\tilde{X}; \mathbb{Z})$, which will now be described, comes from Poincaré duality. Let C be a chain complex of $\tilde{X}$ constructed by means of a Morse function on X. The Poincaré duality isomorphism [M2]

$$H_{n+2-i}(\tilde{X}, \partial\tilde{X}) \to \overline{H^i(C; \Lambda)}$$

is a Λ-isomorphism. Here $H^i(C; \Lambda)$ is a right Λ-module and the bar means that we convert it into a left Λ-module using the standard involution

$$t_i \to t_i^{-1}, \qquad i = 1, \ldots, \mu$$

in Λ. It is clear that

$$H_{n+2-i}(\tilde{X}, \partial\tilde{X}) = H_{n+2-i}(\tilde{X})$$

for $2 \leq i \leq n+1$. On the other hand, it is shown in [F2] that the universal coefficient spectral sequence gives a natural epimorphism

$$H^i(C; \Lambda) \to \mathrm{Hom}_\Lambda(H_{i-1}(\tilde{X}); \Gamma/\Lambda)$$

with kernel equal to the $\mathbb{Z}$-torsion subgroup of $H^i(C; \Lambda)$.

Combining these facts we get pairings (Blanchfield forms)

$$[\,,\,] : H_p(\tilde{X}) \times H_q(\tilde{X}) \to \Gamma/\Lambda, \qquad p + q = n + 1$$

with the following properties:

(1) $[\lambda a, b] = \lambda[a, b]$ for $a \in H_p(\tilde{X})$, $b \in H_q(\tilde{X})$, $\lambda \in \Lambda$;
(2) $[a, \lambda b] = [a, b]\bar{\lambda}$;
(3) the associated map

$$H_q(\tilde{X}) \to \overline{\mathrm{Hom}_\Lambda}\,(H_p(\tilde{X}); \Gamma/\Lambda)$$

is a Λ-isomorphism, provided $H_q(\tilde{X})$ has no $\mathbb{Z}$-torsion.

6.6. Let $(\Sigma, \mathscr{S})$ be an $\mathscr{F}$-link of dimension $n = 2q - 1$ which is $(q - 1)$-simple. Then it follows from [F2], Th. 5.7 that $H_q(\tilde{X})$ has no $\mathbb{Z}$-torsion. Thus, in this case we have a *non-degenerate* pairing

$$[\,,\,] : H_q\tilde{X} \times H_q\tilde{X} \to \Gamma/\Lambda.$$

The following theorem is the main result of the paper.

6.7. THEOREM. *Let $q \geq 3$. The Λ-module $H_q(\tilde{X})$ together with the pairing*

$$[\,,\,] : H_q\tilde{X} \times H_q\tilde{X} \to \Gamma/\Lambda.$$

provide a complete system of invariants for $(q - 1)$-simple $(2q - 1)$-dimensional $\mathscr{F}$-links. In other words, two $(q - 1)$-simple $(2q - 1)$-dimensional $\mathscr{F}$-links are equivalent iff there exists an isomorphism between corresponding modules $H_q\tilde{X}$, preserving the pairing $[\,,\,]$.

The proof of this theorem will be based on Th. 4.7 of [F2] and Th. 5.6 of the present paper.

6.8. Recall some definitions from [F2].

An *ε-symmetric isometry structure of multiplicity μ* is a tuple

$$(A, \langle\,,\,\rangle, z, \pi_1, \ldots, \pi_\mu),$$

where A is a finitely generated free abelian group, $\langle\, , \rangle : A \otimes A \to \mathbb{Z}$ is an ε-symmetric bilinear form, and $z, \pi_1, \ldots, \pi_\mu : A \to A$ are endomorphisms of A satisfying

(i) $\langle\, , \rangle$ is unimodular,
(ii) $\langle za, b\rangle = \langle a, \bar{z}b\rangle$ for $a, b \in A$, where $\bar{z}$ denotes $1 - z : A \to A$;
(iii) $\langle \pi_i a, b\rangle = \langle a, \pi_i b\rangle$;
(iv) $\pi_1 + \pi_2 + \cdots + \pi_\mu = 1_A$;
(v) $\pi_i \circ \pi_j = \delta_{ij}\pi_j$.

6.9. Examples of isometry structures are of both an algebraic and a geometric nature.

Consider first a $\mathbb{Z}$-torsion free module M of type L over the ring $\Lambda = \mathbb{Z}[F_\mu]$ supplied with a non-degenerate $(-\varepsilon)$-hermitian form

$$[\, ,] : M \times M \to \Gamma/\Lambda.$$

Then any self-dual lattice A in M is an isometric structure of multiplicity μ; the form $\langle\, , \rangle : A \times A \to \mathbb{Z}$ is the restriction of the scalar form corresponding to $[\, ,]$ (cf. §5) and $z, \pi_1, \ldots, \pi_\mu$ are restrictions of the corresponding self-maps of M, cf. §1.

