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Autor(en): **Moser, Jürgen / Pöschel, Jürgen**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **59 (1984)**

PDF erstellt am: **27.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-45383

Comment. Math. Helvetici 59 (1984) 39–85 0010-2571/84/010039-47$01.50 + 0.20/0

An extension of a result by Dinaburg and Sinai on quasi-periodic potentials

JÜRGEN MOSER and JÜRGEN PÖSCHEL

§1. Introduction

Floquet representations

We consider the differential operator

$$L = -\frac{d^2}{dx^2} + q(x)$$

on the real line $\mathbb{R}$, where q is a quasi-periodic real valued function. More precisely, we define L as the unique self-adjoint extension of the above operator on $C_{\text{comp}}(\mathbb{R})$, the space of twice continuously differentiable functions on R with compact support. Such a self-adjoint extension is unique, since this problem is in the "limit point case."

A function f is called quasi-periodic with rationally independent frequencies $(\omega_1, \ldots, \omega_d) = \omega$, if it can be written in the form

$$f(x) = F(\omega_1 x, \ldots, \omega_d x) = F(\omega x),$$

where F is a continuous function with period 2π in all d variables. The space of all these functions f is denoted by $\mathfrak{Q}(\omega)$. In $\mathfrak{Q}(\omega)$ we will distinguish the subspaces $\mathfrak{Q}^a$, $\mathfrak{Q}^\infty$ and $\mathfrak{Q}^r$ according to whether in the above representation, F is analytic or of class C^∞, C^r, respectively. Clearly, if for instance $f \in \mathfrak{Q}^a(\omega)$, then f is an analytic function of x and admits a Fourier series expansion

$$f(x) = \sum_{j \in \mathbb{Z}^d} f_j e^{i(j,\omega)x}, \qquad (j, \omega) = \sum_{i=1}^d j_i \omega_i,$$

where

$$|f_j| \leq c e^{-\gamma |j|}, \qquad |j| = \sum_{i=1}^d |j_i|$$

for all $j \in \mathbb{Z}^d$ with some positive constants c, γ. The converse, however, is not true. It may happen, that f is analytic, while F is only continuous [7].

The space $\mathfrak{Q}(\omega)$ is contained in the space $\mathscr{A}$ of all uniformly almost periodic functions in the sense of Bohr. With any $f \in \mathscr{A}$ we can associate a mean value

$$[f] = \lim_{x \to \infty} \frac{1}{x} \int_0^x f(t)\, dt$$

and a frequency module $\mathscr{M} = \mathscr{M}(f)$, which is the smallest module over $\mathbb{Z}$ containing all those frequencies ν for which

$$[fe^{-i\nu x}] \neq 0.$$

For example, $\mathscr{M}(f) = \{(j, \omega) \colon j \in \mathbb{Z}^d\}$ for $f \in \mathfrak{Q}(\omega)$.

In the case $d = 1$, the functions $q \in \mathfrak{Q}(\omega)$ are simply periodic of period $2\pi/\omega_1$, and the operator L gives rise to Hill's equation with the familiar band spectrum. Moreover, the differential equation

$$Ly = -y'' + q(x)y = \lambda y \tag{1.1}$$

has two linearly independent solutions of the form

$$e^{wx}p_1, \qquad e^{-wx}p_2, \quad \text{if} \quad 2iw \notin \omega_1\mathbb{Z}$$

or

$$e^{wx}(p_1 + \varepsilon x p_2), \qquad e^{wx}p_2 = e^{-wx}p_3, \quad \text{if} \quad 2iw \in \omega_1\mathbb{Z}$$

where $\varepsilon = 0, 1$ and p_1, p_2, p_3 are complex valued functions of period $2\pi/\omega_1$. This is the content of Floquet theory [12].

The following is motivated by the question for an analogous representation of the solutions of (1.1) in the quasi-periodic case. To be precise, let $\mathscr{F}$ be one of the subspaces of $\mathfrak{Q}(\omega)$ introduced above. Given $q \in \mathscr{F}$ we say that equation (1.1) admits a *Floquet representation* if it possesses two linearly independent solutions of the form

$$e^{wx}\chi_1, \qquad e^{-wx}\chi_2, \quad \text{if} \quad 2iw \notin \mathscr{M}(q) \tag{1.2$'$}$$

or

$$e^{wx}(\chi_1 + \varepsilon x \chi_2), \qquad e^{wx}\chi_2 = e^{-wx}\chi_3, \quad \text{if} \quad 2iw \in \mathscr{M}(q) \tag{1.2$''$}$$

where $\varepsilon = 0, 1$ and $\chi_1, \chi_2, \chi_3 \in \mathscr{F}$.

