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An acyclic extension of the braid group

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Abstract. We relate Artin's braid group $B_\infty = \lim_\rightarrow B_n$ to a certain group F' of *pl*-homeomorphisms of the interval. Namely, there exists a short exact sequence $1 \to B_\infty \to A \to F' \to 1$, where $H_k A = 0$, $k \geq 1$.

1. Introduction

Recent years have seen a growth of interest in the dynamical, combinatorial and homological aspects of groups of *pl*-homeomorphisms of the real line ([BS], [Br], [BrG], [Gh], [GS], [G1]).

In this paper, we link certain of these groups with the Artin braid groups in an algebraic construction which exploits the geometrical bases of the two.

Let F be the group of *pl*-homeomorphisms of $[0, 1]$ whose derivative, which may be undefined at a finite subset of $\mathbf{Z}[\frac{1}{2}]$, is otherwise an integral power of 2, and let F' be the subgroup of elements of F which agree with the identity near 0 and 1. Let B_n be the braid group on n strings and $B_\infty = \lim_\rightarrow B_n$ the usual infinite braid group. Recall that a group is acyclic provided it's homology with trivial coefficients vanishes in positive dimensions.

Our main result is the following:

THEOREM. *There exists an exact sequence* $1 \to B_\infty \to A \to F' \to 1$, *with* A *an acyclic group.*

Our task in this paper is the construction of the group A and the proof of it's acyclicity. We also indicate the homology of related groups, replacing either F' with a group acting on the circle or B_∞ with the group Σ_∞ of finitely supported permutations. The latter group is connected with the Fredholm permutations as studied by J. Wagoner and S. Priddy ([W], [P]).

The construction of A is quite geometrical. It exploits the idea of an action at infinity on a tree. The critical point is to force $H_1(A; \mathbf{Z}) = 0$. The role played here by the second derivative recalls the discretized Thurston cocycle introduced in [GS].

The initial evidence for the theorem is homological.

PROPOSITION ([G1], [GS], [S1]). *There are maps $BF' \to \Omega S^3$, $BB_\infty \to \Omega^2 S^3$ inducing isomorphisms in homology with integer coefficients.*

Thus, the path fibration $\Omega^2 S^3 \to P\Omega S^3 \to \Omega S^3$ suggests the existence of the group A claimed by the theorem. However, as we will show in Section 6, given a fibration $F \to E \to B$ and groups H and K with the homology of F and B, it is not generally possible to build an extension of groups $1 \to H \to G \to K \to 1$ with the homology of that fibration.

It seems that A is a new kind of acyclic group. Well-known examples of such groups include Higman's finitely presented group (see [BDH] and Section 6) as well as various "large" groups: the group of compactly supported homeomorphisms of $\mathbf{R}^n$ [M2], the group of all permutations of an infinite set or the group of continuous automorphisms of an infinite dimensional Hilbert space. The proof of acyclicity of the Higman group uses a Mayer-Vietoris argument, while for the large groups it requires an infinite repetition device due to Mather and Wagoner.

In contrast, in order to prove that A is acyclic we use a different approach involving a fairly delicate delooping argument.

We note that while a basic theorem of Kan and Thurston embeds any group in an acyclic one, our construction embeds the braid group in A as a *normal* subgroup.

This paper is organized as follows. In Section 2 we introduce a technique to build automorphisms of braid groups starting with the action of a group at infinity on a tree. Section 3 contains the definition of A and of some auxiliary monoids. In Section 4 we use the homological properties of B_∞ and F' and a delooping technique to prove the Theorem. Section 5 contains related results for other groups and Section 6 a relevant example.

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2. Trees and Braids

This section starts with some motivating remarks and an overall idea of our approach. We hope, thus, to make the material in the rest of the paper easier to follow.

As already stated in the Introduction, our main goal is to build an acyclic extension $1 \to B_\infty \to A \to F' \to 1$. The group B_∞ being centerless, it is well-known that this comes down to a certain morphism from F' to the group of outer automorphisms Out B_∞.

Unfortunately, there is no natural way to obtain this morphism using the standard description of B_∞.

Instead, we note that F' acts naturally on the group Σ_∞ viewed as finitely supported permutations of the *dyadic* numbers in $]0, 1[$. Indeed, F' is a group of bijections of the dyadics. This suggests to look for some sort of braiding of the above action.

To avoid braiding a dense set, we first place the dyadic numbers as vertices of the binary tree.

A system of generators for the related braid group can be constructed from the edges of the tree. Note that F' does not act on the whole tree.

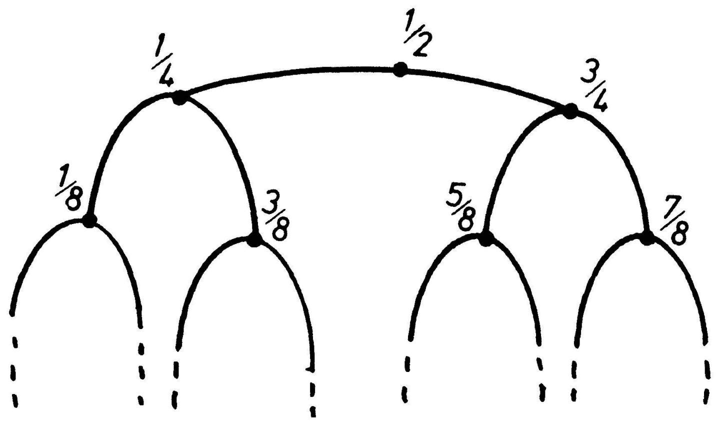


Fig. 2.0

Our basic observation is that nevertheless, each element of F' does act simplicially *outside* a finite subtree. Moreover, this action will *extend* to an automorphism of B_∞, well defined up to inner automorphisms. Thus one gets a morphism from F' to Out B_∞.

We mention that in fact, a slightly more involved construction is needed in order to get the extension A acyclic.

We now proceed to put these remarks in a proper context.

Recalling their relationship with configuration spaces, we define braid groups relative to a discrete set of points in the plane. When this is the set of vertices of a planar tree, the edges provide generators for the corresponding braid group.

We then introduce braid groups associated to cyclically oriented trees. Our setup is appropriate to show that an isomorphism "at infinity" of a tree becomes an outer automorphism of the associated braid group. This fact will be essential for our constructions in the next section.

2.1. DEFINITION. Let S be a discrete, closed subset of $\mathbf{R}^2$. For any open relatively compact, contractible set $D \subset \mathbf{R}^2$, let C_D be the space of injections of the finite set $S \cap D$ into D, modulo the action of the group of permutations of $S \cap D$. Let $B_D = \pi_1 C_D$, and $B_S = \lim_{\rightarrow} B_D$, the limit taken over the directed system of open sets and inclusions. We call B_S the braid group of S.

2.2. *Remarks.* a) The isomorphism class of B_S depends only on the cardinality of S.

b) Consider an embedded arc c in $\mathbf{R}^2$ which intersects S precisely at it's end points.

Pick a contractible neighbourhood D such that $S \cap D$ is the end points of c. Then $B_D \cong \mathbf{Z}$; we let $\underline{c}$ be a generator of B_D given by a counterclockwise exchange of the endpoints of c.

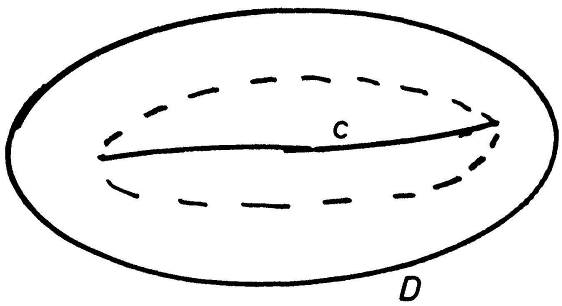


Fig. 2.2

The $\underline{c}$ constitute a "coordinate-free" set of generators of B_S, and satisfy simple relations given by the following propositions, whose proof is left to the reader:

2.3. PROPOSITION (Triangle Rule). *Let S be a discrete closed set of $\mathbf{R}^2$. Let a, b, c be the edges, in clockwise order, of a triangle embedded in $\mathbf{R}^2$. Suppose that a, b, c intersect S precisely at their end points, and the interior of the triangle contains no point of S. Then $\underline{c} = \underline{a}^{-1}\underline{b}\underline{a} = \underline{b}\underline{a}\underline{b}^{-1}$*

We recall some homological facts.

2.4. PROPOSITION. *Let S be a discrete, closed subset of $\mathbf{R}^2$. Then $H_1(B_S) \simeq \mathbf{Z}$, and the map from B_S to $H_1(B_S)$ takes all of the $\underline{c}$ to the same generator.*

It is classical ([CLM] III, App.) that $H_1 B_S \simeq \mathbf{Z}$, generated by some $\underline{c}$. The triangle rule shows that all $\underline{c}$ lie in the same conjugacy class. □

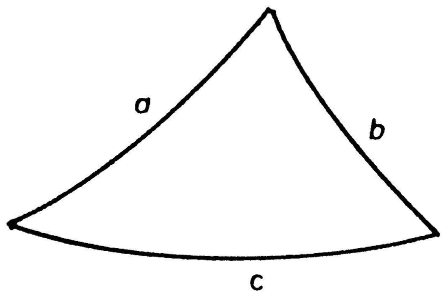


Fig. 2.3

2.5. PROPOSITION. *For any integer $m \geq 0$, there exists a $k > 0$ such that if $S \subseteq S'$ are discrete, closed subsets of $\mathbf{R}^2$ and card $S > k$, then the inclusion $B_S \to B_{S'}$ induces an isomorphism in homology in degrees $\leq m$.*

For S, S' finite, this is ([CLM], III, App]. The general result comes on taking limits. □

In our notation, the usual presentation for the braid group is as B_N, where $N = \{(n, 0) \mid n \in \mathbf{N}\}$; B_N has generators $\underline{e}_i$, $e_i = [i, i+1]$ and relations $\underline{e}_i\underline{e}_{i+1}\underline{e}_i = \underline{e}_{i+1}\underline{e}_i\underline{e}_{i+1}$, $[\underline{e}_i, \underline{e}_j] = 1$ if $|i - j| \geq 2$. One can show that these relations follow from the triangle rule. In general, the edges of any tree in the plane with the set S as vertices provide generators for B_S; the $\underline{c}$, for arcs c which are not edges in the tree, may be easily calculated in terms of the edges using the triangle rule. We proceed to develop this idea.

Let $T = (V, E)$ be a countable, locally finite tree with vertex set V and edges E. We always identify T with it's geometric realization.

2.6. DEFINITION. An orientation of T is an equivalence class of collections $\Phi = \{\varphi_v\}_{v \in V}$ of bijections $\partial_v : E(v) \to \{1, 2 \ldots, \text{card } E(v)\}$, where we set $\Phi \sim \Phi'$ if for all $v \in V$, φ_v and φ'_v differ by a cyclic permutation of $\{1, 2, \ldots, \text{card } E(v)\}$.

An *embedding* $f : T \to \mathbf{R}^2$ is a homeomorphism onto its image, such that $f(V)$ is a discrete, closed subset of $\mathbf{R}^2$. An embedding of a tree determines an orientation of the tree, the clockwise ordering of the edges $E(v)$, $v \in V$. The converse is also true:

2.7. PROPOSITION. *If $T = (V, E)$ is a tree, there is a one-to-one correspondence between orientations and isotopy classes of embeddings to $\mathbf{R}^2$.*

Namely, given an orientation, pick $v \in V$ and define $f(v) = (0, 0) \in \mathbf{R}^2$. The orientation describes how to embed $E(v)$, and there after by induction on subtrees (T being connected) the entire tree, up to an ambient isotopy. □

Let T be an oriented tree. Let $f : T \to \mathbf{R}$ be an orientation preserving embedding. The group $B_{f(V)}$ has $\{\underline{f(e)}, e \in E\}$ as a set of generators. The relations satisfied by the $\underline{f(e)}$ in $B_{f(V)}$ depend only on the isotopy class of f, that is, by 2.7., only on the orientation of T. That is, we can make the following definition.