6.10. We will say that an isometric structure A admits an *embedding* in a module M of type L supplied with a non-degenerate hermitian form

$$[\, ,] : M \times M \to \Gamma/\Lambda$$

if A is isomorphic as an isometry structure to a self-dual lattice in M.

If an isometry structure A admits embeddings in two Λ-modules $(M_i, [\, ,]_i)$, $i = 1, 2$ of type L, then there exists a Λ-isomorphism $f : M_1 \to M_2$ preserving the forms; this follows from Lemma 2.6 and from Theorem 3.2.

6.11. Isometry structures appear geometrically as homology modules of Seifert surfaces of boundary links. Let $V^{n+1} = V_1 \cup \cdots \cup V_\mu \subset S^{n+2}$ be a Seifert manifold with $\partial V = \Sigma^n \subset S^{n+2}$ be a boundary link of multiplicity μ. Assume that $n = 2q - 1$ and $H_q(V)$ has no $\mathbb{Z}$-torsion. Let

$$\langle\, , \rangle : H_q(V) \times H_q(V) \to \mathbb{Z}$$

be the intersection form on V and let $\pi_i : H_q(V) \to H_q(V)$, $i = 1, \dots, \mu$, be the projector corresponding to the direct summand $H_q(V_i) \subset H_q(V)$. In the next paragraph we will define an operator $z : H_q(V) \to H_q(V)$ in such a way that

$$(H_q(V), \langle\ ,\ \rangle, z, \pi_1, \dots, \pi_\mu)$$

would be a $(-1)^q$-symmetric isometric structure of multiplicity μ.

Let Y denote the result of cutting the sphere S^{n+2} along V. Denote by $i_+, i_- : V \to Y$ maps given by small shifts in the directions of positive and negative normals to V, respectively. Then the map $i_{+*} - i_{-*} : H_q(V) \to H_q(Y)$ is an isomorphism (cf. [F1], §1.1) and we define $z(v) \in H_q(V)$ for $v \in H_q(V)$ by

$$(i_{+*} - i_{-*})(z(v)) = i_{+*}(v).$$

In 1.2 of [F1] it has been shown that the endomorphism $z : H_q(V) \to H_q(V)$ satisfies (ii) of 6.8; the other properties of 6.8 are evident. Thus any Seifert manifold defines an isometry structure $(H_q(V), \langle\ ,\ \rangle, z, \pi_1, \dots, \pi_\mu)$ which will be denoted simply as $H_q(V)$.

6.12. An isometric structure $(A, \langle\ ,\ \rangle, z, \pi_1, \dots, \pi_\mu)$ will be called *minimal* iff $z\pi_i a = 0$ or $\bar{z}\pi_i a = 0$ imply $\pi_i a = 0$, for $a \in A$.

Any isometric structure admitting an embedding (cf. 6.10) in a module of type L is minimal.

(Proof: if an isometric structure A is realized as a self-dual lattice in a module M of type L supplied with a form $[\ ,\ ]$, then for $a \in A$ the condition $z\pi_i a = 0$ means $\partial_i a = 0$ and thus $\pi_i a = (t_i - 1)\,\partial_i a = 0$).

We will see later in 7.1 that the converse is also true: any minimal isometry structure can be embedded.

If $V^{2q} \subset S^{2q+1}$ is a Seifert manifold of a boundary link, then $H_q(V)$ is minimal iff the homomorphisms $i_{+*}, i_{-*} : H_q(V) \to H_q(Y)$ map each $H_q(V_i)$ monomorphically, where $V = V_1 \cup \cdots \cup V_\mu$.

Gutiérrez [G] has shown that any $(q-1)$-simple $(2q-1)$-dimensional boundary link admits a $(q-1)$-connected Seifert manifold V with minimal $H_q(V)$. We will say that a Seifert manifold V with these properties is *minimal.*

6.13. THEOREM. *Let $(\Sigma, \mathscr{S})$, $\Sigma^{2q-1} = \Sigma_1 \cup \cdots \cup \Sigma_\mu \subset S^{2q+1}$ be a $(2q-1)$-dimensional $(q-1)$-simple $\mathscr{F}$-link. Let V be a minimal Seifert manifold of Σ, realizing the splitting $\mathscr{S}$ (cf.* 6.2). *Let $\tilde{X}$ be the free covering of $X = S^{2q+1} - \Sigma$, corresponding to $\mathscr{S}$ (cf.* 6.5) *and let*

$$[\ ,\ ] : H_q(\tilde{X}) \times H_q(\tilde{X}) \to \Gamma/\Lambda$$

be the Blanchfield form. Then the isometry structure $H_q(V)$ *admits an embedding in* $(H_q(\tilde{X});[\,,\,])$.

We first show that this Theorem implies Theorem 6.7, the main theorem of this paper.

