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On products of soluble groups of finite rank

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1. Introduction

1.1. If H, K are subgroups of a finite group G, then the set HK contains $|H|\,|K|/|H\cap K|$ elements; thus writing $s(X)=\log|X|$ for each finite group X, we have

$$s(G)+s(H\cap K)\geq s(H)+s(K), \qquad (*)$$

with equality if and only if $G=HK$. More generally, suppose that s is a function mapping groups to elements of $\{x;x\in\mathbb{R},x\geq 0\}\cup\{\infty\}$ and suppose that s is constant on isomorphism classes, additive on extensions, and satisfies $s(Y)\leq s(X)$ whenever Y is a subgroup of X. If H, K are subgroups of a group G and K is normal, then

$$H/(H\cap K)\cong HK/K\leq G/K,$$

and so again $(*)$ holds, with equality if $G=HK$. There are many "rank functions" s which have these properties. Our intention here is to consider some of those which are useful in the study of infinite soluble groups, and to investigate to what extent the above conclusions are valid if the requirement that K be a normal subgroup is relaxed.

The rank function that we shall be mainly concerned with is *minimax rank*. The minimax rank $m(X)$ of a group X is the number of infinite factors in a finite series

$$1=X_0\lhd\cdots\lhd X_n=X$$

for X with all factors finite, cyclic or quasicyclic, if such a series exists, and is infinite otherwise. It follows from Schreier's refinement theorem that $m(X)$ is an invariant of X. The groups with finite minimax rank are just the soluble by finite minimax groups, and a description of many of the properties of these groups can be found in Robinson [7], Chapter 10. Polycyclic groups are minimax, and the minimax rank of a polycyclic group X is just its Hirsch number $h(X)$. We shall say that a group G is *almost* the product of its subgroups H, K if the set HK contains a subgroup of finite index in G. We may now state our first result as follows:

THEOREM 1. *Let H, K be subgroups of a soluble by finite minimax group G. Then*

(a) $m(G)+m(H\cap K)\geq m(H)+m(K)$, *and*

(b) *equality holds in* (a) *if and only if G is almost the product of H and K.*

This is not in the same category as the elementary results mentioned earlier: we shall show later that it implies a weak form of Dirichlet's Unit theorem. The fact that the above equality holds if $G=HK$ was proved for the case when H, K are abelian by Zaičev in [10], and for the case when H, K are nilpotent by Amberg and Robinson in [1].

1.2. Theorem 1 yields information about subgroups of a minimax group which permute with each other, because of the simple

LEMMA 1. *For subgroups H, K of a group G the following conditions are equivalent:*

(i) *G is almost the product of H and K;*

(ii) *H has a subgroup H_0 of finite index such that $H_0K=KH_0$ and $|G:H_0K|$ is finite;*

(iii) *K has a subgroup K_0 of finite index such that $HK_0=K_0H$ and $|G:HK_0|$ is finite.*

Proof. Each of (ii), (iii) clearly implies (i). If (i) holds, there is a subgroup $L\lhd G$ with G/L finite and $L\leq HK$. We define $H_0=H\cap LK$; thus LK is a subgroup, and both $|H:H_0|$ and $|G:LK|$ are finite. However

$$LK=LK\cap HK=(LK\cap H)K=H_0K.$$

This and a similar argument show that (i) implies (ii) and (iii).

We may now deduce

COROLLARY 1. *Let H, K be subgroups of a soluble by finite minimax group G such that $HK=KH$, and let H_1 be a subgroup of finite index in H. Then there exists a subgroup H_2 of finite index in H_1 such that $H_2K=KH_2$.*

To prove this, we apply Theorem 1 twice. First we obtain

$$m(HK)+m(H\cap K)=m(H)+m(K);$$

then, since $m(H\cap K)=m(H_1\cap K)$ and $m(H)=m(H_1)$, we deduce that HK is almost the product of H_1 and K, and the result follows from Lemma 1.

It has long been known that every product of two abelian groups is metabelian (Itô [5]), and one suspects that every product of two abelian by finite groups is metabelian by finite. The next corollary provides further evidence for this conjecture.

COROLLARY 2. *Let $G=HK$, where H, K are abelian by finite. If G is soluble by finite and minimax, then G is metabelian by finite.*

Corollary 1 permits us to deduce this from Itô's theorem; it yields an abelian subgroup of H_2 of finite index in H such that $H_2K=KH_2$ and an abelian subgroup K_2 of finite index in K such that $H_2K_2=K_2H_2$. Thus H_2K_2 is a metabelian subgroup, and it has finite index in HK since $|HK:H_2K|\leq|H:H_2|$ and $|H_2K:H_2K_2|\leq|K:K_2|$.