2.8. DEFINITION. Let T be an oriented tree. Let X be the free group on the set E of edges of T. An orientation preserving embedding $f: T \to \mathbf{R}^2$ induces a surjection $X \to B_{f(V)}$. Define B_T, the braid group of T, to be the quotient of X by the kernel of $X \to B_{f(V)}$, for any orientation preserving f. We write $\underline{f}: B_T \to B_{f(V)}$ for the induced isomorphism.

In [Ser] the relations amongst the generators $e \in E$ which define the group B_T are determined. They are: $[\underline{e}, \underline{e}'] = 1$, if e and e' share no vertex, $\underline{e}\underline{e}'\underline{e} = \underline{e}'\underline{e}\underline{e}'$ if e and e' share a vertex, and $\underline{e}\underline{e}'\underline{e}''\underline{e} = \underline{e}'\underline{e}''\underline{e}\underline{e}' = \underline{e}''\underline{e}\underline{e}'\underline{e}''$ if e, e', e'' share a vertex, and are cyclically oriented.

If $f: T \to \mathbf{R}^2, f': T' \to \mathbf{R}^2$ are orientation preserving embeddings of oriented trees, such that $f(V) = f'(V')$, then $\underline{f}^{-1} \circ \underline{f}'$ defines an isomorphism from $B_{T'}$ to B_T. Our next goal is to define *outer* automorphisms of B_T via embeddings of T in T defined up to a finite subtree. The approach is suggested by a similar construction for permutations due to Wagoner ([W], [P]).

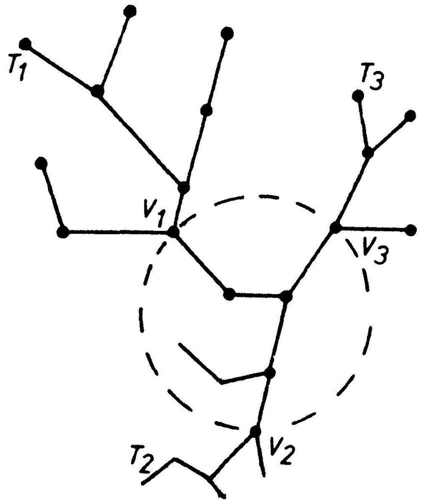
Recall that a forest is a disjoint union of trees. We keep the notation $T = (V, E)$ for a forest. We discuss some aspects of the geometry of forests.

2.9. DEFINITION. Let $T' = (V', E')$ be a subforest of a forest $T = (V, E)$. The complement T'^c of T' is the subforest of T with edges $\{e \in E \mid e \notin E'\}$. Whenever T'^c is finite, T' is called a cofinite subforest. If T' is connected and $T' \cap T'^c$ is a single vertex v, then T' is a rooted subtree whose root is v.

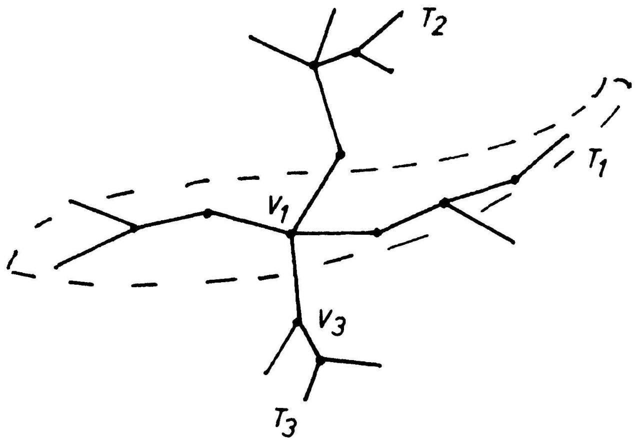
We shall need one more technical notion. Let $T_1, \ldots, T_n$ be rooted infinite disjoint subtrees of an oriented tree T. Let v_i be the root of T_i, $i = 1, \ldots, n$.

2.10. DEFINITION. We say that $T_1 \cup \cdots \cup T_n$ has uncut complement if for some embedding $f: T \to \mathbf{R}^2$ which preserves orientation, there exists an embedding of the closed unit disc $g: D^2 \to \mathbf{R}^2$ such that $g(D^2) \cap T_i = \{v_i\}$. The cyclic counter clockwise order in which the roots v_i occur on $g(S^1) = g(\partial D^2)$ is called the cyclic order of the T_i (we always assume this is $T_1, \ldots, T_n$).

Note that a forest can have a connected complement without having an uncut complement.



T_1, T_2, T_3 in cyclic order



T_1, T_2, T_3 do not have uncut complement

We state a technical lemma:

2.11. LEMMA. (*a*) *Any finite subforest T' of an oriented infinite tree T may be enlarged to a finite subtree $T'' \supseteq T'$ so that T''^c has uncut complement.*

(*b*) *Let T' be a finite subtree of an oriented tree T so that $T'^c = T_1 \cup \cdots \cup T_n$ has uncut complement. Let v_i be the root of $T_i, i = 1, \ldots, n$. Suppose that $f = \amalg f_i : T'^c \to \mathbf{R}^2$ is an embedding, so that each $f_i : T_i \to \mathbf{R}^2$ is orientation preserving. Further, suppose that there is an embedding $g : D^2 \to \mathbf{R}^2$ of the disk, with $g(D^2) \cap f_i(T_i) = \{f_i(v_i)\}$ and such that the $f_i(v_i)$ occur in $g(S^1)$ in the order given by the cyclic ordering of the T_i.*
Then f extends to an orientation preserving embedding $f : T \to \mathbf{R}^2$.

We leave part (a) to the reader, and pass to part (b). Begin with an embedding $h : T \to \mathbf{R}^2$, which we can assume takes T' to $g(D^2)$, and such that $h(v_i) \in g(S^1)$ for each i. By definition of cyclic order, the $h(v_i)$ occurs on $g(S^1)$ in the same cyclic order as the $f_i(v_i)$. Thus we can isotope h, keeping $h(T') \subseteq g(D^2)$, such that $h(v_i) = f_i(v_i)$. Lastly, we isotope h outside $g(D^2)$ until $h = f_i$ on each T_i. □

One has the following key notion.

2.12. DEFINITION. Let T, T' be oriented trees. An oriented Fredholm map φ from T to T' is an isomorphism $\varphi = \{\varphi_i : T_i \to T_i', i = 1, \cdot, n\}$ from a cofinite subforest $T_1 \cup \cdots \cup T_n$ of T with uncut complement to a cofinite subforest $T_1' \cup \cdot \cup T_n'$ of T' with uncut complement. Each φ_i must be orientation preserving, and the cyclic orders of the T_i and T_i' must agree. The index of φ is defined as $\operatorname{ind}(\varphi) = \operatorname{card}\{v \in V, v \notin \cup V_i\} - \operatorname{card}\{v \in V', v \notin \cup V_i'\}$.

The set of oriented Fredholm maps from T to T' is denoted $\mathrm{Fred}^+(T, T')$. We put an equivalence relation on $\mathrm{Fred}^+(T, T')$ by setting $\varphi \sim \varphi'$ if φ and φ' agree on some cofinite subforest. The equivalence classes are called *germs* and form a set $\mathrm{Germ}^+(T, T')$. For a single connected oriented tree T, $\mathrm{Germ}^+(T, T)$ is a group and $\mathrm{ind} : \mathrm{Germ}^+(T, T) \to \mathbf{Z}$ is a homomorphism.

Finally, we show how Fredholm maps produce automorphisms of braid groups.

2.13. PROPOSITION. *Let $T = (V, E)$ and $T' = (V', E')$ be infinite oriented trees, and let $\varphi \in \mathrm{Germ}^+(T, T')$ with index 0. Then there exists an isomorphism $\Phi : B_T \to B_{T'}$ so that $\Phi(\underline{e}) = \underline{\varphi(e)}$ for all but a finite number of edges $e \in E$. Further, Φ is well defined up to composition with an inner automorphism of B_T.*

First we show that Φ exists. Pick $\Psi \in \mathrm{Fred}^+(T, T')$, $\Psi = \{\Psi_i : T_i \to T'_i, i = 1, \ldots, n\}$ in the equivalence class φ. Let $f : T \to \mathbf{R}^2$ be an orientation preserving embedding. Define $f'_i : T'_i \to \mathbf{R}^2$ by $f'_i = f \circ \Phi_i^{-1}$. Because Ψ preserves the cyclic order of the T_i, by 2.11 (b) we can extend f'_i to an embedding $f' : T' \to \mathbf{R}^2$. Since $\mathrm{ind}\, \varphi = 0$, we can choose f' so that $f'(V') = f(V)$. Then define $\Phi = \underline{f'}^{-1}\underline{f}$. Clearly $\Phi(\underline{e}) = \underline{\varphi(e)}$ for any $e \in \cup E_i$.

Suppose that $\Psi : B_T \to B_{T'}$ is an isomorphism and that $\Psi(\underline{e}) = \underline{\varphi(e)}$ for all but a finite number of edges $e \in E$. Then $\Psi^{-1}\Phi : B_T \to B_T$ is the identity on $\underline{e}$ for all but a finite number of edges $e \in E$. Thus there exists some finite subtree $T'' = (V'', E'')$ of T such that $\Psi^{-1}\Phi$ induces an automorphism $B_{T''}$ and $\Psi^{-1}\Phi^{-1}(\underline{e}) = \underline{e}$, $e \notin E''$. By this latter condition and 2.4., $\Psi^{-1}\Phi$ induces the identity on H_1 of $B_{T''}$. The theorem of Dyer and Grossman ([DG], 4) thus implies that $\Psi^{-1}\Phi$, restricted to $B_{T''}$, is an inner automorphism conjugation of some $g \in B_{T''}$. Enlarging T'' if necessary, it follows that after possibly multiplying with an element in the center of $B_{T''}$, g commutes with any $\underline{e}$, $e \notin E''$. Thus $\Psi^{-1}\Phi : B_T \to B_T$ is conjugation by g, q.e.d. □

A mild extension of 2.13 is the following.

2.14. PROPOSITION. *Let T, T' be infinite, oriented trees, and $\varphi \in \mathrm{Germ}^+(T, T')$ with index 0. Let τ, τ' be finite subforests of T, T' respectively, and $\theta : \tau \to \tau'$ an orientation preserving isomorphism of trees. Then there exists an isomorphism $\Phi : B_T \to B_{T'}$ such that $\Phi(\underline{e}) = \underline{\varphi(e)}$ for all but a finite number of $e \in T$, and such that $\Phi(\underline{e}) = \underline{\theta(e)}$ for all $e \in \tau$.*

First, pick Ψ as in 2.13, but so that $\tau \cap T_i = \varnothing$. One can then pick f', as in the proof of 2.13, so that $f'|_{\tau'} = f \circ \theta^{-1}$. □

2.15. COROLLARY. *Let T be an infinite, oriented tree, and $\theta : \tau \to \tau'$ an orientation preserving isomorphism between finite subforests of T. Then there exists an inner automorphism Φ of B_T so that $\Phi(\underline{e}) = \underline{\theta(e)}$ for all $e \in \tau$.*

Take φ to be the identity in 2.14. □

2.16. COROLLARY. *Let T be an infinite, oriented tree, τ a finite subforest, and $\varphi \in \mathrm{Germ}^+(T, T)$ with index 0. Then there exists an automorphism Φ of B_T so that $\Phi(\underline{e}) = \underline{e}$, $e \in \tau$ and $\Phi(\underline{e}) = \underline{\varphi(e)}$ for all but a finite number of edges e of T.*

Take θ to be the identity in 2.14. □

3. Constructions

In this section we perform the main construction of the paper. We will define the group A, to be shown acyclic in Section 4, and a related group A_G. Also, monoids used in the proof of acyclicity of A are constructed. We begin by defining some oriented trees.