6.14. *Proof of Theorem* 6.7. Assume that $(\Sigma_i, \mathscr{S}_i)$, $i = 1, 2$ are two $(q-1)$-simple $(2q-1)$-dimensional $\mathscr{F}$-links, having isomorphic Blanchfield forms $(H_q(\tilde{X}_i), [\,,\,]_i)$, $i = 1, 2$. Denote $M = H_q(\tilde{X}_1)$, $[\,,\,] = [\,,\,]_1$ and fix an isomorphism

$$f : H_q(\tilde{X}_2) \to M$$

with

$$[a, b]_2 = [f(a), f(b)]$$

for $a, b \in H_q(\tilde{X}_2)$.

Let V_i, $i = 1, 2$, be a minimal Seifert manifold of $(\Sigma_i, \mathscr{S}_i)$, $i = 1, 2$.

Denote by $A \subset M$ the image of an embedding of $H_q(V_1)$ in $H_q(\tilde{X}_1) = M$, which exists by Theorem 6.13. Let $B \subset M$ be the image under f of a self-dual lattice in $H_q(\tilde{X}_2)$ isomorphic to $H_q(V_2)$. Thus, A and B are embedded as self-dual lattices in M.

By Theorem 5.6 there exists a sequence $C_1, C_2, \ldots, C_s$ of self-dual lattices in M such that $C_1 = A$, $C_s = B$ and C_i, C_{i+1} are adjacent for all $i = 1, \ldots, s-1$. Considering C_i and C_{i+1} as isometry structures, (cf. 6.9) we see that they are *contiguous* (cf. 4.6 of [F2]): if $zC_i \subset C_{i+1}$ then $\bar{z}C_{i+1} \subset C_i$ and one may define $\varphi : C_i \to C_{i+1}$ and $\psi : C_{i+1} \to C_i$ by $\varphi(a) = za$, $\psi(b) = \bar{z}b$ for $a \in C_i$, $b \in C_{i+1}$, satisfying the definition of contiguity in subsection 4.6 in [F2]. By Theorems 2.6 and 4.2 of [F2] there exists a sequence of minimal $(q-1)$-connected Seifert manifolds $W_1, W_2 \cdots W_s$ of $(\Sigma_1, \mathscr{S}_1)$ with:

(i) $W_1 = V_1$;
(ii) the isometry structure $H_q(W_i)$, determined by W_i, is isomorphic to C_i, $i = 1, \ldots, s$.

Thus, W_s is a Seifert manifold of $(\Sigma_1, \mathscr{S}_1)$ having the same isometry structure as V_2, the Seifert manifold of $(\Sigma_2, \mathscr{S}_2)$. Now Theorem 4.7 of [F2] implies that $(\Sigma_1, \mathscr{S}_1)$ and $(\Sigma_2, \mathscr{S}_2)$ are equivalent.

This completes the proof.

As another corollary of Theorem 6.13 we shall prove:

6.15. COROLLARY. *The Blanchfield pairing*

$$[\,,\,] : H_q(\tilde{X}) \times H_q(\tilde{X}) \to \Gamma/\Lambda$$

of a $(2q-1)$*-dimensional* $(q-1)$*-simple link* $(\Sigma, \mathcal{S})$ *is* $(-1)^{q+1}$*-hermitian*:

$$[a, b] = (-1)^{q+1}\overline{[b, a]}$$

for $a, b \in H_q(\tilde{X})$.

Proof. It follows from Theorems 3.3 and 6.13.

6.16. The rest of the paper is devoted to the proof of Theorem 6.13.

Let $(\Sigma, \mathcal{S})$ be a $(2q-1)$-dimensional $\mathcal{F}$-link of μ components and let $V = V_1 \cup \cdots \cup V_\mu$ be its minimal Seifert manifold, realizing the splitting $\mathcal{S}$. Denote by X the complement $S^{2q+1} - \Sigma^{2q-1}$ and by $p : \tilde{X} \to X$ the free covering corresponding to $\mathcal{S}$.

Consider the $(2q+1)$-dimensional manifold Y obtained by cutting $S^{2q+1} - \Sigma$ along V. The boundary of Y is the disjoint union of manifolds

$$\partial_1^+ Y \cup \partial_1^- Y \cup \partial_2^+ Y \cup \partial_2^- Y \cup \cdots \cup \partial_\mu^+ Y \cup \partial_\mu^- Y,$$

where each component $\partial_i^\varepsilon Y$, $(\varepsilon = \pm)$ is homeomorphic to V_i. There is also a natural identification map $\psi : Y \to S^{2q+1} - \Sigma^{2q-1}$ which is a homeomorphism on int Y and maps $\partial_i^+ Y$ and $\partial_i^- Y$ homeomorphically on V_i. The internal normal on $\partial_i^+ Y$ corresponds under ψ to the positive normal on V_i.