Necessary and sufficient for (1.1) to admit a Floquet representation is that the first order system

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(x)-\lambda & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix} \tag{1.3}$$

can be transformed into a system

$$z' = C(\lambda)z$$

with constant coefficients by a transformation

$$\begin{pmatrix} y \\ y' \end{pmatrix} = T(x, \lambda)z,$$

where the coefficients of T and T^{-1} belong to $\mathcal{F}$, and $d/dx \arg\det T$ has mean value 0. Then $\operatorname{tr} C = 0$, and the eigenvalues of C are of the form $\pm w \bmod (i\mathcal{M})$, where w is the exponent in (1.2).

In contrast to the periodic case, however, such a Floquet representation is generally not possible for all λ for almost periodic potentials, as examples in $\mathfrak{Q}(\omega)$ with point eigenvalues show [7]. For other examples, see also [6, 13, 22].

The rotation number

Even if such a Floquet representation is not available, one can still define the analogue of the Floquet exponent $w = w(\lambda)$ for any almost periodic potential. The following was shown in [7]. If $\operatorname{Im}\lambda > 0$ and $G(x, y; \lambda)$ is the kernel (Green's function) of the resolvent $(L-\lambda)^{-1}$, then $G(x, x; \lambda)$ and $G^{-1}(x, x; \lambda)$ are almost periodic, and

$$w(\lambda) = \lim_{x\to\infty} \frac{-1}{x}\int_0^x \frac{dt}{2G(t, t; \lambda)}$$

defines a holomorphic function on $\operatorname{Im}\lambda > 0$. This function satisfies

$$\operatorname{Im} w > 0, \qquad \operatorname{Re} w < 0.$$

Moreover, the harmonic function

$$\alpha(\lambda) = \operatorname{Im} w(\lambda)$$

is continuous in $\operatorname{Im} \lambda \geq 0$, which on the real line can also be defined by

$$\alpha(\lambda) = \lim_{x \to \infty} \frac{1}{x} \arg(\varphi'(x, \lambda) + i\varphi(x, \lambda)), \tag{1.4}$$

where φ is any real solution of (1.1). This function α is called the *rotation number* for q, which will play an important role in the following.

On the real axis, α is a continuous, monotone increasing function which is constant precisely on the intervals I of $\rho(L) \cap \mathbb{R}$, where $\rho(L)$ is the resolvent set of L. Moreover, on any such interval, 2α belongs to the frequency module of q. In particular, if $q \in \mathfrak{Q}(\omega)$, then

$$\alpha(\lambda) = \tfrac{1}{2}(j, \omega) \quad \text{for} \quad \lambda \in I$$

for some $j \in \mathbb{Z}^d$. This is the "gap labelling theorem."

For further interesting properties of the rotation number see also the paper by Kotani [10]. More references can be found in the review article by Simon [21].

Results

In order to find those λ for which (1.1) admits a Floquet representation, we require that

$$|(j, \omega)|^{-1} \leq \Omega(|j|), \qquad 0 \neq j \in \mathbb{Z}^d, \tag{1.5}$$

where Ω is some not too rapidly increasing approximation function. The precise properties of Ω are stated in Section 3.[1] For instance, almost all $\omega \in \mathbb{R}^d$ satisfy the Diophantine inequalities

$$|(j, \omega)|^{-1} \leq c\,|j|^{\beta}, \qquad 0 \neq j \in \mathbb{Z}^d \tag{1.6}$$

with some constants $c > 0$ and $\beta > d - 1$.

THEOREM 1.1. *If $q \in \mathfrak{Q}^a(\omega)$ and ω satisfies* (1.5), *then* (1.1) *admits a Floquet representation for every $\lambda \in \rho(L)$, the resolvent set of L. If $q \in \mathfrak{Q}^\infty(\omega)$, then the same holds, if ω satisfies* (1.6).

This follows from the work of many authors, and one finds references, for example, in [8, 19]. The proof is based on the hyperbolic character of the flow of

[1] Note that this function Ω is the *inverse* of the functions Ω in [5] and [18].

(1.3), since $\operatorname{Re} w \neq 0$ for $\lambda \in \rho(L)$. For the sake of completeness, we include a proof of Theorem 1.1 in Section 2.