1.3. The other rank functions that we consider are *torsion-free rank* and C_{p^∞}-*rank*. The torsion-free rank $r_0(X)$ of a group X is the number of infinite cyclic factors in a finite series for X each of whose factors is either infinite cyclic or a torsion group; and for each prime p the C_{p^∞}-rank $r_p(X)$ is the number of factors quasicyclic of type C_{p^∞} in a finite series each of whose factors either is of type C_{p^∞} or has no sections of type C_{p^∞}. Of course these are understood to be infinite if no series of the required type exist, and, if finite, they are invariants by Schreier's refinement theorem. It is easy to see that the class of soluble by finite groups having no infinite abelian sections of finite exponent contains the class of soluble by finite minimax groups and is contained in the class of groups of finite torsion-free rank. We shall prove the following two results:

THEOREM 2. *If G is a soluble by finite group having no infinite abelian sections of finite exponent and if G is almost the product of its subgroups H, K, then*

$$r_0(G)+r_0(H\cap K)=r_0(H)+r_0(K).$$

THEOREM 3. *Let G be a soluble by finite minimax group and let $H, K\leq G$. Then for each prime p*

(a) $r_p(G)+r_p(H\cap K)\geq r_p(H)+r_p(K)$, *and*

(b) *equality holds in* (a) *if G is almost the product of H and K.*

If X is a minimax group, then $m(X)=r_0(X)+\sum r_p(X)$, the sum being taken over all primes p, and so Theorem 2 and Theorem 3 provide one of the implications in Theorem 1(b). In conjunction with Theorem 1, they yield a little more information than Theorem 1(b).

COROLLARY 3. *If G is a soluble by finite minimax group and H, K are subgroups of G, then G is almost the product of H and K if and only if*

$$r_p(G)+r_p(H\cap K)=r_p(H)+r_p(K)$$

for $p=0$ and for every prime p.

1.4. Let R be a commutative ring and U a subgroup of its group of units, and write G for the group of matrices

$$\begin{pmatrix} u & b \\ 0 & 1 \end{pmatrix},$$

with $u\in U$, $b\in R$. Let A, H be respectively the group of all upper unitriangular matrices in G and the group of all diagonal matrices in G, and let K be the group of all matrices

$$\begin{pmatrix} u & u-1 \\ 0 & 1 \end{pmatrix},$$

with $u\in U$. Thus A is normal in G and is isomorphic as an abelian group to R, while both H and K are complements to A in G and are isomorphic to U.

Our theorems are proved by first reducing to the case of groups with structure rather like this group G, and similar reductions would apply for any sufficiently well behaved rank functions. It is in the consideration of groups with the above structure that the choice of rank function becomes important, and we can best illustrate this with some examples.

EXAMPLE 1. First we take $R=\mathbb{Z}[1/s]$, where s is the product of distinct primes $p_1,\dots,p_n$, and we take U to be the multiplicative group generated by $p_1,\dots,p_n$. Thus G is minimax and

$$m(G)=m(A)+m(H)=2n+1.$$

However

$$r_0(G)=n+1 \quad \text{and} \quad H\cap K=1,$$

so that

$$r_0(G)+r_0(H\cap K)-r_0(H)-r_0(K)=n+1-2n=1-n.$$

Therefore no inequality holds in Theorem 2 corresponding to the inequalities in Theorem 1(a) and Theorem 3(a).

EXAMPLE 2. More generally, we take for R any ring which is minimax regarded as an abelian group, and we take for U any finitely generated subgroup of its group of units. Then

$$m(G)+m(H\cap K)-m(H)-m(K)=m(A)-m(H)=m(R)-m(U).$$

According to Theorem 1 we therefore have $m(U)\leq m(R)$. This applies in particular if R is the ring of integers of an algebraic number field F, and it shows that no abelian subgroup of the group of units of R has torsion-free rank greater than $\dim_{\mathbb{Q}} F$. This is the weak form of Dirichlet's Unit theorem mentioned earlier. In fact Theorem 3(a) implies that the group of units of R is itself finitely generated. A form of the Unit theorem plays a crucial part in the proof of Theorem 1.

EXAMPLE 3. It is well known that, if H, K are closed subgroups of an arbitrary linear algebraic group G, then the set HK is closed and its dimension is $\dim H+\dim K-\dim(H\cap K)$. Therefore

$$\dim(G)+\dim(H\cap K)\geq \dim(H)+\dim(K),$$

with equality if $G=HK$. For our final example we take G of the type constructed above and we take $R=\mathbb{C}$ and $U=\mathbb{C}\setminus\{0\}$. Then G is a linear algebraic group, and

$$\dim(G)+\dim(H\cap K)-\dim(H)-\dim(K)=0.$$

However G is not almost the product of H and K: if it were, then since A is divisible we would have $A\leq HK$, and an easy calculation shows that the element

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

of A does not lie in HK.

In §2, Theorems 1, 2 and 3 are shown to follow from two results concerning groups G and subgroups H, K such that H, K are complements to an abelian normal subgroup A. These results, Propositions 1 and 2, are proved in §3. In Proposition 1, A is a divisible abelian q-group of finite rank for some prime q, and the proof is straightforward. Proposition 2 lies deeper, and the Unit theorem enters its proof through Lemma 5 in §3.2.