3.1. DEFINITION. The trees T_N. For each integer $N \geq 1$, we construct an oriented tree $T_N = (V_N, E_N)$. The set of vertices of T_N is

$$V_N = \left\{ v_n^d;\, n \geq 0,\, d \in \mathbf{Z}\left[\frac{1}{2}\right] \cap (0, N) \right\}$$

and the set of edges of T_N is $E_N = C_N \cup \bigcup_d F_N^d$, the latter union over $d \in \mathbf{Z}[\frac{1}{2}] \cap (0, N)$, where

$$F_N^d = \{e_n^d;\, n \geq 0\} \quad \text{and} \quad \partial e_n^d = \{v_n^d, v_{n+1}^d\}$$

$$C_N = \left\{ e_l^d, e_r^d,\, d \in \mathbf{Z}\left[\frac{1}{2}\right] \cap (0, N) \right\}$$

and

$$\partial e_l^d = \{v_0^d, v_0^{l(d)}\}, \quad \partial e_r^d = \{v_0^d, v_0^{r(d)}\}$$

where if $d = k/2^m$, k odd and $m \geq 1$, or $m = 0$ then $l(d) = (2k-1)/2^{m+1}$, $r(d) = (2k+1)/2^{m+1}$.

One should think of F_N^d as a fiber over d, and the elements of C_N as connecting the fibers. The orientation of T_N is that induced by the following embedding into $\mathbf{R}_+^2 = \{(x, y), y > 0\}$:

For example, the cyclic ordering of the edges $E(v_0^{3/2})$ is as follows: $e_r^1, e_l^{3/2}, e_0^{3/2}, e_r^{3/2}, e_l^2$.

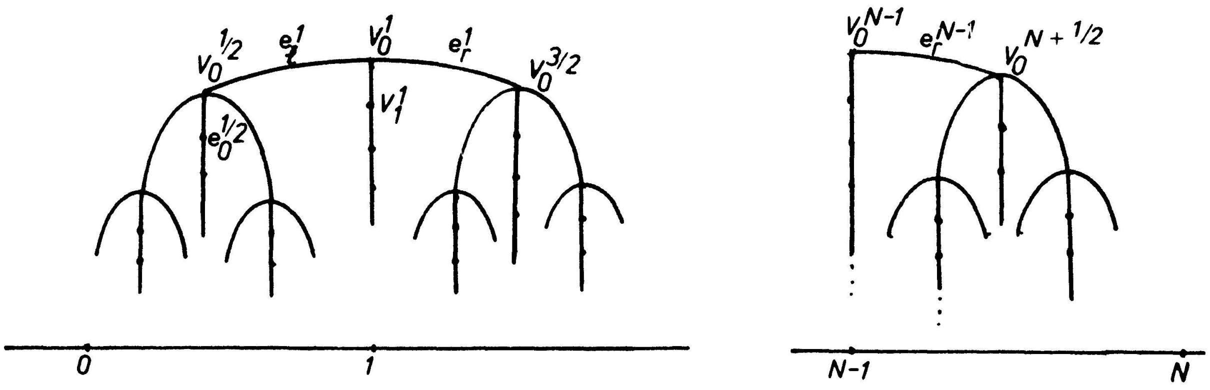


Fig. 3.2

3.3. DEFINITION. T_G is the oriented subtree of T_2 with edges

$$E_G = \bigcup_{d \leq 1} F_2^d \cup \{e_l^d; d \geq 1\} \cup \{e_r^d; d > 1\}.$$

We write $i : T_N \to T_{N+k}$ for the obvious embedding of T_N as a subtree of T_{N+k}. Other useful embeddings are the translation maps $\tau : T_N \to T_{N+k}$ defined by $\tau(v_n^d) = v_n^{d+k}$, $\tau(e_*^d) = e_*^{d+k}$, $(* = r, l, n)$.

3.4. DEFINITION. We write $B_{(N)}$ for the braid group B_{T_N} of the tree T_N, and B_G for the braid group of T_G.

3.5. LEMMA. *The map $H_*(i) : H_*(B_{(N)}) \to H_*(B_{(N+k)})$ is an isomorphism for all N, k and $H_*(i) = H_*(\tau)$.*

The first assertion is a consequence of 2.5. We show that $H_*(i) = H_*(\tau)$. Let $x \in H_*(B_{(N)})$. Then, by definition of $B_{(N)}$, x comes from B_T, for some finite subtree $T \subseteq T_N$. The trees $i(T)$ and $\tau(T)$ are isomorphic subtrees of T_{N+k}. Applying 2.15, there is an inner automorphism of $B_{(N+k)}$ taking $H_*(i)(x)$ to $H_*(\tau)(x)$, □

To define the groups A and A_G, and certain monoids, we make some remarks on the geometry of the trees T_N and T_G.

3.6. DEFINITION. A dyadic interval is an open interval of the form $I = (k/2^n, (k+1)/2^n)$, $n \geq k \in \mathbf{Z}$. If I and J are dyadic intervals, γ_{IJ} denotes the unique element of the dyadic affine group such that $\gamma_{IJ}(J) = I$. If I is a dyadic interval and $N \geq 1$ so that $I \subseteq (0, N)$, let T_I be the subtree of T_N with vertices $V_I = \{v_n^d; d \in I\}$, $E_I = \{e \in E_N, \partial e \subset V_I\}$.

3.7. LEMMA. *Let I and J be dyadic intervals. Then $G_{IJ}(v_n^d) = v_n^{\gamma_{IJ}(d)}$, $G_{IJ}(e_*^d) = e_*^{\gamma_{IJ}(d)}(* = n, r, l)$ defines an isomorphism $G_{IJ} : T_J \to T_I$ of trees.*

Routine verification □

3.8. *Construction of A.*

We now define, for any piecewise affine dyadic homeomorphism $g : [0, n] \to [0, k]$, an element φ_g of $\mathrm{Germ}^+ (T_n, T_k)$. Let M be an integer large enough so that for every dyadic interval of the form $J = (k/2^M, (k+1)/2^M)$, $g|_J$ is affine, and $g(J)$ is a dyadic interval.

3.9. DEFINITION. Let $d \in (0, n)$. Define $g''(d) = \log_2 g^r(d) - \log_2 g^l(d)$, where $g^r(d) = \lim_{\varepsilon \searrow 0} g'(d + \varepsilon)$, $g^l(d) = \lim_{\varepsilon \searrow 0} g'(d - \varepsilon)$.

Let $s_g = \max_{d \in (0,n)} | g''(d) |$. Define a cofinite subtree of T_n to be the union of the T_J, $J = (k/2^M, (k+1)/2^M)$, with subtrees $F^{d'}$ of the fibers F^d, $d = k/2^M$; $F^{d'}$ has edges $\{e_n^d, n \geq s_g + 1\}$. A glance at 3.2 shows that this cofinite subtree has uncut complement.

We define a representative $\tilde{\varphi}_g \in \mathrm{Fred}^+ (T_n, T_k)$ of $\varphi_g \in \mathrm{Germ}^+ (T_n, T_k)$ as follows: $\tilde{\varphi}_g \mid T_J = G_{g(J)J}$, for any $J = (k/2^M, (k+1)/2^M)$. For each $d = k/2^M$, define $\tilde{\varphi}_g(e_n^d) = e_{n+g''(d)}^{g(d)}$, so that $\tilde{\varphi}_g(F^{d'})$ is a subtree of $F^{g(d)}$. It is clear that the image of $\tilde{\varphi}_g$ is a cofinite subtree with uncut complement, and that $\tilde{\varphi}_g$ preserves the cyclic order of the components of the subtree.

3.10. PROPOSITION. *If $g : [0, n] \to [0, k]$, $h : [0, k] \to [0, m]$ are piecewise affine dyadic homeomorphisms, then $\varphi_{hg} = \varphi_h \varphi_g$.*

This follows in a straightforward way from the following two facts. First, if I, J, K are dyadic intervals, then $G_{IJ}G_{JK} = G_{IK}$. Second, the derivative defined in 3.9 satisfies the chain rule: $h''(g(d)) + g''(d) = (hg)''(d)$. □

3.11. PROPOSITION. *Let $g : [0, n] \to [0, k]$ be a piecewise affine dyadic homeomorphism, such that $g^r(0) = g^l(n) = 1$. Then* $\operatorname{ind} \varphi_g = 0$.

Since $\tilde{\varphi}_g$ takes fibers to fibers, $\operatorname{ind} \varphi_g = \Sigma_{d \in (0,n)} g''(d)$. Since $g^r(0) = g^l(n) = 1$, by the "fundamental theorem of calculus" $\Sigma_{d \in (0,n)} g''(d) = 0$, so $\operatorname{ind} \varphi_g = 0$. □

3.12. DEFINITION. A is the group of automorphisms h of $B_{(1)}$, such that there exists a $g \in F'$ such that $h(\underline{e}) = \underline{\varphi_g(e)}$ for all but a finite number of edges $e \in E_1$. We say that h lies over g, and define $\rho : A \to F'$ by $\rho(h) = g$.

3.13. THEOREM. *There is an exact sequence* $1 \to B_{(1)} \to A \xrightarrow{\rho} F' \to 1$.

Definition 3.12 gives $\rho : A \to F'$, which is a homomorphism by 3.10, and surjective by 3.11. The kernel of ρ contains all inner automorphisms of $B_{(1)}$; by definition, $\ker(\rho)$ is the set of all automorphisms of $B_{(1)}$ which fix $\underline{e}$ for almost all $e \in E_1$. By the argument of 2.13, $\ker(\rho)$ consists of exactly the inner automorphisms of $B_{(1)}$. But $B_{(1)}$ has trivial center, and can be identified with its group of inner automorphisms. Hence $\ker(\rho) = B_{(1)}$. □

3.14 PROPOSITION. *The action of* F' *on* $H_* B_{(1)}$, *coming from the exact sequence* $1 \to B_{(1)} \to A \xrightarrow{\rho} F' \to 1$ *is trivial.*

Let $g \in F'$, $x \in H_* B_{(1)}$. Then x is the image in $H_* B_{(1)}$ of an $\bar{x} \in H_* B_T$, for T a finite subtree of T_1. By corollary 2.16, we can pick an $h \in A$ over g such that h fixes T. Thus g fixes x. □

3.15. *Construction of* A_G

We briefly describe the construction of the extension $B_G \to A_G \to G$. Here, G is the group of orientation preserving piecewise affine dyadic homeomorphisms of S^1, thought of as $[0, 1]$ with 0 and 1 identified, see [GS]. As in 3.8, to every $g \in G$ we associate a $\varphi_g \in \text{Germ}^+ (T_G, T_G)$ with ind $\varphi_g = 0$. Then A_G is defined as the group of automorphisms h of the group B_G, such that for some $g \in G$, $h(\underline{e}) = \underline{\varphi_g}(\underline{e})$ for all but a finite number of edges $e \in T_G$. As in 3.13 and 3.14, we obtain an exact sequence $B_G \to A_G \to G$, and G acts trivially on the homology of B_G. Note that we have a commutative diagram, whose vertical arrows are inclusions:

$$\begin{array}{ccccc} B_{(1)} & \to & A & \to & F' \\ \downarrow & & \downarrow & & \downarrow \\ B_G & \to & A_G & \to & G \end{array} \tag{3.16}$$

The proof of acyclicity of A requires the construction of strictly associative topological monoids and continuous homomorphisms $M_B \to M_A \xrightarrow{\rho} M_F$. The construction of M_F is due, in essence, to Quillen ([Q], §.8).