It remains to study the case $\lambda \in \sigma(L)$, where $\sigma(L) = \mathbb{C} \setminus \rho(L) \subset \mathbb{R}$ is the spectrum of L. The rotation number $\mu = \alpha(\lambda)$ is an increasing function of $\lambda \in \mathbb{R}$, which is constant precisely in the spectral gaps of L. Thus, if $\mu \neq \frac{1}{2}(j, \omega)$ for all $j \in \mathbb{Z}^d$, then $\alpha^{-1}(\mu)$ is a unique point in the spectrum, and we speak of the *non-resonant case.* In the *resonant case,* we have $\mu = \frac{1}{2}(k, \omega)$ for some $k \in \mathbb{Z}^d$. Then $\alpha^{-1}(\mu)$ is a – possibly collapsed – interval $[\lambda_-, \lambda_+]$, whose endpoints belong to $\sigma(L)$.

Dinaburg and Sinai [5] considered rotation numbers μ not too close to resonances $\frac{1}{2}(j, \omega)$. Precisely, they considered the set

$$\mathcal{N}(\Omega) = \{\mu \in \mathbb{R} : |\mu - \tfrac{1}{2}(j, \omega)|^{-1} \leq \Omega(|j|), j \in \mathbb{Z}^d\},$$

where

$$\Omega(s) = c \exp\left(\frac{s}{\log^{1+\varepsilon} s}\right), \qquad s \geq s_0$$

with positive constants c, ε. This is a Cantor set, which leaves out in $\mathbb{R}$ a set of small measure.

THEOREM 1.2 [Dinaburg–Sinai]. *Suppose $q \in \mathfrak{Q}^a(\omega)$ and ω satisfies* (1.5). *If $\mu \in \mathcal{N}(\Omega)$ is sufficiently large, then* (1.1) *admits a Floquet representation for $\lambda = \alpha^{-1}(\mu)$. That is,* (1.1) *possesses two linearly independent solutions of the form*

$$e^{i\mu x}\chi, \qquad e^{-i\mu x}\bar{\chi}, \qquad \chi \in \mathfrak{Q}^a(\omega).$$

In particular, these solutions belong to $\mathfrak{Q}^a(\omega, \mu)$.

This result gives rise to a Cantor set contained in the upper part of the spectrum, in fact, in the absolutely continuous spectrum. This theorem was sharpened by Rüssmann [18] who also enlarged the class of approximation functions Ω.

The purpose of this paper is to derive a similar result in the resonant case, that is, for rotation numbers $\mu = \frac{1}{2}(k, \omega)$. However, not all those μ are accessible to our technique, and we have to restrict ourselves to the set

$$\mathcal{R}(\Omega) = \{\mu = \tfrac{1}{2}(k, \omega) : |\mu - \tfrac{1}{2}(j, \omega)|^{-1} \leq \Omega(|j|),\ k \neq j \in \mathbb{Z}^d\}. \tag{1.7}$$

The points in $\mathcal{R}(\Omega)$ can be considered as resonances not too close to other resonances. Note that $\mathcal{R}(\Omega)$ and $\mathcal{N}(\Omega)$ are disjoint.

THEOREM 1.3. *Suppose* $q \in \mathcal{Q}^a(\omega)$ *and* ω *satisfies* (1.5). *If* $\mu \in \mathcal{R}(\Omega)$ *is sufficiently large, then the interval*

$$I(\mu) = [\lambda_-, \lambda_+] = \alpha^{-1}(\mu)$$

is either collapsed to a point, in which case (1.1) *has two linearly independent solutions*

$$e^{i\mu x}\chi, \qquad e^{-i\mu x}\bar{\chi}, \qquad \chi \in \mathcal{Q}^a(\omega), \tag{1.8$'$}$$

or $I(\mu)$ *has positive length, in which case* (1.1) *has two linearly independent solutions*

$$e^{i\mu x}(\chi_1 + x\chi_2), \qquad e^{i\mu x}\chi_2, \qquad \chi_1, \chi_2 \in \mathcal{Q}^a(\omega) \tag{1.8$''$}$$

at each endpoint of $I(\mu)$. *Moreover, if* $\mu = \frac{1}{2}(k, \omega)$, *then*

$$|I(\mu)| \leq ce^{-|k|\gamma}, \qquad |\lambda_\pm - \mu^2| \leq c$$

with positive constant c, γ, *which are independent of* k.

Actually we will construct a whole family of Floquet representations for $\lambda \in [\lambda_-, \lambda_+]$, thereby continuing the hyperbolic solutions in the interior of the gap to its endpoints (see Section 3).