It is clear that the map $\psi : Y \to S^{2q+1} - \Sigma = X$ can be lifted into $\tilde{X}$ and any lifting $\tilde{\psi} : Y \to \tilde{X}$ is an embedding with

$$\bigcup_{g \in F_\mu} g\tilde{\psi}(Y) = \tilde{X}.$$

One can find a lifting $\tilde{\psi} : Y \to \tilde{X}$ such that

$$\tilde{\psi}(Y) \cap g\tilde{\psi}(Y) = \begin{cases} \partial_i^- Y & \text{for } g = t_i, \\ \partial_i^+ Y & \text{for } g = t_i^{-1} \\ \phi & \text{for other } g \in F_\mu, g \neq 1. \end{cases}$$

We will identify Y with its image under $\tilde{\psi}$.

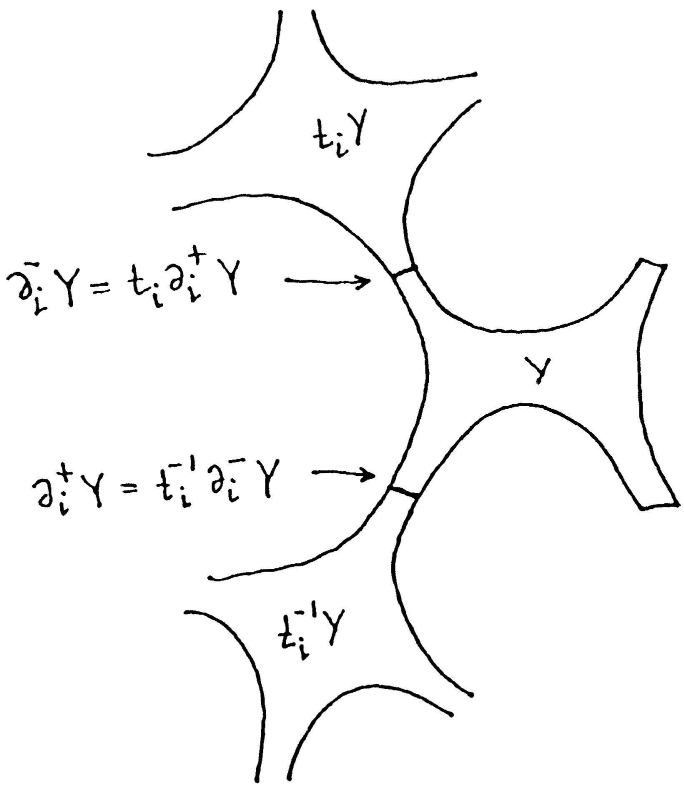

Figure 1. The picture of $\tilde{X}$ in the neighbourhood of Y

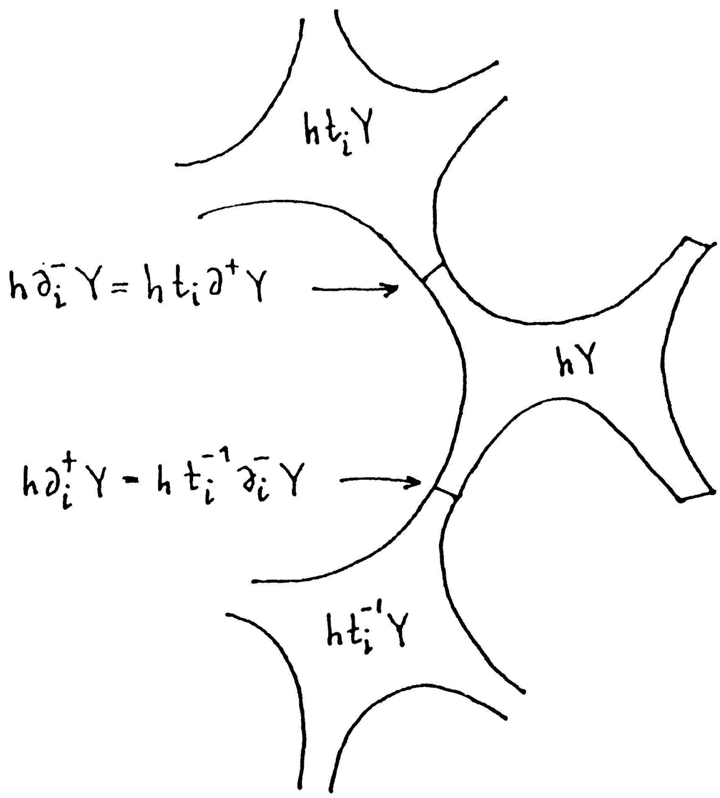

Figure 2. The picture of $\tilde{X}$ in the neighbourhood of another lifting hY of Y

6.17. Let us show that the inclusion $i: Y \to \tilde{X}$ induces a monomorphism $i_*: H_q(Y) \to H_q(\tilde{X})$. Assuming the contrary, suppose that α is a cycle in Y bounding a chain β in $\tilde{X}$. Because of compactness, β lies in a finite union

$$\bigcup_{i=1}^{m} g_i Y, \qquad g_i \in F_\mu.$$

Consider the graph Γ_μ of F_μ; vertices of Γ_μ are labeled by elements $h \in F_\mu$ and each vertex h is joined by an edge with vertices ht_i, ht_i^{-1}, $i = 1, \ldots, \mu$. It is well known that Γ_μ is a tree.