3.17 *Construction of* M_F

M_F is the (thin) geometrical realization $B\mathscr{C}_F$ of a category $\mathscr{C}_F$. The objects of $\mathscr{C}_F$ *are* $\underline{0}, \underline{1}, \underline{2}, \ldots$. The set of morphisms $\mathscr{C}_F(\underline{n}, \underline{k})$ is null if $n = 0$ unless $k = 0$, and

$\mathscr{C}_F(\underline{0}, \underline{0}) = \{id\}$. For $n, k \geq 1$, $\mathscr{C}_F(n, k)$ is the set of piecewise dyadic affine homeomorphisms $g : [0, n] \to [0, k]$ such that $g^r(0) = g^l(n) = 1$. Composition in $\mathscr{C}_F$ is via composition of homeomorphisms. Note that $M_F = B\mathscr{C}_F$ is the union of a point $\underline{0}$ with a $K(F', 1)$, and the inclusion $F' \xrightarrow{=} \mathscr{C}_F(\underline{1}, \underline{1})$ defines a homotopy equivalence $* \amalg BF' \to M_F$.

The product $\mu_F = M_F \times M_F \to M_F$ is defined via a functor, also denoted $\mu_F : \mathscr{C}_F \times \mathscr{C}_F \to \mathscr{C}_F$. On objects, $\mu_F(\underline{k}, \underline{n}) = \underline{k+n}$. If $g \in \mathscr{C}_F(\underline{k}, \underline{n})$ and $g' \in \mathscr{C}_F(\underline{k}', \underline{n}')$, define $\mu_F(g, g') \in \mathscr{C}_F(\underline{k+k'}, \underline{n+n'})$ by

$$\mu_F(g, g')(x) = \begin{cases} g(x) & x \leq k \\ g'(x-k') + n & x \geq k \end{cases}$$

Composing with the canonical homeomorphism $B\mathscr{C}_F \times B\mathscr{C}_F \to B(\mathscr{C}_F \times \mathscr{C}_F)$, we obtain $\mu_F : M_F \times M_F \to M_F$, one checks that μ_F is strictly associative with unit $* = \underline{0}$.

3.18. *Construction of M_B*

M_B is the geometrical realization $B\mathscr{C}_B$ of a category $\mathscr{C}_B$. The objects of $\mathscr{C}_B$ are $\underline{0}, \underline{1}, \ldots$. The set of morphisms $\mathscr{C}_B(\underline{n}, \underline{k})$ is empty unless $n = k$, and $\mathscr{C}_B(\underline{0}, \underline{0}) = \{id\}$. For $n \geq 1, \mathscr{C}_B(\underline{n}, \underline{n})$ is the set of elements of $B_{(n)}$, and composition in $\mathscr{C}_B$ is a composition in the $B_{(n)}$. Thus $M_B = B\mathscr{C}_B = \underline{0} \amalg_{k \geq 1} BB_{(k)}$.

The product $\mu_B : M_B \times M_B \to M_B$ is defined via a function $\mu_B : \mathscr{C}_B \times \mathscr{C}_B \to \mathscr{C}_B$. On objects, $\mu_B(\underline{n}, \underline{k}) = \underline{n+k}$. If $g \in \mathscr{C}_B(\underline{n}, \underline{n})$ and $h \in \mathscr{C}_B(\underline{k}, \underline{k})$ then $\mu_B(g, h) \in \mathscr{C}_B(\underline{n+k}, \underline{n+k})$ is defined by

$$\mu_B(g, h) = i(g)\tau(h)$$

where $i : T_n \to T_{n+k}, \tau : T_k \to T_{n+k}$ are as defined below 3.3. Again, one checks that μ_B is strictly associated, with unit $\underline{0}$.

3.19. *Construction of M_A*

M_A is the geometrical realization $B\mathscr{C}_A$ of a discrete category $\mathscr{C}_A$. The objects of $\mathscr{C}_A$ are $\underline{0}, \underline{1}, \ldots$. The set of morphisms $\mathscr{C}_A(\underline{k}, \underline{n})$ is the set of isomorphisms $h : B_{(k)} \to B_{(n)}$, such that there exists some $g \in \mathscr{C}_F(\underline{k}, \underline{n})$, and $h(\underline{e}) = \underline{\varphi_g(e)}$ for all but a finite number of edges $e \in E_k$ (recall 3.8). $\mathscr{C}_A(\underline{0}, \underline{k})$ is void if $k \neq 0$, and $\mathscr{C}_A(\underline{0}, \underline{0}) = \{\mathrm{id}\}$. Composition in $\mathscr{C}_A$ is via composition of isomorphisms. The isomorphism $A \xrightarrow{=} \mathscr{C}_A(\underline{1}, \underline{1})$ induces a homotopy equivalence $* \amalg BA \to M_A$; we send $*$ to $\underline{0}$.

As before, $\mu_A : M_A \times M_A \to M_A$ is defined via a functor $\mu_A : \mathscr{C}_A \times \mathscr{C}_A \to \mathscr{C}_A$. On objects, $\mu_A(\underline{k}, \underline{n}) = \underline{k+n}$. If $h \in \mathscr{C}_A(\underline{n}, \underline{k}), h' \in \mathscr{C}_A(\underline{n}', \underline{k}')$ lie over $g \in \mathscr{C}_F(\underline{n}, \underline{k}), g' \in \mathscr{C}_F(\underline{n}', \underline{k}')$ respectively, then $\mu_A(h, h') \in \mathscr{C}_A(\underline{n+n'}, \underline{k+k'})$, lying over $\mu_F(g, g')$ is defined on the generators $\underline{e}$, $e \in E_{n+n'}$ by

$$\mu_A(h, h')(\underline{e}^n_m) = \underline{e}^k_m \quad n \geq 0$$

$$\mu_A(h, h')(\underline{e}) = h(\underline{e}) \quad e \in T_n$$

$$\mu_A(h, h')(\underline{\tau e}) = \tau h'(\underline{e}) \quad e \in T_k$$

where $\tau : T_k \to T_{n+k}$ is the translation map defined after 3.3. The above formulae determine $\mu_A(h, h')(\underline{e}^n_l)$ and $\mu_A(h, h')(\underline{e}^n_r)$. One checks that $\mu_A : M_A \times M_A \to M_A$ is strictly associative, with unit $\underline{0}$.

3.20. *Homomorphisms*

The homomorphisms $M_B \to M_A \xrightarrow{p} M_F$ are defined via functors $\mathscr{C}_B \to \mathscr{C}_A \xrightarrow{p} \mathscr{C}_F$. On objects, the functors send $\underline{n}$ to $\underline{n}$. The map $\mathscr{C}_B(\underline{n}, \underline{n}) \to \mathscr{C}_A(\underline{n}, \underline{n})$ is the isomorphism of $B_{(n)}$ with its group of inner automorphisms. The map $\mathscr{C}_A(\underline{n}, \underline{k}) \to \mathscr{C}_F(\underline{n}, \underline{k})$ assigns to an $h : B_{(n)} \to B_{(k)}$ the element of $\mathscr{C}_F(\underline{n}, \underline{k})$ over which it lies. It is straightforward that these functors define homomorphisms of monoids $M_B \to M_A \xrightarrow{p} M_F$.

4. Proof of Acyclicity

We will now prove that A is acyclic, quoting propositions 4.1.–4.3. which will be proven later in this section.

Essentially, 4.1. and 4.2. show us how to "deloop" the sequence $BB_{(1)} \to BA \to BF'$. Acyclicity of A then reduces to proving (4.3.) that $H_1(A; \mathbf{Z}) = 0$. It is here that all details of our construction come into play.

To begin, recall that if M is a (strictly) associative topological monoid, we can construct BM, namely as the realization of the simplicial space

$$* \mathrel{\substack{\longleftarrow \\ \longleftarrow}} M \mathrel{\substack{\longleftarrow \\ \longleftarrow \\ \longleftarrow}} M \times M \cdots$$

and that this construction is functorial for monoids and continuous homomorphisms. The evident map $\Sigma M \to BM$ has as adjoint a map $M \to \Omega BM$ which is not, in general, a homotopy equivalence. We will use the group completion theorem [McS] (see also [D] for a very detailed treatment) to compare the homology of M and ΩBM.

In the previous section we constructed monoids and homomorphisms $M_B \to M_A \xrightarrow{p} M_F$. Therefore we have spaces and maps $BM_B \to BM_A \xrightarrow{Bp} BM_F$, whence $\widetilde{BM}_B \to BM_A \to BM_F$ ($\sim$ denotes universal cover).

4.1. PROPOSITION. (*a*) *There is a map $BA \to \Omega BM_A$ inducing isomorphism in homology.*

(*b*) *$\widetilde{BM}_B \to BM_A \xrightarrow{Bp} BM_F$ is homotopically a fibration. There is a weak homotopy equivalence from $\widetilde{BM}_B$ to the homotopy fiber of Bp.*

4.2. PROPOSITION. *There are weak homotopy equivalences $\Omega S^3 \sim \widetilde{BM}_B$, $BM_F \sim S^3$.*

4.3. PROPOSITION . $H_1 A = 0$.

4.4. *Proof of Acyclicity*

By 4.1. (a) and 4.2, we have, up to homotopy, a fibration $\Omega S^3 \to BM_A \to S^3$. By 4.3. and 4.1. (a) $H_2 BM_A = 0$. This, and an easy Serre cohomology spectral sequence argument, show that BM_A is contractible. Thus ΩBM_A is contractible, and by 4.1. (a), A is acyclic. □

The proof of 4.1. begins with lemmas 4.6. and 4.7. below. Recall from Section 3 the maps $BA \to M_A$, $BF' \to M_F$ induced by the identifications $A = \mathscr{C}_A(\underline{1}, \underline{1})$, and consider the diagram:

$$\begin{tikzcd} BA \arrow[r] \arrow[d, "Bp"] & M_A \arrow[r] \arrow[d] & \Omega BM_A \arrow[d, "\Omega Bp"] \\ BF' \arrow[r] & M_F \arrow[r] & \Omega BM_F \end{tikzcd} \tag{4.5}$$

4.6. LEMMA. *The horizontal compositions in* (4.5.) *induce isomorphisms in homology*:

$$H_* BA \xrightarrow{\simeq} H_*(\Omega BM_A)$$

$$H_* BF' \xrightarrow{\simeq} H_*(\Omega BM_F)$$

We shall use the group completion theorem of [McS]. Recall that $BA \to M_A$, $BF' \to M_F$ induce homotopy equivalences $*\amalg BA \to M_A$, $*\amalg BF' \to M_F$. Thus, for $M = M_A$ or M_F, $\pi_0 M$ is a multiplicatively closed subset of $H_* M$ with two elements (the multiplication in $H_* M$ is induced by the product $M \times M \to M$). We consider the ring $H_*(M)[\pi_0 M^{-1}]$ obtained by inverting $\pi_0 M$ in $H_* M$; one checks easily that $H_* BA = H_*(M_A)[\pi_0 M_A^{-1}]$, $H_* BF' \simeq H_*(M_F)[\pi_0 M_F^{-1}]$. □

The map $H_* M \to H_* \Omega BM$ factors through $H_*(M)[\pi_0 M^{-1}]$, because $\pi_0 \Omega BM$ is a group. The lemma follows if we can show that $H_*(M)[\pi_0 M^{-1}] \to H_*(\Omega BM)$ is an isomorphism for $M = M_A$ and M_F. But this is exactly what the group completion theorem does for us, provided that we show that $\pi_0 M$ is in the center of $H_* M$. Following lines of Quillen ([Q], §8) we provide this for M_A. For M_F, the proof is parallel.