The case of Theorem 1 in which G is polycyclic (and in which minimax length becomes Hirsch number) is of special interest, and its proof is somewhat easier than the general case. Firstly, there are fairly obvious simplifications in the proof that it follows from Proposition 2. The use of Lemma 5 in Proposition 2(a) can be replaced by an appeal to a weak form of the Unit theorem: that if F is an algebraic number field then the group of units of its ring of integers has torsion-free rank less than $\dim_{\mathbb{Q}} F$. Finally, for the proof of Proposition 2(a) in this case, the first paragraph of the proof of Lemma 6 is unnecessary.

2. Reductions

2.1. In Lemmas 2 and 3, s is a function mapping groups to elements of $\{x;\, x\in\mathbb{R},\, x\geq 0\}\cup\{\infty\}$ and G is a group such that $s(G)$ is finite; s is assumed to be constant on isomorphism classes, additive on extensions, and to satisfy $s(Y)\leq s(X)$ whenever Y is a subgroup of X.

LEMMA 2. *Suppose that $H, K\leq G$ and $A\lhd G$, and suppose in addition that one of H, K contains A. Consider the following statements:*

(i) $s(G/A)+s((HA/A)\cap(KA/A))\geq s(HA/A)+s(KA/A)$;

(ii) $s(G)+s(H\cap K)\geq s(H)+s(K)$.

Then (i) *and* (ii) *are equivalent*; *and equality holds in* (i) *if and only if equality holds in* (ii).

This is a straightforward consequence of the modular law.

LEMMA 3. *Suppose that $H, K\leq G$ and $A\lhd G$, and define $H_1=H\cap KA$, $K_1=K\cap HA$ and $G_1=HA\cap KA$. (Thus $G_1=(H\cap KA)A=H_1A$ and $G_1=K_1A$). Consider the following statements* (i)–(iv):

(i) $s(G)+s(HA\cap K)\geq s(HA)+s(K)$;

(ii) $s(HA)+s(KA\cap H)\geq s(HA\cap KA)+s(H)$;

(iii) $s(G_1)+s(H_1\cap K_1)\geq s(H_1)+s(K_1)$;

(iv) $s(G)+s(H\cap K)\geq s(H)+s(K)$.

(a) *If* (i), (ii) *and* (iii) *hold, then so does* (iv).

(b) *If equality holds in* (i), (ii) *and* (iii), *then equality holds in* (iv).

(c) *If* (i), (ii) *and* (iii) *hold and equality holds in* (iv), *then equality holds in* (i), (ii) *and* (iii).

This follows on adding (i), (ii) and (iii).

LEMMA 4. *Let* $H, K \leq G$ *and* $A \lhd G$, *and suppose that* G *is almost the product of* H *and* K. *Write* $H_1 = H \cap KA$, $K_1 = K \cap HA$ *and* $G_1 = HA \cap KA$. *Then*

(a) G *is almost the product of* HA *and* K;

(b) HA *is almost the product of* $HA \cap KA$ *and* H; *and*

(c) G_1 *is almost the product of* H_1 *and* K_1.

Proof. (a) is clear. Let L be a subgroup of finite index in G such that $L \leqslant HK$. Then $|HA : HA \cap L| < \infty$ and

$$HA \cap L \leq HA \cap HK = H(HA \cap K) \leq H(HA \cap KA).$$

Similarly $|(HA \cap KA):(HA \cap KA) \cap L| < \infty$ and

$$(HA \cap KA) \cap L \leq (HA \cap HK) \cap KA = HK_1 \cap KA = (H \cap KA)K_1 = H_1K_1.$$

2.2. We are now ready for the

FIRST REDUCTION STEP. *It is sufficient to prove Theorems* 1, 2, *and* 3 *for groups* G *and subgroups* H, K *satisfying the following additional conditions:*

(i) G *has an abelian normal subgroup* A *such that*

$$G = HA = KA, \qquad H \cap A = K \cap A = 1, \quad \text{and} \quad C_G(A) = A;$$

(ii) *either* A *is a torsion-free group with* $r_0(A)$ *finite on which* H *acts rationally irreducibly by conjugation, or, for some prime* q, A *is a divisible* q*-group with* $r_q(A)$ *finite, all of whose proper* H*-invariant subgroups are finite.*

Proof. We begin with Theorem 2 and argue by induction on $t(G) = r_0(G) + l$, where l is the smallest integer such that G has a series of normal subgroups of length l with each factor finite, torsion-free abelian or torsion abelian. Let N be the first non-trivial term in such a series of length l. The result is certainly true if $r_0(G) = 0$ and so the induction begins. We define A to be a non-trivial torsion-free abelian normal subgroup if such a subgroup exists and to be N otherwise. In the first case we may add to our induction hypothesis the assumption that

Theorem 2 holds for every group having a torsion-free abelian normal subgroup B with $0<r_0(B)<r_0(A)$.

By Lemma 4, the hypotheses of Theorem 2 are satisfied by each of the triples (G, HA, K), $(HA, (HA\cap KA), H)$ and (G_1, H_1, K_1), where G_1, H_1 and K_1 are defined as in the lemma. Using induction on $t(G)$ and working modulo A, we see from Lemma 2 that the first two triples satisfy the conclusion of the theorem. Thus by Lemma 3 it suffices to show that the theorem holds for (G_1, H_1, K_1). Replacing (G, H, K) by this, we may therefore suppose that $G=HA=KA$.