Let $T \subset \Gamma_\mu$ be the subgraph of Γ_μ spanned by $g_1, \ldots, g_m$ (one of the g_i's is $1 \in F_\mu$). We may assume that T is connected and so T is also a tree. Thus there exists a vertex g_{i_0}, $g_{i_0} \neq 1$ of T with only one edge of T incident to g_{i_0}. Let this edge join g_{i_0} with $g_j = g_{i_0} t_k^\varepsilon$, $\varepsilon = \pm 1$, $k \in \{1, \ldots, \mu\}$. The chain β defines an element of

$$H_{q+1}\Bigg(\bigcup_{i=1}^{m} g_i Y, \bigcup_{\substack{i=1 \\ i \neq i_0}}^{m} g_i Y\Bigg) \approx H_{q+1}(g_{i_0} Y, g_{i_0} \partial_k^{-\varepsilon} Y) \approx H_{q+1}(Y, \partial_k^{-\varepsilon} Y).$$

Since V is minimal, the boundary homomorphism

$$H_{q+1}(Y, \partial_k^{-\varepsilon} Y) \to H_q(\partial_k^{-\varepsilon} Y)$$

vanishes and thus β can be changed in a neighbourhood of $g_{i_0} Y$ in such a way that the new chain β_1 lies in

$$\bigcup_{\substack{i=1 \\ i \neq i_0}}^{m} g_i Y$$

and $\partial \beta_1 = \alpha$ as well. Proceeding in this way, we will get a chain β_s in Y with $\partial \beta_s = \alpha$.

6.18. Let

$$f: H_q(V) \to H_q(\tilde{X})$$

be the composite of $i_{+*} - i_{-*}: H_q(V) \to H_q(Y)$ and $i_*: H_q(Y) \to H_q(\tilde{X})$, where $i: Y \to \tilde{X}$ denotes the inclusion. f is a monomorphism, since $i_{+*} - i_{-*}$ is an isomorphism and i_* is a monomorphism. We can complete the proof of Theorem 6.11 by showing that

(i) $f(\pi_i a) = \pi_i f(a)$ for $a \in H_q(V)$, $i \in \{1, \ldots, \mu\}$;
(ii) $f(za) = zf(a)$ for $a \in H_q(V)$;
(iii) the image of f generates $H_q(\tilde{X})$ over Λ;
(iv) the Blanchfield form on im (f) is given by the formula

$$[f(a), f(b)] = \sum_{i=1}^{\mu} \sum_{\alpha} x^{\alpha} \langle \partial^{\alpha} a, \pi_i b \rangle x_i \quad (\text{mod } \Lambda),$$

where ∂^{α} denotes $\partial_{i_s} \partial_{i_{s-1}} \cdots \partial_{i_1}$ for $\alpha = (i_1, \ldots, i_s)$; cf. §3.

In fact, (i), (ii), (iii) would mean that im (f) is a lattice in $H_q(\tilde{X})$. Let us prove that im (f) is quasiminimal (cf. 4.1), that is $\pi_k H_q(V)/\pi_k z H_q(V)$ and $\pi_k H_q(V)/\pi_k \bar{z} H_q(V)$ are finite. If for instance, the first group is infinite then there exists an element $a \in \pi_k H_q(V)$, $a \neq 0$ with $\langle a, \pi_k z H_q V \rangle = 0$. Then $\langle \bar{z} a, H_q(V) \rangle = 0$ and $a = 0$ (since V is minimal).

From (iv) and Theorem 3.2 it would now follow that im (f) is a self-dual lattice isomorphic (as an isometry structure) to $H_q(V)$.

6.19. Let us prove (i) and (ii) of 6.18. For $a \in H_q(V)$, $a = \pi_1 a + \pi_2 a + \cdots + \pi_\mu a$ and each $\pi_i a$ can be represented by a cycle in V_i, the i-th component of V. Since

$$\partial_i^- Y = t_i \partial_i^+ Y$$

we have

$$\begin{aligned} f(\pi_i a) &= i_*[i_+(\pi_i a) - i_-(\pi_i a)] \\ &= i_* i_+(\pi_i a) - t_i i_* i_+(\pi_i a) \\ &= (1 - t_i) i_* i_+(\pi_i a). \end{aligned}$$

Thus,

$$f(a) = \sum_{i=1}^{\mu} f(\pi_i a) = \sum_{i=1}^{\mu} (1 - t_i) i_* i_+(\pi_i a).$$

Now we can compute

$$\pi_i f(a) = (1 - t_i) i_* i_+ (\pi_i a) = f(\pi_i a),$$

$$zf(a) = \sum_{i=1}^{\mu} i_* i_+ (\pi_i a) = i_* i_+ (a)$$

$$= i_*((i_+ - i_-)(za)) = f(za).$$

Here the definitions of π_i and z (given in 1.3) have been used; in the last stage we have used the identity

$$i_+ (a) = (i_+ - i_-)(za)$$

which defines the operator z in $H_q(V)$, cf. 6.11.