Recall that A may be identified with $\mathscr{C}_A(\underline{1}, \underline{1})$; let A_2 denote the set $\mathscr{C}_A(\underline{2}, \underline{2})$ with the obvious group structure whose composition is composition of automorphisms. Proving that $\pi_0 M_A$ is in the center of $H_* M_A$ comes down to proving that the homomorphisms $L, R : A \to A_2$ defined by

$$L(g) = \mu_A(g, id_{\underline{1}}), \quad R(g) = \mu_A(id_{\underline{1}}, g)$$

induce the same map in homology.

Let A_ε be the subgroup of A consisting of elements $g \in A$ lying over elements of F' whose support is contained in $(\varepsilon, 1-\varepsilon)$ and such that $g(e_*^d) = e_*^d (* = l, r, m)$ if $d < \varepsilon/2$ or $d > 1 - \varepsilon/2$. Clearly, A is the direct limit of the A_ε, so it suffices to show that L and R, restricted to an A_ε, induce the same map on homology. Let $g_\varepsilon \in \mathscr{C}_F(\underline{2}, \underline{2})$ such that g_ε restricted to $(1 + \varepsilon/4, 2 - \varepsilon/4)$ is translation by -1, and pick $h_\varepsilon \in A_2$, h_ε lying over g_ε such that $h_\varepsilon(e_*^d) = e_*^{d-1}$ if $d \in (1 + \varepsilon/4, 2 - \varepsilon/4)$, $* = l, r, m$. Then for any $g \in A_\varepsilon$, $h_\varepsilon R(g) h_\varepsilon^{-1} = L(g)$. Hence R and L induce the same map on $H_* A_\varepsilon$. □

4.7. LEMMA. *There is a map $BB_{(1)} \to \Omega B\widetilde{M}_B$ inducing an isomorphism in homology, so that the square*

$$\begin{array}{ccc} BB_{(1)} & \to & \Omega B\widetilde{M}_B \\ \downarrow & & \downarrow \\ BA & \to & \Omega BM_A \end{array}$$

commutes up to homotopy.

We first apply the group completion theorem of [McS] to prove that $H_*(M_B)[\pi_0 M_B^{-1}] \to H_*(\Omega BM_B)$ is an isomorphism. Thus we need that $\pi_0 M_B \simeq \mathbf{N}$ is in the center of $H_* M_B$. But this is a consequence of 3.5. $\square$

Now $H_*(M_B)[\pi_0 M_B^{-1}] = H_*(\mathbf{Z} \times \lim_\to BB_{(n)})$, where $i : B_{(n)} \to B_{(n+1)}$ is the usual inclusion. Since by 3.5, these inclusions induce isomorphism in homology, and since $BB_{(1)} \to \Omega BM_B$ takes $BB_{(1)}$ to the component $(\Omega BM_B)_1$ of $1 \in \pi_1 BM_B \simeq \mathbf{Z}$, we have an isomorphism in homology $BB_{(1)} \to (\Omega BM_B)_{(1)}$. Composing with a representative of $-1 \in \pi_1 BM_B$ gives a homotopy equivalence $(\Omega BM_B)_1 \to (\Omega BM_B)_0$. Composing with the homeomorphism $(\Omega BM_B)_0 \to \Omega B\widetilde{M}_B$, we obtain

$$\begin{tikzcd} BB_{(1)} \arrow[r] \arrow[d] & (\Omega BM_B)_1 \arrow[r, "o-1"] \arrow[d] & (\Omega BM_B)_0 \arrow[r] & \Omega B\widetilde{M}_B \\ BA \arrow[r] & \Omega BM_A & & \end{tikzcd}$$

which commutes up to homotopy, because $1 = 0$ in $\pi_1 \Omega BM_A$.

4.8. Proof of 4.1.

4.1. (a) is part of lemma 4.6. We pass to the proof of 4.1. (b).

Let $* \in BM_F$ be the canonical basepoint arising from the definition of BM_F as the geometrical realization of the simplicial space $* \leftarrow M_F \leftarrow \cdots$. Let $\mathrm{Fib}_*(Bp)$ be the homotopy fiber [Sp] of $Bp : BM_A \to BM_F$ over $*$. The natural numbers $\mathbf{N}$ are a submonoid of M_F, coming from the objects $\underline{0}, \underline{1}, \ldots$ of $\mathscr{C}_F$, and the image of $B\widetilde{M}_B$ in BM_F is $B\mathbf{N}$. Now $B\mathbf{N}$ is contractible in BM_F. Picking a contraction defines a map $\beta : B\widetilde{M}_B \to \mathrm{Fib}_*(Bp)$. We aim to show that β is a weak equivalence. It suffices to show that $\Omega\beta : (\Omega BM_B)_0, \to \mathrm{Fib}_*(\Omega Bp)$ is a homotopy equivalence; here we identify $\Omega B\widetilde{M}_B = (\Omega BM_B)_0$, and $\Omega\mathrm{Fib}_*(Bp)$ with $\mathrm{Fib}_*(\Omega Bp)$, the homotopy fiber of $\Omega Bp : \Omega BM_A \to \Omega BM_F$, over the constant loop at $* \in BM_F$.

Add a whisker to BF', so that $BF' \to \Omega BM_F$ takes the new basepoint to the constant loop at $*$. As before, denote by $B\rho : BA \to BF'$ the map from BA to (the new) BF'. The whisker gives an obvious homotopy equivalence $BB_{(1)} \to \mathrm{Fib}_*(B\rho)$, and naturality of homotopy fibers gives a map $\mathrm{Fib}_*(B\rho) \to \mathrm{Fib}_*(\Omega Bp)$. Now $\Omega BM_A \xrightarrow{\Omega Bp} \Omega BM_F$ is the + construction [Be] of $B\rho : BA \to BF'$, by 4.6. Further, F' acts trivially on the homology of $B_{(1)}$, by 3.14. Thus by ([Be], 6.4), $\mathrm{Fib}_*(B\rho) \to \mathrm{Fib}_*(\Omega Bp)$ is an isomorphism in homology.

We have a square

$$\begin{array}{ccc} BB(1) & \xrightarrow{\circ -1} & (\Omega BM_B)_0 \\ \downarrow & & \downarrow \Omega\beta \\ \mathrm{Fib}_*(B\rho) & \to & \mathrm{Fib}_*(\Omega Bp) \end{array} \tag{4.9}$$

which, one checks by hand, is homotopy commutative. Further, every arrow, except possibly $\Omega\beta$, induces isomorphism in homology, and so $\Omega\beta$ must as well.

So $\Omega\beta$ is a map between loop spaces, which induces isomorphism in homology. We can apply Whitehead's theorem to see that $\Omega\beta$, and hence β, is a weak equivalence. □

The proof of proposition 4.2 is divided between lemmas 4.10 and 4.12.

4.10. LEMMA. *There is a weak equivalence $S^3 \to BM_F$.*

Let Γ be the pseudogroup of orientation preserving, piecewise affine dyadic homeomorphisms between open subsets of $\mathbf{R}$. In [GS], using techniques of [G1], it is shown tht $B\Gamma \simeq S^3$. Results of [G2] extending a theorem of Mather show that there is a homology equivalence $BF' \to \Omega B\Gamma$, hence $BF' \to \Omega S^3$. But by 4.6, there is a homology isomorphism $BF' \to \Omega BM_F$. Further, $\pi_1 \Omega BM_F = 0$. Thus (see e.g. [Be]) ΩBM_F and ΩS^3 are both the plus construction of BF' with respect to $\pi_1 BF' = F'$, and 5.1. of [Be] implies that ΩBM_F and ΩS^3 are weakly equivalent. □

4.11. LEMMA. *Let X be a space such that ΩX is weakly equivalent to ΩS^3. Then X is weakly equivalent to S^3.*

By the Hurewicz theorem, it suffices to show that X has the homology of S^3. Consider the Serre cohomology spectral sequence of $\Omega X \to PX \to X$, where PX is contractible. Let $a \in H^2\Omega X = E_2^{0,2}$ be a generator. Let $b \in H^3X = \mathbf{Z}$ a generator so that $d_3^{3,0}(b) = a$. A little work with the multiplicative structure shows that $d_3^{3,2n} : E_3^{3,2n} \to E_3^{0,2n+2}$ is an isomorphism for all n. Suppose that for some $k > 3$, $H^kX \neq 0$, and let $y \in H^k(X)$, $y \neq 0$, for smallest such k. Then y must survive to E_∞, a contradiction. □

4.12. LEMMA. *$\widetilde{BM}_B$ is weakly homotopy equivalent to ΩS^3.*

It suffices to show that $BM_B \sim \Omega S^2$, since $\Omega S^2 = S^1 \times \Omega S^3$. Let $M = *\amalg_{k \geq 1} BB_k$ be the disjoint union of the classifying spaces of the finite braid groups, considered as a monoid as in [S1]. We will define a homomorphism $a : M \to M_B$, and prove that Ba is a weak equivalence. Since by [S1] (see also [CLM], III, 3, for an alternative approach), $BM \sim \Omega S^2$, the lemma follows. □

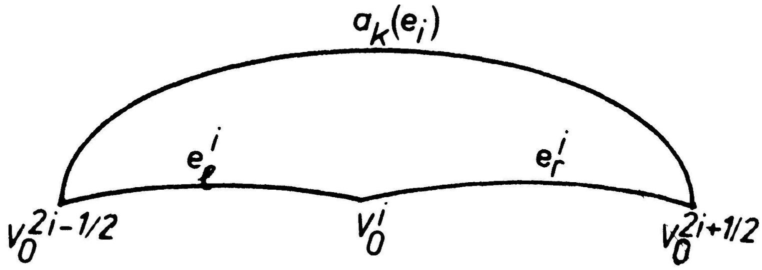


Fig. 4.13

Let $B_k = \langle e_1, \ldots, e_{k-1} \mid [e_i, e_j] = 1, \mid i-j \mid \geq 2, e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} \rangle$ as usual. We define $a : M \to M_B$ by $a(*) = \underline{0}$, and with homomorphisms $a_k : B_k \to B_{(k)}$ defined by "braiding the $v_0^{1/2}, \ldots, v_0^{2k-1/2}$." That is (see Figure 4.13), using the triangle rule we define $a_k(e_i) = (\underline{e}_r^i)^{-1} \underline{e}_0^i \underline{e}_r^i$. One checks that $M \to M_B$ thus defined is a homomorphism of monoids. Applying group completion to both monoids, we obtain a diagram:

$$\begin{array}{ccccc}
H_*(\mathbf{Z} \times \lim_{\rightarrow} BB_k) & \simeq & H_*(M)\,[\pi_0 M^{-1}] & \to & H_* \Omega BM \\
\downarrow & & \downarrow & & \downarrow \\
H_*(\mathbf{Z} \times \lim_{\rightarrow} BB_k) & \simeq & H_*(M_B)[\pi_0 M_B^{-1}] & \to & H_* \Omega BM_B
\end{array}$$

By 2.5, $\lim_{\rightarrow} BB_k \to \lim_{\rightarrow} BB_{(k)}$ induces an isomorphism in homology. Therefore $\Omega BM \to \Omega BM_B$ is a homology isomorphism and a loop map, and hence a weak equivalence, whence $BM \to BM_B$ is a weak equivalence. □

4.14. *Proof of* 4.3.