Suppose now that A is a torsion group. We must show that $r_0(H\cap K)=r_0(K)$. For any chain (L_λ) of subgroups we have $r_0(\bigcup L_\lambda)=\max(r_0(L_\lambda))$; thus if $r_0(H\cap K)<r_0(K)$ then Zorn's Lemma yields a subgroup $H_1\geq H$ maximal subject to $r_0(H_1\cap K)<r_0(K)$. Choose $a\in A\setminus H_1$ and let $B=\langle a^G\rangle$, so that $H_1B>H_1$. Because of the hypothesis on elementary abelian sections of G, the subgroup B is finite. It follows that $|H_1B\cap K:H_1\cap K|$ is finite and that $r_0(H_1B\cap K)=r_0(H_1\cap K)$, in contradiction to the choice of H_1.

Suppose instead that A is torsion-free. By Lemma 2 we may pass to factor groups modulo $\bigcap(H^g;\, g\in G)$, and so we may assume that $\bigcap(H^g;\, g\in G)=1$. Thus since $C_H(A)\lhd G$ we have $C_H(A)=1$, so that

$$C_G(A)=A \quad\text{and}\quad H\cap A\leq C_H(A)=1.$$

If $K\cap A>1$ then the result holds by induction and Lemma 2, and so we may assume that $K\cap A=1$. Finally our induction ensures that the result holds if G has an abelian normal subgroup $B\leq A$ with $0<r_0(B)<r_0(A)$, and so we may assume A rationally irreducible as an H-module. This completes our reduction of Theorem 2.

The reduction of Theorem 1 and Theorem 3 is rather similar. In these results G is a minimax group and we argue by induction on $m(G)$. We may assume $m(G)>0$, so that G has an abelian normal subgroup A which is either torsion-free or a divisible q-group for some prime q, and we may add to our induction hypothesis the assumption that Theorem 1 and Theorem 3 hold for every group having an abelian normal subgroup B with $0<m(B)<m(A)$.

We defer the reduction of the proof that if $m(G)+m(H\cap K)=m(H)+m(K)$ then G is almost the product of H and K, and we complete the reduction of the proofs of all the other assertions of Theorem 1 and Theorem 3 simultaneously. If the triple (G, H, K) is such that G is almost the product of H and K, then each of the triples (G, HA, K), $(HA, (HA\cap KA), H)$ and (G_1, H_1, K_1) has the corresponding property by Lemma 4. Thus, after using induction and Lemma 2, and after appealing to Lemma 3, we are left to consider the triple (G_1, H_1, K_1). In other words, we may assume $G=HA=KA$. By Lemma 2 we may assume that

$\bigcap(H^g; g \in G) = 1$, so that $H \cap A = C_H(A) = 1$. Induction and Lemma 2 allow us to assume that $m(K \cap A) = 0$ and that $K \cap A$ is finite. If A is a divisible q-group then only the identity automorphism of A induces the identity automorphism of $A/(K \cap A)$, so that the centralizer in G of $A/(K \cap A)$ is $C_G(A) = A$. Further we have

$$(H(K \cap A)) \cap A = (H \cap A)(K \cap A) = K \cap A.$$

Therefore, if we now pass to the quotient group $G/(K \cap A)$, all conditions (including the conditions that $C_G(A) = A$ and $H \cap A = 1$) are preserved and so we may also assume that $K \cap A = 1$. Finally the induction hypothesis on abelian normal subgroups of G allows us to assume that H acts rationally irreducibly on A if A is torsion-free and that A has no proper infinite H-invariant subgroups if A is a divisible q-group.

Theargument for the remaining assertion of Theorem 1 is slightly different. The hypothesis on (G, H, K) is that

$$m(G) + m(H \cap K) = m(H) + m(K).$$

Since Theorem 1(a) has already been proved for groups $\tilde{G}$ with $m(\tilde{G}) \leq m(G)$, Lemma 3(c) shows that the triple (G_1, H_1, K_1) inherits this hypothesis. If the conclusion holds for this triple then some subgroup of finite index in G_1 lies in H_1K_1, and so for some n we have $A^n \leq H_1K_1$. However, applying Lemma 3(c) again with A^n in place of A, we find that (G, HA^n, K) inherits the hypothesis. By Lemma 2, so does $(G/A^n, HA^n/A^n, KA^n/A^n)$; thus by induction some subgroup of finite index in G lies in $(HA^n)(KA^n)$. Since

$$(HA^n)(KA^n) = HA^nK \leq HH_1K_1K = HK,$$

the result follows.