6.20. Let us prove (iii) of 6.18. Assume that $\alpha \in H_q(\tilde{X})$; we want to show that α can be represented in the form

$$\alpha = \sum_{j=1}^{N} g_j \beta_j,$$

where $g_j \in F_\mu$ and $\beta_j \subset i_* H_q(Y)$. Represent α by a cycle c in $\tilde{X}$. There are a finite number of elements $g_j \in F_\mu$, $j = 1, \ldots, N$ such that c lies in

$$\bigcup_{j=1}^{N} g_j Y.$$

Consider g_j as vertices of the graph Γ_μ of F_μ (cf. 6.17) and join g_{j_1} and g_{j_2} by an edge if there is an edge joining g_{j_1} and g_{j_2} in Γ_μ. We shall get a graph T each component of which is a tree. Thus, there is a vertex g_{j_0} in T which is incident to only one edge of T. Let it be the edge joining g_{j_0} with $g_{j_0} t_k^{\varepsilon}$, $\varepsilon = \pm 1$, $k \in \{1, \ldots, \mu\}$. The cycle c determines an element α_1 of

$$H_q\left(\bigcup_{j=1}^{N} g_j Y, \bigcup_{\substack{j=1 \\ i \neq j_0}}^{N} g_j Y\right) \approx H_q(g_{j_0} Y, g_{j_0} \partial_k^{-\varepsilon} Y) \approx H_q(Y, \partial_k^{-\varepsilon} Y).$$

Now, since V is $(q-1)$-connected, α_1 can be represented by an absolute cycle of the form $g_{j_0}c_1$, with c_1 lying in Y. Let $\beta_1 \in H_q(Y)$ denote the homology class of c_1. It follows that $\alpha - g_{j_0}\beta_1$ can be represented by a cycle lying in

$$\bigcup_{\substack{j=1 \\ j \neq j_0}}^{N} g_j Y$$

and so we can obtain (iii) by induction.

6.21. All that remains to be proven is (iv) of 6.18; that is, we have to compute the Blanchfield form on the image of $H_q(Y)$.

Recall the definition of the Blanchfield form, cf. [F2].

Fix a triangulation of X and consider the corresponding equivariant triangulation of $\tilde{X}$ and the simplicial chain complex $C_*(\tilde{X})$. Let X^1 denote the dual triangulation of X and let $C_*(\tilde{X}^1)$ denote the similar chain complex. $C_*(\tilde{X})$ and $C_*(\tilde{X}^1)$ are complices of free finitely generated left Λ-modules, where $\Lambda = \mathbb{Z}[F_\mu]$. There is an intersection pairing, see [M2],

$$C_{q+1}(\tilde{X}^1) \times C_q(\tilde{X}) \to \Lambda, \qquad (\alpha, \beta) \mapsto \alpha \cdot \beta,$$

with the following properties:

(i) bilinear over $\mathbb{Z}$;
(ii) $(g\alpha) \cdot \beta = g(\alpha \cdot \beta)$,
$\alpha \cdot (g\beta) = (\alpha \cdot \beta)g^{-1}$

for $g \in F_\mu$, $\alpha \in C_{q+1}(\tilde{X}^1)$, $\beta \in C_q(\tilde{X})$. Consider the "completed" chain complex

$$C'_*(\tilde{X}^1) = \Gamma \otimes_\Lambda C_*(\tilde{X}^1),$$

where $\Gamma = \mathbb{Z}[[x_1, \ldots, x_\mu]]$ is the ring of formal power series (the completion of Λ with respect to the augmentation ideal). There is an intersection pairing

$$C'_{q+1}(\tilde{X}^1) \otimes C_q(\tilde{X}) \to \Gamma,$$

with similar properties.

Assume that cycles $\alpha \in C_q(\tilde{X}^1)$ and $\beta \in C_q(\tilde{X})$ represent classes $[\alpha], [\beta] \in H_q(\tilde{X})$. As was shown in §5 of [F2], the cycle α is a boundary in $C'_*(\tilde{X}^1)$ and if

$c \in C'_{q+1}(\tilde{X}^1)$ is a chain with $\partial c = \alpha$ then the intersection $c \cdot \beta \in \Gamma$, viewed modulo Λ, is equal to the value of the Blanchfield form,

$$c \cdot \beta = [a, b] \in \Gamma/\Lambda, \qquad a = [\alpha], \qquad b = [\beta] \in H_q(\tilde{X}).$$

We are going now to construct some special chains in Y and in $\tilde{X}$ in order to find the precise form of the chain $c \in C'_{q+1}(\tilde{X}^1)$ with $\partial c = \alpha$.