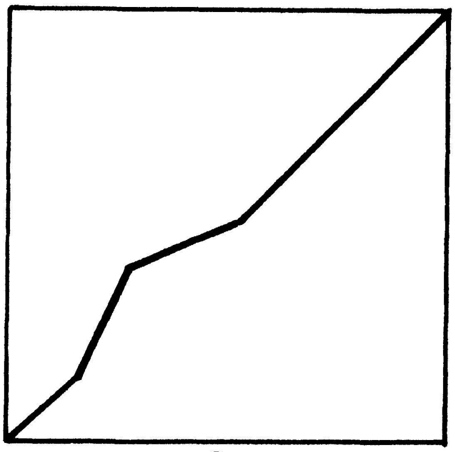
We have an exact sequence $B_{(1)} \to A \to F'$, and F' acts trivially on the homology of $B_{(1)}$. Further, since BF' has the homology of ΩS^3, we know that $H_1 B_{(1)} \simeq H_2 F' \simeq \mathbf{Z}$, $H_1 F' = 0$. Thus, to prove that $H_1 A = 0$ it suffices to show that the differential $d_1 : H_2 F' \to H_1 B_{(1)}$ in the Leray-Serre spectral sequence is an isomorphism. We will explicitly calculate the image in $H_1 B_{(1)}$ of a generator of $H_2 F'$ as follows.

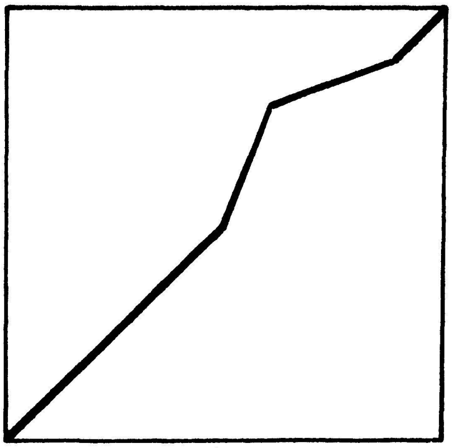
$H_2 F'$ is generated (via Hopf's formula) by the relation $[g, h] = 1$, for $g, h \in F'$ described below. To calculate the image $k \in H_1 B_{(1)}$, we lift g, h to $\tilde{g}$, $\tilde{h} \in A$, and compute the commutator $[\tilde{g}, \tilde{h}]$ which will lie in $B_{(1)}$. The image of the commutator in $H_1 B_{(1)}$ is k. Indeed, we shall see that $[\tilde{g}, \tilde{h}]$, thought of as an element of A, is an inner automorphism of $B_{(1)}$ by an element $\underline{e}$, for a certain edge in $T_{(1)}$. By 2.4, the homology class of $\underline{e}$ generates $H_1 B_{(1)}$.

It is found in in [GS] that the commutator of the following $g, h \in F'$ generates H_2F':

$$g(x) = \begin{cases} x & x \le 1/8 \\ 2x - 1/8 & 1/8 \le x \le 1/4 \\ 1/2x + 1/4 & 1/4 \le x \le 1/2 \\ x & x \geqslant 1/2 \end{cases}$$

$$h(x) = \begin{cases} x & x \leqslant 1/2 \\ 2x - 1/2 & 1/2 \le x \le 5/8 \\ 1/2x + 7/16 & 5/8 \le x \le 7/8 \\ x & 7/8 \le x \end{cases}$$


g


h

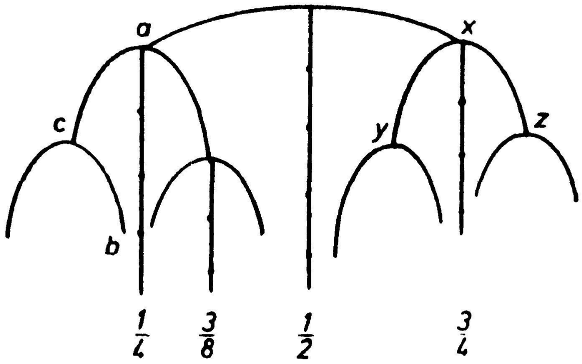
Recall (Figure 4.15(a)) the standard embedding $\sigma : T_1 \to \mathbf{R}^2$. As in the proof of 2.13, $\tilde{g}$ and $\tilde{h}$ are defined via embedings $G, H : T_1 \to \mathbf{R}^2$ (Figure 4.15 (b), (c)) which agrees with $\sigma \circ \varphi_g, \sigma \circ \varphi_h$ near infinity; the $\varphi_{g,}\ \varphi_h$ being defined in 3.8. Namely, $\tilde{h} = \underline{H}^{-1}\underline{\sigma}, \tilde{g} = \underline{G}^{-1}\underline{\sigma}$. Note that, restricted to $T_{(1/2,1)}$, $G \equiv \sigma$. Similarly, $H \equiv \sigma$ restricted to $T_{(0,1/2)}$. Also, $G(e_r^{1/2}) = \sigma(e_r^{1/2})$, $H(e_l^{1/2}) = \sigma(e_l^{1/2})$. It follows that $\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}(\underline{e}) = \underline{e}$, for $e = e_l^{1/2}, e_r^{1/2}, e \in E_{(0,1/2)} \cup E_{(1/2,1)}$. Further for $n \ge 2$, $G(e_n^{1/2}) = H(e_n^{1/2}) = e_{n-1}^{1/2}$. Hence, for $n \ge 3$, $\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}(\underline{e}_n^{1/2}) = \underline{e}_n^{1/2}$. It remains to calculate $\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$ on $\underline{e}_n^{1/2}, n = 0, 1, 2$.

Figure 4.16 is useful for the application of the triangle rule to the calculation of $\tilde{g}^{-1}(\underline{e}_0^{1/2}), \tilde{g}^{-1}(\underline{e}_1^{1/2}), \tilde{h}^{-1}(\underline{e}_0^{1/2}), \tilde{h}^{-1}(\underline{e}_1^{1/2})$. From 4.16 (a), writing $\alpha = (\underline{e}_r^{1/4})^{-1}\underline{e}_l^{1/2}\underline{e}_r^{1/4}$, we have

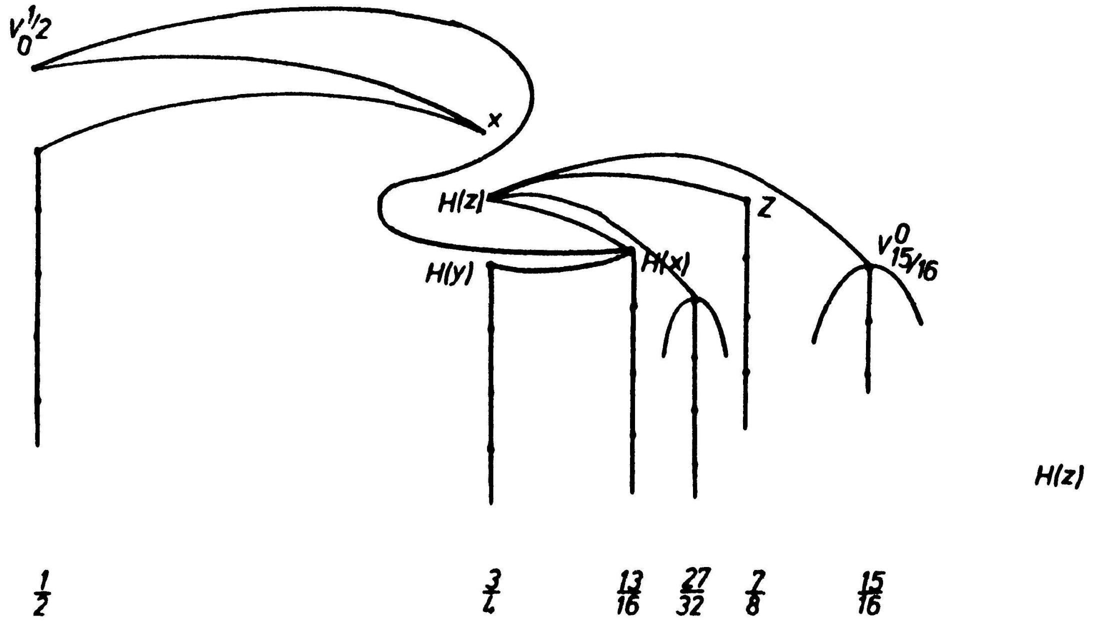
$$\tilde{g}^{-1}(\underline{e}_0^{1/2}) = (\underline{e}_0^{3/8})^{-1}\underline{\alpha}\underline{e}_0^{3/8} \tag{4.17}$$

and it follows that

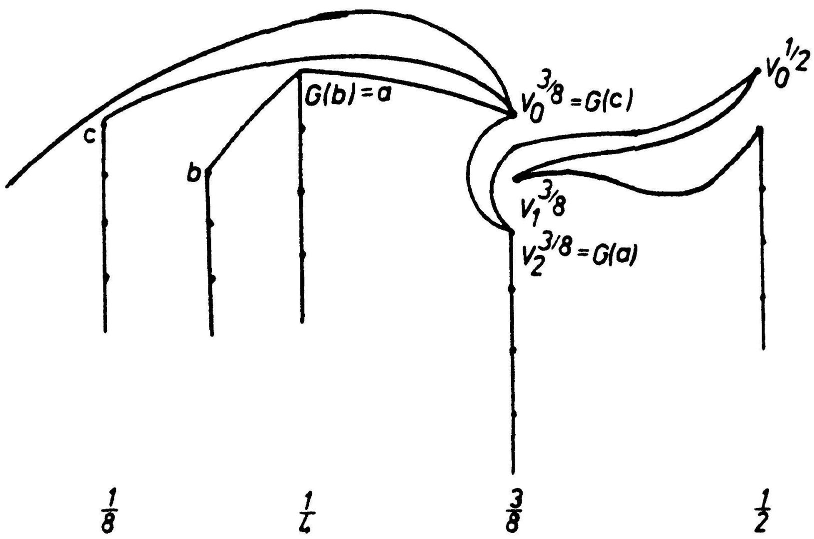
$$\tilde{h}^{-1}(\tilde{g}^{-1}(\underline{e}_0^{1/2})) = \tilde{g}^{-1}(\underline{e}_0^{1/2}.) \tag{4.18}$$



(a) $\sigma : T_1 \to \mathbb{R}^2$ standard embedding.

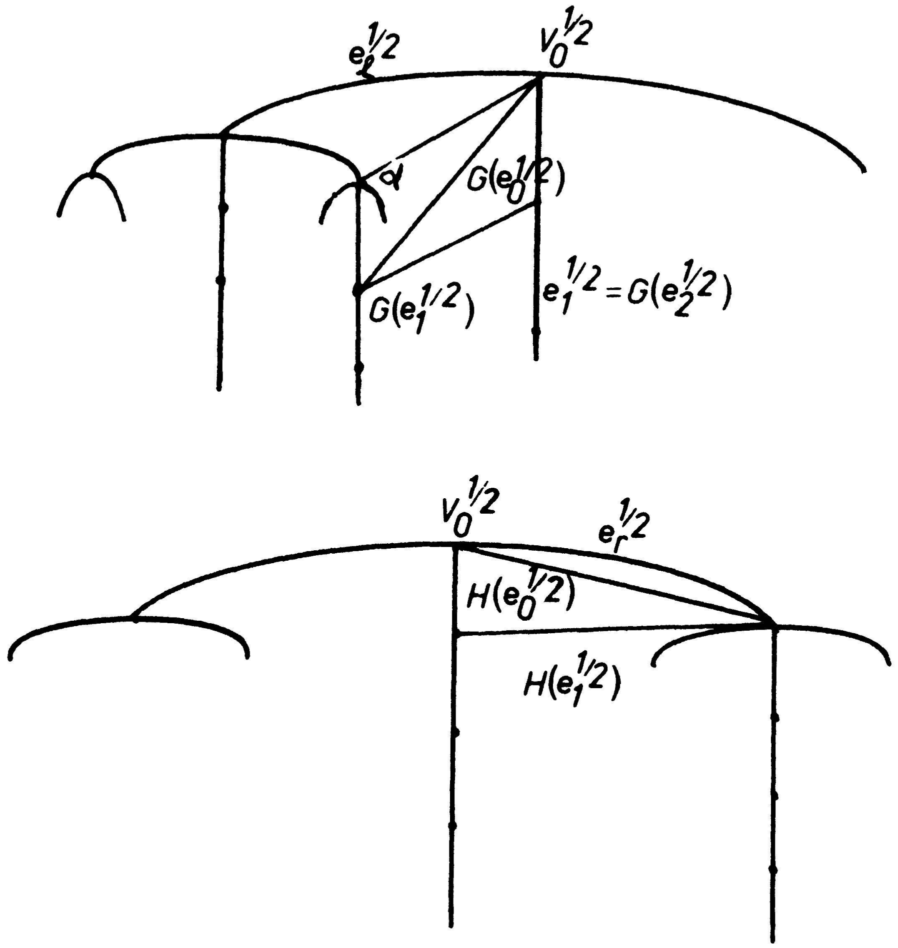


(b) $G : T_1 \to \mathbb{R}^2$ (only G restricted to $T_{(0,\frac{1}{2})}$ pictured)