Thus we must investigate the triple (G_1, H_1, K_1), and we replace (G, H, K) by this, so that we have $G = HA = KA$. Suppose that $N \lhd G$ and N is contained in H or K. By Lemma 2 the hypothesis is satisfied by $(G/N, HN/N, KN/N)$, and if the conclusion holds for this triple then it clearly holds for (G, H, K). This observation permits us to follow the argument above. First we may assume that $\bigcap(H^g; g \in G) = 1$, so that $A \cap H = C_H(A) = 1$. Next, by induction we may assume that $K \cap A$ is finite, and passing to $G/(K \cap A)$ we also ensure that $K \cap A = 1$. Finally, an appeal to the hypothesis on abelian normal subgroups completes the reduction.

2.3. The following two Propositions will be proved in §3:

PROPOSITION 1. *Let $G = HA = KA$, where A is a non-trivial divisible abelian normal q-subgroup of finite rank, and where*

$$H \cap A = K \cap A = 1 \quad \text{and} \quad C_G(A) = A.$$

Suppose that H is abelian by finite and has finite torsion-free rank, and suppose that all proper H-invariant subgroups of A are finite. Then $\langle H, K\rangle \cap A$ is finite. In particular

(a) *all of the indices $|\langle H, K\rangle : H|$, $|\langle H, K\rangle : K|$, $|H : H \cap K|$ and $|K : H \cap K|$ are finite, and $|G : \langle H, K\rangle|$ is infinite;*

(b) *G is not almost the product of H and K.*

PROPOSITION 2. *Let $G = HA = KA$, where A is a non-trivial torsion-free abelian normal subgroup of finite torsion-free rank, and where*

$$H \cap A = K \cap A = 1 \quad \text{and} \quad C_G(A) = A.$$

Suppose that H is abelian by finite and has finite torsion-free rank, and suppose that H acts rationally irreducibly on A. Then

(a) *$r_p(G) + r_p(H \cap K) \geq r_p(H) + r_p(K)$ for each prime p, and moreover $m(G) + m(H \cap K) > m(H) + m(K)$ if $m(G) < \infty$;*

(b) *G is not almost the product of H and K.*

SECOND REDUCTION STEP. *Theorems* 1, 2 *and* 3 *follow Propositions* 1 *and* 2.

First we note that in Proposition 1 and 2 the hypothesis that H is abelian by finite is implied by the other hypotheses and the assumption that H is soluble by finite. To see this we use the well known fact that soluble by finite irreducible linear groups are abelian by finite (cf. Wehrfritz [9], Corollary 3.4 supplemented by Theorems 1.7 and 1.19). In the case of Proposition 2, H acts on faithfully and irreducibly on the $\mathbb{Q}$-vector space $A \otimes_{\mathbb{Z}} \mathbb{Q}$. For Proposition 1 it is most convenient to use duality (see Hartley [4]): $A^* = \mathrm{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is a free $\mathbb{Z}_p$-module of finite rank on which H acts faithfully according to the rule $a(fh) = (ah^{-1})f$, for $a \in A$, $f \in A^*$, $h \in H$; moreover $A^* \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is an irreducible $\mathbb{Q}_p H$-module by Lemma 2.1 of [4].

It follows that in the cases of Theorem 1, 2 and 3 that remain to be studied, the hypotheses of either Proposition 1 or Proposition 2 are satisfied. Propositions 1(a) and 2(a) give the inequalities required in Theorem 1(a) and Theorem 3(a). The remaining implications of Theorems 1, 2 and 3 hold vacuously in these cases:

Propositions 1(a) and 2(a) show that equality cannot arise in Theorem 1(b) while Propositions 1(b) and 2(b) show that G cannot be almost the product of H and K.

3. Proof of Proposition 1 and Proposition 2

3.1. First we give the rather elementary proof of Proposition 1.

We suppose the first assertion of Proposition 1 false; thus $\langle H, K\rangle \cap A$ is an infinite H-invariant subgroup of A, and so $\langle H, K\rangle \cap A = A$ and $\langle H, K\rangle = G$. Since H is abelian by finite and $r_0(H)$ is finite, there is a finitely generated normal subgroup $H_1 = \langle h_1, \ldots, h_r\rangle$ of H such that H/H_1 is periodic. For each i, write $h_i = a_i k_i$ with $a_i \in A$ and $k_i \in K$. The normal subgroup A_1 of G generated by $a_1, \ldots, a_r$ is finite. We have $H_1 \leq A_1 K$, and therefore H_1 has a subgroup of finite index, which may be chosen characteristic in H_1, which lies in K. Replacing H_1 by this, we have $H_1 \leq K$ and we still have $H_1 \lhd H$. Thus

$$AH_1 \lhd AH = G \quad \text{and} \quad H_1 = AH_1 \cap K \lhd K.$$

Since $\langle H, K\rangle = G$ we conclude that $H_1 \lhd G$. However $H_1 \cap A = 1$ and $C_G(A) = A$, and therefore $H_1 = 1$.

It follows that H is a torsion group. Since torsion subgroups of Aut A are finite (see Robinson [7], Corollary to Lemma 3.28), H is therefore finite, and $G = AH$ is locally finite. However

$$K \cong AK/A = AH/A \cong H,$$

so that K is also finite, and G is finitely generated. The resulting contradiction completes the proof of the first assertion of the proposition.