6.22. LEMMA. *Let*

$$\Delta : H_{q+1}(Y, \partial Y) \to H_q(V)$$

be the composite of $\partial : H_{q+1}(Y, \partial Y) \to H_q(\partial Y)$ *with the identifying map* $\psi_* : H_q(\partial Y) \to H_q(V)$ *(note that* ∂Y *consists of two copies of* V *and* ψ *sends each of them homeomorphically onto* V*). Then*

(1) Δ *is an isomorphism*;

(2) $\langle x, (i_{+*} - i_{-*})(y) \rangle_Y = -\langle \Delta(x), y \rangle_V$
for $x \in H_{q+1}(Y, \partial Y)$ *and* $y \in H_q(V)$, *where* $\langle\ ,\ \rangle_Y$, $\langle\ ,\ \rangle_V$ *denote the intersection forms in* Y *and* V *respectively.*

Proof. (1) follows from (2) and the above-mentioned fact that $i_{+*} - i_{-*} : H_q(V) \to H_q(Y)$ is an isomorphism.

To prove (2), note that $i_+(V) \subset \partial Y, i_-(V) \subset \partial Y$, the orientation of $i_-(V)$ coincides with the orientation of ∂Y, and the orientation of $i_+(V)$ is opposite to the orientation of ∂Y. Thus,

$$\langle x, i_+(y) \rangle_Y = -\langle \partial^+(x), y \rangle_V,$$

$$\langle x, i_-(y) \rangle_Y = \langle \partial^-(x), y \rangle_V,$$

where $\partial^+(x)$ and $\partial^-(x)$ denote the corresponding parts of $\partial(x)$, lying on $i_+(V)$ and $i_-(V)$, respectively. These formulas imply (2). This completes the proof.

6.23. Identify Y with its image under $i : Y \to \tilde{X}$. Identify V with its image under $V \approx \partial^+ Y \xrightarrow{i} \tilde{X}$.

Let v be a q-dimensional cycle in V and let $\pi_i z v$ be a cycle representing $\pi_i z[v] \in H_q(V)$, $i = 1, \ldots, \mu$. According to the definition of z (cf. 6.11), there exists a $(q+1)$-dimensional chain c_v in Y with $\partial c_v \subset \partial Y$ such that

$$\partial c_v = i_+(v) - (i_+ - i_-)(zv).$$

In other words, identifying $i_+(v)$ with v, and $i_+(zv)$ with zv we should identify $i_-(zv)$ with

$$\sum_{i=1}^{\mu} t_i(\pi_i zv)$$

and the equality above gives

$$\partial c_v = v + \sum_{i=1}^{\mu} (t_i - 1)(\pi_i zv)$$

or

$$\partial c_v = v + \sum_{i=1}^{\mu} x_i(\pi_i zv),$$

where $x_i = t_i - 1 \in \Lambda$.

6.24. The chain c_v is a cycle modulo ∂Y and we obviously have

$$\Delta([c_v]) = [v] \in H_q(V),$$

where $[c_v] \in H_{q+1}(Y, \partial Y)$ denotes the corresponding homology class.

6.25. Assume that we are given a homology class $a \in H_q(V)$. For each multi-index $\alpha = (i_1, \ldots, i_s)$ and each number $i \in \{1, \ldots, \mu\}$ define

$$a^i_\alpha = \pi_{i_s} z \pi_{i_{s-1}} z \cdots \pi_{i_1} z \pi_i a \in H_q(V)$$

and let a q-dimensional cycle v^i_α realize a^i_α. Note that

$$a^i_\phi = \pi_i a.$$

According to the construction of 6.23, for each multi-index α and for each $i \in \{1, \ldots, \mu\}$ there is a $(q+1)$-dimensional chain c^i_α in $i(Y) \subset \tilde{X}$ with boundary lying on $i(\partial Y) \subset \tilde{X}$ such that

$$\Delta([c^i_\alpha]) = a^i_\alpha,$$

$$\partial c^i_\alpha = v^i_\alpha + \sum_{j=1}^{\mu} x_j v^i_{\alpha j},$$

where $\alpha j = (i_1, \ldots, i_s, j)$ for $\alpha = (i_1, \ldots, i_s)$.

Denote

$$c = \sum_{\alpha} \sum_{i=1}^{\mu} (-1)^{|\alpha|+1} x_i x^{\alpha} c^i_{\alpha},$$

where α runs over all multi-indices $\alpha = (i_1, \ldots, i_s)$, $|\alpha| = s$, $x^{\alpha} = x_{i_1}, \ldots, x_{i_s}$. It is clear that c is an infinite chain representing an element of $C'_{q+1}(\tilde{X}^1)$. Computing ∂c one gets

$$\begin{aligned}\partial c &= \sum_{\alpha} \sum_{i=1}^{\mu} (-1)^{|\alpha|+1} x_i x^{\alpha} \left[v^i_{\alpha} + \sum_{j=1}^{\mu} v^i_{\alpha j} \right] \\ &= \sum_{\alpha} \sum_{i=1}^{\mu} (-1)^{|\alpha|+1} x_i x^{\alpha} v^i_{\alpha} + \sum_{\alpha} \sum_{i=1}^{\mu} \sum_{j=1}^{\mu} (-1)^{|\alpha|+1} x_i x^{\alpha} x_j v^i_{\alpha j} \\ &= -\sum_{i=1}^{\mu} x_i v^i_{\phi}.\end{aligned}$$

Since v^i_{ϕ} represents the homology class $a^i_{\phi} = \pi_i a$, the cycle

$$-\sum_{i=1}^{\mu} x_i v^i_{\phi}$$

represents $f(a) \in H_q(\tilde{X})$, see 6.19.