(c) $H : T_1 \to \mathbb{R}^2$ (only H restricted to $T_{(\frac{1}{2},1)}$ pictured)

Fig. 4.15



(a)

(b)

Fig. 4.16

We also see from 4.16 (a) that

$$\tilde{g}^{-1}(\underline{e}_1^{1/2}) = [\tilde{g}^{-1}(\underline{e}_0^{1/2})]^{-1}\underline{e}_0^{1/2}\tilde{g}^{-1}(\underline{e}_0^{1/2}) \tag{4.19}$$

and consequently

$$\tilde{g}(\underline{e}_1^{1/2}) = (\underline{e}_1^{1/2})^{-1}\underline{e}_0^{1/2}\underline{e}_1^{1/2} \tag{4.20}$$

From 4.16 (b) we see $\tilde{h}^{-1}(\underline{e}_0^{1/2}) = \underline{e}_r^{1/2}$, and

$$\tilde{h}^{-1}(\underline{e}_1^{1/2}) = (\underline{e}_0^{1/2})^{-1}\underline{e}_r^{1/2}\underline{e}_0^{1/2} \tag{4.21}$$

Further, from 4.16 (b), $\underline{e}_0^{1/2} = \tilde{h}^{-1}((\underline{e}_0^{1/2})^{-1})\tilde{h}^{-1}(\underline{e}_1^{1/2})\tilde{h}^{-1}(\underline{e}_0^{1/2})$, and so

$$\tilde{h}(\underline{e}_0^{1/2}) = (\underline{e}_0^{1/2})^{-1}\underline{e}_1^{1/2}\underline{e}_0^{1/2} \tag{4.22}$$

Now, let us calculate. Let $c = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$. Then

$$c(\underline{e}_0^{1/2}) = \tilde{g}\tilde{h}(\underline{e}_r^{1/2}) = \tilde{g}(\underline{e}_0^{1/2}) = (\underline{e}_1^{1/2})^{-1}\underline{e}_0^{1/2}\underline{e}_1^{1/2} \tag{4.23}$$

$$\begin{aligned} c(\underline{e}_2^{1/2}) &= \tilde{g}(\tilde{g}^{-1}(\underline{e}_1^{1/2})^{-1}\underline{e}_1^{1/2}\tilde{g}^{-1}(\underline{e}_1^{1/2})) \\ &= (\underline{e}_1^{1/2})^{-1}\underline{e}_2^{1/2}\underline{e}_1^{1/2} \end{aligned} \tag{4.24}$$

Lastly, $c(\underline{e}_1^{1/2}) = \tilde{g}\tilde{h}\tilde{g}^{-1}((\underline{e}_0^{1/2})^{-1}\underline{e}_r^{1/2}\underline{e}_0^{1/2})$. Using 4.18 and $\tilde{h}(\underline{e}_r^{1/2}) = \underline{e}_0^{1/2}$,

$$\tilde{h}\tilde{g}^{-1}((\underline{e}_0^{1/2})^{-1}\underline{e}_r^{1/2}\underline{e}_0^{1/2}) = \tilde{g}^{-1}(\underline{e}_1^{1/2}),$$

and thus

$$c(\underline{e}_1^{1/2}) = \underline{e}_1^{1/2} \tag{4.25}$$

Now, (4.23) – (4.25) affirm that c is conjugation by $\underline{e}_1^{1/2}$, hence, by 2.4 a generator of $H_1 B_{(1)}$.

5. Related Groups

In this section we describe the homology of two groups closely related to A.

In will be convenient to make use of the plus construction of Quillen ([Q], [Be]). Recall that if X is a space, and $N \subseteq \pi_1 X$ is the maximal perfect subgroup, there exists a space X_+ and a map $X \to X_+$, well defined up to homotopy, such that $\pi_1 X_+ = \pi_1 X/N$, and $X \to X_+$ is an equivalence in homology. We will often invoke the fact ([Be], 6.4)] that if $1 \to H \to G \to K \to 1$ is an exact sequence of groups, such that BH_+ is a nilpotent space and such that $\pi_1 K$ acts trivially on $H_*(H; \mathbf{Z})$, then $BH_+ \to BG_+ \to BK_+$ is a quasifibration.

5.1. *The group A_G*

Recall (3.15) the group A_G which was constructed as an extension $B_G \to A_G \to G$. We will prove:

5.2. PROPOSITION. *The cohomology ring $H^*(A_G; \mathbf{Z})$ is the free graded $\mathbf{Z}$-algebra with generators in dimensions* 2 *and* 3.

We do not know whether BA_{G+} is homotopy equivalent to $S^3 \times \mathbf{C}P^\infty$.

The proof of 5.2 will involve an auxiliary group $A_{\tilde{G}}$. Let $\tilde{G}$ be the group of homeomorphisms of $\mathbf{R} = \tilde{S}^1$ which are lifts of elements of G. Let $B_G \to A_{\tilde{G}} \to \tilde{G}$ and $B_G \to A^{\mathrm{aug}} \to F'$ be the extensions obtained by pullback over the natural maps $F' \to \tilde{G} \to G$. We have the following diagram:

$$\begin{array}{ccccccc} B_{T_1} & \to & B_G & \to & B_G & \to & B_G \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A & \to & A^{\mathrm{aug}} & \to & A_{\tilde{G}} & \to & A_G \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ F' & \to & F' & \to & \tilde{G} & \to & G \end{array} \tag{5.3}$$

The inclusion $B_{T_1} \to B_G$ induces isomorphisms in homology, and thus A^{aug} is an acyclic group.

Let LS^3 denote the space of unbased maps of a circle to S^3, and let $\mathscr{L}S^3 = ES^1 \times_{S^1} LS^3$ denote the homotopy quotient of LS^3 by S^1, acting by reparametrization of loops. One can apply the plus construction to (5.3), obtaining the following diagram commuting up to homotopy, whose vertical arrows are fibrations.

$$\begin{array}{ccccc} \Omega^2 S^3 & \xrightarrow{=} & \Omega^2 S^3 & \xrightarrow{=} & \Omega^2 S^3 \\ \downarrow & & \downarrow & & \downarrow \\ * & \to & BA_{\tilde{G}+} & \to & BA_{G+} \\ \downarrow & & \downarrow & & \downarrow \\ \Omega S^3 & \to & LS^3 & \to & \mathscr{L}S^3 \end{array} \tag{5.4}$$

The map $\Omega S^3 \to LS^3$ has homotopy fiber ΩS^3, because it is simply the inclusion $\Omega S^3 \to S^3 \times \Omega S^3 = LS^3$. Further, it is not hard to see that the plus construction commutes with pullbacks of surjective homomorphisms. We thus obtain a fibration $\Omega S^3 \to * \to BA_{\tilde{G}+}$.

5.5. PROPOSITION. *There is a homotopy equivalence* $S^3 \to BA_{\tilde{G}+}$.

Since $\pi_1 BA_{\tilde{G}+} = 0$, it suffices to show that $BA_{\tilde{G}+}$ has the integral cohomology of S^3. This follows from an easy argument on the cohomology spectral sequence of the fibration $\Omega S^3 \to * \to BA_{\tilde{G}+}$. □

5.6. LEMMA. *The homomorphism* $A_{\tilde{G}} \to \tilde{G}$ *induces an isomorphism* $\mathbf{Z} \simeq H^3(\tilde{G}) \to H^3(A_{\tilde{G}}) \simeq \mathbf{Z}$.

Consider the fibration $\Omega^2 S^3 \to S^3 \to LS^3$, arising, as an application of 5.5, from the middle column of 5.4. The identification of LS^3 with $\Omega S^3 \times S^3$ gives an element of $\pi_3 \Omega S^3 \times \pi_3 S^3 = \mathbf{Z}/2 \times \mathbf{Z}$ which is either 0×1 or 1×1. In either case, the map $H^3(LS^3) \to H^3(S^3)$ is an isomorphism. □

The short exact sequence $\mathbf{Z} \to \tilde{G} \to G$ lifts to a short exact sequence $\mathbf{Z} \to A_{\tilde{G}} \to A_G$. Let $e \in H^2(A_G; \mathbf{Z})$ be the Euler class of this extension, and consider the associated Gysin sequence:

$$\begin{aligned} 0 \to H^3(A_G; \mathbf{Z}) \to H^3(A_{\tilde{G}}; \mathbf{Z}) \xrightarrow{f} H^2(A_G; \mathbf{Z}) \xrightarrow{\cup e} H^4(A_G; \mathbf{Z}) \to \cdots \to H^n(A_{\tilde{G}}; \mathbf{Z}) \\ \to H^{n-1}(A_G; \mathbf{Z}) \xrightarrow{\cup e} H^{n+1}(A_G; \mathbf{Z}) \to H^{n+1}(A_{\tilde{G}}; \mathbf{Z}) \to \cdots \end{aligned} \tag{5.7}$$

Proposition 5.2 follows from the following:

5.8. PROPOSITION. *$H^3(A_{\tilde{G}}; \mathbf{Z}) \to H^2(A_G; \mathbf{Z})$ is the zero map.*

Proof of 5.2. Since $BA_{\tilde{G}+} \simeq S^3$, we have isomorphisms $H^3(A_G; \mathbf{Z} \simeq H^3(A_{\tilde{G}}; \mathbf{Z})$, and $\cup e : H^n(A_G; \mathbf{Z}) \to H^{n+2}(A_G : \mathbf{Z}), n \geq 2$. This implies the proposition; indeed, if $y \in H^3(A_G : \mathbf{Z})$ is a generator, $H^*(A_G : \mathbf{Z}) = \mathbf{Z}[e, y]/(y^2 = 0)$.

Proof of 5.8. Considering the Gysin sequences arising from the extensions $0 \to \mathbf{Z} \to A_{\tilde{G}} \to A_G \to 1$ and $0 \to \mathbf{Z} \to \tilde{G} \to G \to 1$, we obtain a commuting square:

$$\begin{array}{ccc} H^3(A_{\tilde{G}}; \mathbf{Z}) & \xrightarrow{f} & H^2(A_G; \mathbf{Z}) \\ \uparrow & & \uparrow \\ H^3(\tilde{G}; \mathbf{Z}) & \xrightarrow{f} & H^2(G; \mathbf{Z}) \end{array} \tag{5.9}$$

Let $\beta \in H^3(\tilde{G}; \mathbf{Z})$ be a generator, and $\alpha = f(\beta)$. By lemma 5.6, it suffices to show:

5.10. ASSERTION. The image of α in $H^2(A_G; \mathbf{Z})$ is 0.

Consider the differential $d_2 : H_2(G; \mathbf{Z}) \to H_1(B_G; \mathbf{Z}) = \mathbf{Z}$ in the homology spectral sequence of the extension $B_G \to A_G \to G$. It is not hard to see that the kernel of d_2 is $H_2(A_G; \mathbf{Z})$. Let $\gamma \in H^2(G; \mathbf{Z})$ be the cohomology class defined by d_2. We will show that $\gamma = \alpha$, proving 5.10.