The remaining assertions now follow immediately: we have for example

$$|\langle H, K\rangle : H| = |\langle H, K\rangle A : HA|\,|\langle H, K\rangle \cap A : H \cap A| = |\langle H, K\rangle \cap A : H \cap A| < \infty$$

and $|K : H \cap K| \leq |\langle H, K\rangle : H|$. Clearly G is not almost the product of H and K because H and K fail to generate a subgroup of finite index in G.

3.2. Next we turn to the proof of Proposition 2(a). The crucial information about ranks is given by the following lemma, the first assertion of which is well known.

LEMMA 5. *Let A be a torsion-free abelian group of finite torsion-free rank and* *let H be an abelian group which acts faithfully and rationally irreducibly on A.*

(a) *If $r_0(H)$ is finite, then H is finitely generated.*

(b) *If further A is a minimax group, then $r_0(H) < m(A)$.*

Proof. The action of H on A makes $V = A \otimes_{\mathbb{Z}} \mathbb{Q}$ an irreducible $\mathbb{Q}H$-module, and the centralizer ring $F = \mathrm{End}_{\mathbb{Q}H} V$ is a division ring by Schur's Lemma. Because H is abelian, its image $\hat{H}$ in $\mathrm{End}_{\mathbb{Q}} V$ lies in and spans F, so that F is an algebraic number field and $\dim_F V = 1$. Since $H \cong \hat{H}$, assertion (a) now follows from the theorem of Skolem [8] that the multiplicative group of an algebraic number field is a direct product of a finite cyclic group and a free abelian group. However we shall give a proof of (a) since the arguments are needed in the proof of (b). We shall use the Unit theorem (see for example Cassels [3], p. 72) together with some facts about valuation rings and Dedekind rings; a convenient reference is Bourbaki [2], Chapters VI and VII.

Since $\hat{H}$ has finite torsion-free rank, it has a finitely generated subgroup $L = \langle h_1, \ldots, h_m \rangle$ such that $\hat{H}/L$ is a torsion-group. Each element of F lies in all but finitely many of the valuation rings of F (see [2], proof of Proposition 12 on p. 487). Let $V_1, \ldots, V_k$ be the valuation rings not containing the set $\{h_1, h_1^{-1}, h_2, h_2^{-1}, \ldots, h_m, h_m^{-1}\}$, let S_1 be the family of non-archimedean absolute values on F corresponding to $V_1, \ldots, V_k$, and let S be the union of S_1 and the set of archimedean absolute values on F. Thus if V is a valuation ring of F and $V \notin \{V_1, \ldots, V_k\}$ then the group of units U of V contains L; indeed, since V is integrally closed, all roots of elements of L lie in U, and so $\hat{H} \leq U$. It follows that $\hat{H}$ is contained in the group of S-units of F, so that the Unit theorem may be applied: it yields that $\hat{H}$ is a direct product of a finite cyclic group and a free abelian group of rank at most $|S| - 1$, and (a) follows.

To prove (b) we must show that $|S| \leq m(A)$. First we note that the number of archimedian absolute values on F is $r + s$, where r is the number of real embeddings of F and s is the number of (pairs of conjugate) complex embeddings; and since

$$r + 2s = \dim_{\mathbb{Q}} F = r_0(A),$$

we certainly have $|S| - |S_1| \leq r_0(A)$. It will therefore suffice to show that $|S_1| \leq m(A) - r_0(A)$.

Let M be the subring of F generated by $\hat{H}$. If $a \in A \setminus 0$ then the map $x \mapsto ax$ from M to A is injective; thus M is minimax and $m(M) \leq m(A)$. Let I be the ring of integers of F and let N be the subring generated by I and M. Since $|I : M \cap I|$ is finite we have $eI \leq M$ for some integer $e > 0$, and therefore $eN \leq M$. It follows that N/M is a finite group and that $m(N) \leq m(A)$; and because

$$m(I) = r_0(I) = r_0(A)$$

the result will follow if we prove that $|S_1| \leq m(N/I)$.

Since I is noetherian and the ring N is generated by I and $\hat{H}$, N is noetherian. Let P be a maximal ideal of N; then $I \cap P$ is a non-zero prime ideal of I and the localization $I_{I \cap P}$ of I at $I \cap P$ is a discrete valuation ring (for example by Theorem 1 on p. 494 of [2]). However

$$I_{I \cap P} \leq N_P \leq F,$$

and because $N_P \neq F$ we have $N_P = I_{I \cap P}$. It follows, again from Theorem 1 on p. 494 of [2], that N is a Dedekind domain.