If $b \in H_q(V)$ is another homology class represented by a cycle w then the homology class $f(b) \in H_q(\tilde{X})$ is represented by the cycle $i_+(w) - i_-(w)$ lying in $i(Y) \subset \tilde{X}$ and

$$[f(a), f(b)] = c \cdot (i_+(w) - i_-(w)) \mod \Lambda,$$

see 6.21. Thus

$$\begin{aligned}[f(a), f(b)] &= \sum_{\alpha} \sum_{i=1}^{\mu} (-1)^{|\alpha|+1} x_i x^{\alpha} \langle [c^i_{\alpha}], (i_{+*} - i_{-*})(b) \rangle_Y \\ &= \sum_{\alpha} \sum_{i=1}^{\mu} (-1)^{|\alpha|} x_i x^{\alpha} \langle \Delta([c^i_{\alpha}]), b \rangle_V \\ &= \sum_{\alpha} \sum_{i=1}^{\mu} (-1)^{|\alpha|} x_i x^{\alpha} \langle a^i_{\alpha}, b \rangle_V \\ &= \sum_{\substack{(i, \ldots, i_s) \\ s \geq 1}} (-1)^{s-1} x_{i_1} x_{i_2} \cdots x_{i_s} \langle \pi_{i_s} z \pi_{i_{s-1}} x \cdots z \pi_{i_1} a, b \rangle\end{aligned}$$

$$= \sum_{\substack{(i_1,\ldots,i_s)\\ s\geq 1}} x_{i_1}x_{i_2}\cdots x_{i_s}\langle \partial_{i_{s-1}}\partial_{i_{s-1}}\cdots\partial_{i_1}a, \pi_{i_s}b\rangle$$

$$= \sum_{\alpha}\sum_{i=1}^{\mu} x^{\alpha}\langle \partial^{\alpha}a, \pi_i b\rangle x_i \quad (\mathrm{mod}\ \Lambda).$$

This proves formula (iv) in 6.18 and thus completes the proof of Theorem 6.13.

§7. Some remarks

7.1. *Any minimal isometry structure A of multiplicity μ admits an embedding (cf. 6.10) in a $\Lambda = \mathbb{Z}[F_\mu]$-module M of type L supplied with a non-degenerate form*

$$[\,,] : M \times M \to \Gamma/\Lambda.$$

Proof. Assume A is ε-symmetric and let q be an integer with $(-1)^q = \varepsilon$, $q \geq 3$. By Theorem 4.7 of [F2] there exists a $(q-1)$-simple μ-component link $\Sigma^{2q-1} \subset S^{2q+1}$ and a $(q-1)$-connected Seifert manifold V of Σ such that the associated isometry structure

$$(H_q(V), \langle\,,\rangle, z, \pi_1, \ldots, \pi_\mu)$$

(cf. 6.11) is isomorphic to A. Let $\tilde{X}$ be the free covering of the complement of Σ. Consider the Blanchfield form

$$[\,,] : H_q(\tilde{X}) \times H_q(\tilde{X}) \to \Gamma/\Lambda.$$

By Theorem 6.13 the isometry structure $H_q(V)$, which is isomorphic to A, admits an embedding in $H_q(\tilde{X})$.

This completes the proof.

7.2. Let $q \geq 3$ be an integer. *Any $\Lambda = \mathbb{Z}[F_\mu]$-module M of type L supplied with a non-degenerate $(-1)^{q+1}$-hermitian form*

$$M \times M \to \Gamma/\Lambda$$

can be realized as the Blanchfield form

$$H_q(\tilde{X}) \times H_q(\tilde{X}) \to \Gamma/\Lambda$$

of a $(q-1)$-simple $(2q-1)$-dimensional μ-component link Σ in S^{2q-1}.

Proof. Let $\Lambda \subset M$ be a self-dual lattice (cf. 5.6). Consider Λ as an isometry structure and realize it by a Seifert manifold of $(q-1)$-simple μ-component link $\Sigma^{2q-1} \subset S^{2q+1}$ as in the previous remark 7.1. Now, Λ embeds both in M and in $H_q(\tilde{X})$ and by 6.10 there exists a Λ-isomorphism $M \to H_q(\tilde{X})$ preserving the forms. This completes the proof.

A similar realization theorem was proved by Duval [D].

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