Write $\gamma = m\alpha + ne$, where $e \in H^2(G ; \mathbf{Z})$ is the Euler class of the extension. Evaluating both sides of this equality on the image of a generator of $H_2(F'; \mathbf{Z}) \simeq \mathbf{Z}$, and using the computation in the proof of 4.3, we see that $m = 1$. The proof of 5.10 is complete when we show that $n = 0$.

Now G is a group of homeomorphisms of the circle, and contains the cyclic subgroups $\mathbf{Z}/2^r, r \geq 1$ generated by rotations $R_r(x) = x + 2^{-r}, x \in \mathbf{R}/\mathbf{Z}$. Suppose that the inclusion $\mathbf{Z}/2^r \to G$ lifts to A_G. Since $H^2(\mathbf{Z}/2^r; \mathbf{Z}) \simeq \mathbf{Z}/2^r$, generated by the pullbacks of the Euler class $e \in H^2(G; \mathbf{Z})$, one would obtain that $n = 0 \bmod 2^r$, $r \geq 1$, and thus that $n = 0$.

It thus remains to show that the subgroups $\mathbf{Z}/2^r$ lift to A_G. We do this explicitly for $r = 1$; the general case follows similarly, but is more intricate.

The rotation R_1 acts naturally on T_G away from the edges $e_l^{1/2}, e_r^{1/2}$ (see Figure 5.11). We define a lift $\tilde{R}_1 \in A_G$ of R_1 by sending $\underline{e}_l^{1/2}$ to $\underline{\delta}$, and $\underline{e}_r^{1/2}$ to $\underline{\varepsilon}$, where δ and ε are as shown in Figure 5.11.

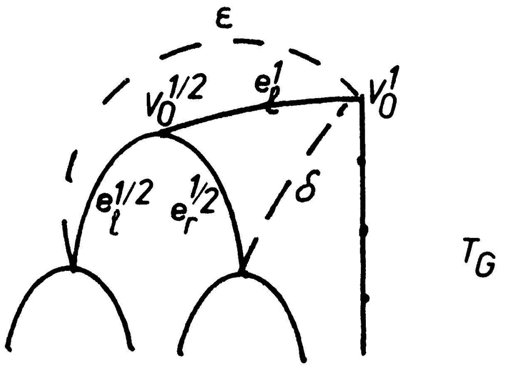


Fig. 5.11

Let us verify that $\tilde{R}_1^2$ is the identity.
This is clear away from $\underline{e}_r^{1/2}$ and $\underline{e}_l^{1/2}$. Using the triangle rule, we find:

$$\tilde{R}_1^2(\underline{e}_r^{1/2}) = \tilde{R}_1(\underline{e}_l^{1/2}\underline{e}_l^1(\underline{e}^{1/2})_l^{-1}) = \underline{\delta e_l^1 \delta}^{-1} = \underline{e}_r^{1/2}$$

$$\tilde{R}_1^2(\underline{e}_l^{1/2}) = \tilde{R}_1((\underline{e}^{1/2})_r^{-1}\underline{e}_l^1\underline{e}_r^{1/2}) = \underline{\varepsilon}^{-1}\underline{e}_l^1\underline{\varepsilon} = e_l^{1/2}$$

So $\tilde{R}_1^2$ is the identity, as claimed.

5.11. *The group A_Σ*

We now introduce a second group related to A. Recall the tree T_1 fudamental in the construction of A, and let $V(T_1)$ be the set of vertices of T_1. Let A_Σ be the set of bijections $\varphi : V(T_1) \to V(T_1)$ such that there is some $g \in F'$, such that $\varphi(v_n^d) = v_{n+g''(d)}^{g(d)}$, $v_n^d \in V(T_1)$ except for a finite number of points.

Clearly, A_Σ surjects to F' with kernel Σ_∞, the group of finitely supported permutations. If we consider A_Σ as embedded in the natural way in $\mathrm{Aut}(\Sigma_\infty)$, we have a map of exact sequences

$$\begin{array}{ccccccccc}
1 & \to & B_\infty & \to & A & \to & F' & \to & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \to & \Sigma_\infty & \to & A_\Sigma & \to & F' & \to & 1
\end{array}$$

Thus the group A_Σ is an analogue of A. In the rest of §5, we shall identify the space $BA_{\Sigma+}$. Two auxiliary groups introduced by Wagoner [W] will be useful.

5.12. DEFINITION. Let P_∞ be the group of bijections φ of $V(T_1)$ such that for some $\varepsilon > 0$, $\varphi(v_n^d) = v_n^d$ for $d < \varepsilon$, $1 - \varepsilon < d$. Set $F'_\infty = P_\infty / \Sigma_\infty$.

5.13. *Identifying the space* $BA_{\Sigma+}$

It is clear that $A_\Sigma \subset P_\infty$ and that this inclusion induces an inclusion $F' \subset F'_\infty$, so that we have a pullback

$$\begin{array}{ccccccccc} 1 & \to & \Sigma_\infty & \to & A_\Sigma & \to & F' & \to & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \to & \Sigma_\infty & \to & P_\infty & \to & F'_\infty & \to & 1 \end{array}$$

Passing to the plus construction, we obtain a pullback of fibrations:

$$\begin{array}{ccccc} B\Sigma_{\infty+} & \to & BA_{\Sigma+} & \to & BF'_+ \\ \| & & \downarrow & & \downarrow \\ B\Sigma_{\infty+} & \to & BP_{\infty+} & \to & BF'_{\infty+} \end{array}$$

As $BP_{\infty+}$ is contractible [W], we see that $BA_{\Sigma+}$ is the homotopy fibre of the map $BF'_+ \to BF'_{\infty+}$. Now, we have already used the fact that $BF'_+ \simeq \widetilde{\Omega S}^3 = \Omega S^2$, and a theorem of Priddy [P] identifies $BF'_{\infty+}$ as $\Omega^{\widetilde{\infty - 1}} S^\infty$. We conclude this section with a sketch of the following:

5.14. *Assertion*

$BA_{\Sigma+}$ is the homotopy fibre of the inclusions $\Omega S^2 \to \Omega^{\infty-1} S^\infty$.

i) Let M_B be the monoid, associated to the braid groups, that we considered in §3, and let M_Σ be its analogue for permutations. The results of Cohen ([C] p. 106–108) imply the existence of a homotopy commutative diagram

$$\begin{array}{ccc} \widetilde{BM}_B & \to & \widetilde{BM}_\Sigma \\ \downarrow \simeq & & \downarrow \simeq \\ \widetilde{\Omega S}^2 & \to & \Omega^{\widetilde{\infty - 1}} S^\infty \end{array} \tag{5.15}$$

whose vertical arrows are homotopy equivalences.

ii) Recall (3.19.) the monoid M_A associated to the group A, and let M_{P_∞} the monoid constructed in the same way for P_∞. Further, let $M_{F'_\infty}$ be the monoid constructed analogously to M_F. As we have seen in (4.1. (b)), we have a quasi-fibration $\widetilde{BM}_B \to BM_A \to BM_F$; the same is true for $\widetilde{BM}_\Sigma \to BM_{P_\infty} \to BM_{F'_\infty}$. The total spaces of these quasifibrations are contractible (cf. (4.4) for BM_A). We therefore have a homotopy commutative square.

$$\begin{array}{ccc} \Omega BM_F & \to & \Omega BM_{F_\infty} \\ \downarrow \simeq & & \downarrow \simeq \\ \widetilde{BM}_B & \to & \widetilde{BM}_\Sigma \end{array} \tag{5.16}$$

whose vertical arrows are homotopy equivalences.

iii) As we have seen (4.6), there is a homology equivalence $BF' \to \Omega BM_F$ and thus a homotopy equivalence $BF'_+ \to \Omega BM_F$. Similarly, we have a homotopy equivalence $BF'_{\infty +} \to \Omega BM_{F_\infty}$. Further, the square

$$\begin{array}{ccc} BF'_+ & \to & BF'_{\infty +} \\ \downarrow \simeq & & \downarrow \simeq \\ \Omega BM_F & \to & \Omega BM_{F_\infty} \end{array} \tag{5.17}$$

is homotopy commutative.

iv) Assembling the diagrams 5.15, 5.16 and 5.17, we can identify the map $BF'_+ \to BF'_{\infty +}$ with the inclusions $\widetilde{\Omega S}^2 \to \Omega^{\widetilde{\infty - 1}} S^\infty$, thus establishing the assertion. □

6. An example

In this section we provide the example referred to in the introduction. We construct a fibration $F \to E \to B$, and groups L and K with the homology of F and B, such that there is no exact sequence $1 \to L \to P \to K \to 1$ so that $BL_+ \to BP_+ \to BK_+$ is equivalent to the original fibration.

6.1. *The idea of the construction*

Start with the fibrations $S^1 \times S^1 \to E \to S^1$ whose monodromy is the involution $(x, y) \to (y, x)$. The exact sequence of fundamental groups $\mathbf{Z} \times \mathbf{Z} \to \pi_1 E \to \mathbf{Z}$ has plus construction the initial fibration. We will enlarge $\mathbf{Z} \times \mathbf{Z}$ to a group L with the same homology, such that the involution does not extend, and this leads to our example.

6.2. *Main construction*

Let $\mathscr{H}$ be Higman's acyclic group [BDH] : $\mathscr{H} = \langle a, b, c, d \mid aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle$. Let α be the automorphism of $\mathscr{H}$ which cyclically permutes a, b, c, d. This automorphism determines an extension $1 \to \mathscr{H} \to H \xrightarrow{\pi} \mathbf{Z} \to 0$ where π is a homology equivalence since $\mathscr{H}$ acyclic, and $[H, H] = \mathscr{H}$.

Let $L = \mathbf{Z} \times H$. Obviously $BL_+ \simeq S^1 \times S^1$. We shall show that there is no automorphism φ of L such that $B\varphi_+ : S^1 \times S^1 \to S^1 \times S^1$ is homotopic to the involution $(x, y) \to (y, x)$. This proves that no exact sequence $1 \to L \to P \to \mathbf{Z} \to 0$ induces a sequence $BL_+ \to BP_+ \to S^1$ equivalent to the fibration $S^1 \times S^1 \to E \to S^1$.

The nonexistence of such a φ is established by the following three claims.

6.3. CLAIM. The automorphism α is not inner. Clearly, α is of order 4. If $\alpha(x) = \omega^{-1}x\omega$, then ω^4 is an element of the center of $\mathscr{H}$. But $\mathscr{H}$ is an iterated amalgamated free product, starting with a centerless torsion free group. The center theorem and torsion theorem for amalgamated products ([MKS], [LS]) show that $\omega = e$, a contradiction.

6.4. CLAIM. No element $y \in H$ such that $\pi(y) = 1$ commutes with $\mathscr{H}$. Indeed, such an element allows us to identify H with $\mathscr{H} \times Z$, contradicting the fact that α is not inner.

6.5. CLAIM. No automorphism φ of L induces on $H_1(L) = \mathbb{Z} \oplus \mathbb{Z}$ the involution $(n, m) \to (m, n)$.

If such a φ exists, then $\varphi(1, e) = (0, y)$, for some $y \in H$ such that $\pi(y) = 1$. Moreover, $\varphi(0, x) = (\pi(x), \chi(x))$ where $\chi : H \to H$ is a morphism such that $y\chi(x) = \chi(x)y$. But $\chi(\mathscr{H}) = \mathscr{H}$ as $\varphi([Z \times H, Z \times H]) = \varphi(0 \times [H, H]) = 0 \times [H, H]$. Thus y and $\mathscr{H}$ commute, contradicting claim 6.4.

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