Now each integrally closed subring of F is the intersection of the valuation rings which contain it ([2], Theorem 3 on p. 378); since I lies in all valuation rings of F it follows that N is the intersection of all valuation rings of F apart from $V_1, \ldots, V_k$ and that $I = N \cap V_1 \cap \cdots \cap V_k$. We define

$$W_i = N \cap \bigcap_{j \neq i} V_j \quad \text{for} \quad i \leqslant k.$$

Each W_i/I is a subgroup of N/I and clearly the sum of these subgroups is their direct sum. By Proposition 2 on p. 497 of [2] if $f \notin V_i$ there are elements $x \in W_i$ with $x - f \in V_i$. Thus each W_i/I is non-trivial, and so is infinite since I is integrally closed. It follows that $m(N/I) \geq k = |S_1|$, and the proof of Lemma 5 is complete.

It is now an easy matter to prove Proposition 2(a). Let G, H, K and A be as in the statement of Proposition 2, and let H_0 be an abelian normal subgroup of finite index in H. Since H acts rationally irreducibly on A, the tensor product $A \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}H$-module, and so by Clifford's theorem it is a direct sum of irreducible $\mathbb{Q}H_0$-submodules. The intersections $B_1, \ldots, B_n$ of these with A are acted on rationally irreducibly by H_0, and the quotient of A by their product is a torsion-group. For $i = 1, \ldots, n$ let $C_i = C_{H_0}(B_i)$. Thus

$$\bigcap_{i=1}^{n} C_i = C_{H_0}(B_1 \times \cdots \times B_n) = C_{H_0}(A) = 1,$$

so that H_0 may be embedded in $(H_0/C_1) \times \cdots \times (H_0/C_n)$.

We now apply Lemma 5, regarding B_i as an (H_0/C_i)-module for each i.

Assertion (a) shows that each H_0/C_i is finitely generated. Therefore H_0 is finitely generated, so that $r_p(H)=0$ for each prime p. Since $K \cong H$ we also have $r_p(K)=0$ and so the assertion of Proposition 2(a) concerning C_{p^∞}-rank follows. Moreover for each i we have $r_0(H_0/C_i)<m(B_i)$, so that

$$\begin{aligned} r_0(H) = r_0(H_0) &\le r_0((H_0/C_1)\times\cdots\times(H_0/C_n)) \\ &= \sum r_0(H_0/C_i) < \sum m(B_i) \\ &= m(B_1\times\cdots\times B_n) \le m(A). \end{aligned}$$

Thus

$$\begin{aligned} &m(G)+m(H\cap K)-m(H)-m(K) \\ &\quad = m(A)+m(K)+m(H\cap K)-m(H)-m(K) \\ &\quad \ge m(A)-m(H)>0. \end{aligned}$$

This concludes the proof of Proposition 2(a).

3.3. Finally we must prove Proposition 2(b). The following lemma is essentially the special case in which H is abelian.

LEMMA 6. *Let H be a finitely generated abelian group and let A be a torsion-free abelian group of finite rank on which H acts rationally irreducibly and non-trivially. If $\delta : H \to A$ is a non-zero derivation and $a_1, \ldots, a_m$ are finitely many elements of A then $\{x\delta + a_i x;\ x \in H,\ i = 1, \ldots, m\}$ is a proper subset of A.*

Proof. We suppose the result false. If B is a $\mathbb{Z}H$-submodule containing $x\delta$ and $y\delta$ with $x, y \in H$, then

$$(xy^{-1})\delta = (x\delta - y\delta)y^{-1} \in B;$$

thus if $H = \langle h_1, \ldots, h_s\rangle$ and if we define $a_{i+m} = h_i\delta$ for $i = 1, \ldots, s$, then the submodule generated by $a_1, \ldots, a_{m+s}$ contains $x\delta$ for each $x \in H$ and so equals A. Let $E = \langle e_1, \ldots, e_r\rangle$ be a free abelian subgroup of A of rank $r = r_0(A)$. There are equations

$$a_i = \sum_{j=1}^{r} \lambda_{ij} e_j \quad (i \le m+s)$$

and

$$e_i h_k^{\varepsilon} = \sum_{j=1}^{r} \mu_{ijk\varepsilon} e_j \quad (i \le r, \quad k \le s, \quad \varepsilon \in \{0, 1\}),$$

with rational coefficients λ_{ij} and $\mu_{ijk\varepsilon}$. Writing R for the subring of $\mathbb{Q}$ generated by the λ_{ij} and $\mu_{ijk\varepsilon}$, we see that $E \otimes_{\mathbb{Z}} R$ is a $\mathbb{Z}H$-module containing A. However $E \otimes_{\mathbb{Z}} R$ is free as an R-module, and it follows easily from this that for each $a \in A \setminus 0$ there are only finitely many primes p for which $p^{-1}a \in A$.

Choose $y \in H \setminus C_H(A)$. Since H is abelian, the map $a \mapsto a(y-1)$ is a $\mathbb{Z}H$-module endomorphism of A, and since H acts rationally irreducibly on A, this map is injective and $A/A(y-1)$ is a torsion group. It follows that

$$n(y\delta) = a_0(y-1),$$

for some $a_0 \in A$ and some positive integer n. Since H is abelian we have

$$(x\delta)(y-1) = (y\delta)(x-1),$$

for all $x \in H$, so that

$$(n(x\delta))(y-1) = (n(y\delta))(x-1) = a_0(x-1)(y-1)$$

and

$$n(x\delta) = a_0(x-1)$$

for all $x \in H$. Therefore $a_0 \neq 0$ since $\delta \neq 0$, and writing $b_i = na_i + a_0$ for $1 \le i \le m$ we have

$$nA = \bigcup_{i=1}^{m} \{na_i x + a_0(x-1);\, x \in H\} = \bigcup_{i=1}^{m} \{b_i x - a_0;\, x \in H\}.$$

For $i = 1, \dots, m$ we define U_i to be the set of integers u such that $nua_0 = b_i x - a_0$ for some $x \in H$. Thus $\mathbb{Z} = U_1 \cup \cdots \cup U_n$. If

$$nua_0 = b_i x - a_0 \quad \text{and} \quad nva_0 = b_i y - a_0$$

with $x, y \in H$, then

$$(nu+1)b_i y = (nu+1)(nv+1)a_0 = (nv+1)b_i x,$$

so that

$$(nu+1)(b_i yx^{-1}) = (nv+1)b_i.$$

Thus if $v \in U_i$ then $(nv+1)b_i$ is divisible in A by $nu+1$ for all $u \in U_i$. If $(nv+1)b_i \neq 0$ for some $v \in U_i$ we can therefore conclude that there are only finitely many primes which divide integers in $\{nu+1;\, u \in U_i\}$; and the same conclusion holds if $(nv+1)b_i = 0$ for all $v \in U_i$ since then U_i has at most one element. Because $\mathbb{Z} = U_1 \cup \cdots \cup U_m$ it follows that the prime divisors of the integers in $\{nu+1;\, u \in \mathbb{Z}\}$ are finite in number. However this is absurd: if p is any prime not dividing n, then $-n$ is invertible modulo p so that p divides an integer of form $nu+1$. The result follows.

We may now prove Proposition 2(b). Let G, H, K and A be as in Proposition 2, and suppose that G is almost the product of H and K. Thus $A^n \leq HK$ for some integer $n>0$. We write $H_1 = H \cap A^nK$, so that $|H:H_1|$ is finite. If $x \in H_1$ then $xa \in K$ for some $a \in A^n$; on the other hand if $xa_i = k_i$ with $a_i \in A^n$ and $k_i \in K$ for $i=1,2$, then $a_1^{-1}a_2 = k_1^{-1}k_2 \in A \cap K = 1$ so that $a_1 = a_2$. It follows that for $x \in H_1$ there is a unique $x\theta \in A^n$ with $x(x\theta) \in K$. If $x, y \in H_1$, then K contains

$$x(x\theta)y(y\theta) = (xy)((x\theta)^y(y\theta)),$$

and so the map $\theta: H_1 \to A^n$ is a derivation. Moreover θ is surjective because $A^n \leq HK$.

Let H_0 be an abelian normal subgroup of finite index in H such that $H_0 \leq H_1$, let D be an H_0-invariant subgroup of A^n with $r_0(D)$ as large as possible subject to $r_0(D) < r_0(A)$, and write B/D for the torsion subgroup of A^n/D. Thus B is H_0-invariant, and $\bar{A} = A^n/B$ is rationally irreducible, regarded as an H_0-module. Let $C = C_{H_0}(\bar{A})$. If T is a transversal to H_0 in H, then $H_0/C^t \cong H_0/C$ for each $t \in T$, and

$$\bigcap_{t\in T} C^t = \bigcap_{t\in T} C_{H_0}(A^n/B^t) = C_{H_0}\Big(A^n\Big/\bigcap_{t\in T} B^t\Big) = C_{H_0}(A^n) = 1.$$

Since obviously $H_0 \neq 1$ we conclude that H_0 acts non-trivially on $\bar{A}$. Lemma 6 shows that H_0/C is finitely generated, and so we also conclude that H_0 is finitely generated.

We define δ to be the derivation $x \mapsto (x\theta)B$ from H_0 to $\bar{A}$ and we choose a transversal $\{t_1, \ldots, t_m\}$ to H_0 in H_1. Since $\theta: H_1 \to A^n$ is surjective we have

$$A^n = \bigcup_{i=1}^{m} \{(t_i x)\theta;\, x \in H_0\} = \bigcup_{i=1}^{m} \{(t_i\theta)^x(x\theta);\, x \in H_0\}$$

and

$$\bar{A} = \bigcup_{i=1}^{m} \{a_i^x(x\delta);\, x \in H_0\},$$

where $a_i = (t_i\theta)B$ for each i.

If δ were trivial we would have $H_0\theta \leq B$, and so

$$A^n = \bigcup_{i=1}^{m} \{(xt_i)\theta;\, x \in H_0\} \leq \bigcup_{i=1}^{m} B^{t_i}(t_i\theta).$$

Thus one of the B^{t_i} would have finite index in A^n, by a well known result of Neumann [6], and this is not the case. Therefore δ is non-trivial, and we may apply Lemma 7, with $\bar{A}$ in place of A and H_0 in place of H. We obtain a contradiction, and the proof of Proposition 2(b) is complete.

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