Gaps

We will show that the first alternative in Theorem 1.3, where $I(\mu)$ degenerates to a point, is exceptional. In this case, according to (1.8)$'$, all solutions are quasi-periodic, in fact, are in $\mathcal{Q}^a(\omega/2)$ since $\mu = \frac{1}{2}(k, \omega)$. This is analogous to the periodic case: if λ is an endpoint of a spectral gap, then all solutions are periodic with twice the period of the potential, if and only if the gap degenerates to a point. One refers to this as "coexistence of periodic solutions" [12]. Thus, for the gaps $\alpha^{-1}(\mu)$ with sufficiently large μ in $\mathcal{R}(\Omega)$ one has the analogous phenomenon in the quasi-periodic case.

In the case of a collapsed gap, the squares of all solutions belong to $\mathcal{Q}^a(\omega)$ – and not only to $\mathcal{Q}^a(\omega/2)$. To show that the collapse of such a gap is exceptional, we prove a more general result about almost periodic potentials q. If for some $\lambda = \lambda_0$ the squares of all solutions of (1.1) are almost periodic with frequency module $\mathcal{M}$, then λ_0 corresponds to a collapsed gap, which can be opened up by an arbitrarily small perturbation in $\mathscr{A}(\mathscr{M})$, the space of all almost periodic functions, whose frequency module is contained in $\mathscr{M}$.

THEOREM 1.4. *Let q be almost periodic with arbitrary frequency module, and assume that for $\lambda = \lambda_0$ the squares of all solutions of* (1.1) *are almost periodic with frequency module $\mathscr{M}$. Then there exists a real trigonometric polynomial $\hat{q} \in \mathscr{A}(\mathscr{M})$ such that for all sufficiently small $\varepsilon \neq 0$ the potential*

$$\tilde{q} = q + \varepsilon \hat{q}$$

has a nondegenerate gap $\tilde{\alpha}^{-1}(\mu)$, where $\mu = \alpha(\lambda_0)$, and $\tilde{\alpha}$ is the rotation number for $\tilde{q}$. More generally, this holds for all $\hat{q} \in \mathscr{A}(\mathscr{M})$ outside a subspace of codimension 2.

COROLLARY 1.5. *In any ball*

$$B_{r,\gamma} = \left\{ q(x) = Q(\omega x) : \sup_{|\operatorname{Im} \theta| < \gamma} |Q(\theta)| < r \right\} \subset \mathfrak{Q}^a(\omega)$$

the set of those q, for which all gaps $\alpha^{-1}(\mu)$ with $\mu \in \mathscr{R}$ and $\mu \geqslant \mu_*(r, \gamma)$ are not collapsed, is generic.

One may expect that generically *all* gaps are open. But our technique assures this only for those gaps corresponding to $\mu \in \mathscr{R}(\Omega)$ sufficiently large. Actually, in the class of limit periodic potentials, generically all gaps are open [2]. But this situation is easier, since such potentials can be approximated by periodic ones. Other cases of almost periodic potentials with gaps clustering at infinity were studied by Levitan [11].

We remark that by changing the frequency module, every point $\lambda = \alpha^{-1}(\mu)$, $\mu \in \mathscr{N}$, in the spectrum provided by the theorem of Dinaburg–Sinai can become a nondegenerate gap by a small perturbation $\hat{q} \in \mathfrak{Q}^a(\omega, 2\mu)$, that is, by replacing q by $q + \varepsilon\hat{q}$, where $\hat{q}$ is a real trigonometric polynomial in $\mathfrak{Q}^a(\omega, 2\mu)$. Indeed, this requires just the observation that according to Theorem 1.2 the squares of all solutions of (1.1) belong to $\mathfrak{Q}^a(\omega, 2\mu)$.

Extension to the Dinaburg–Sinai set

There is another connection with the Dinaburg–Sinai set $\mathscr{N}$. In fact, their theorem can be obtained from our result as a limit case. To explain this, one has to study the set $\mathscr{R} = \mathscr{R}(\Omega)$ of resonance points more closely. Clearly, none of the points of $\mathscr{R}$ are cluster points of $\mathscr{R}$. However, we shall show in Theorem 7.1 that the set $\mathcal{R}'$ of cluster points of $\mathcal{R}$ satisfies

$$\mathcal{N}(\Omega/3) \subset \mathcal{R}'(\Omega) \subset \mathcal{N}(\Omega).$$

If we consider the solutions described by Theorem 1.3 for a sequence $\lambda_\nu \in \alpha^{-1}(\mu_\nu)$, $\mu_\nu \in \mathcal{R}$, for which $\lambda_\nu \to \lambda_*$, $\mu_\nu \to \mu_* = \alpha(\lambda_*) \in \mathcal{N}$, then one verifies that they converge to solutions of (1.1) for $\lambda = \lambda_*$ of just the form given in Theorem 1.2. The details of this argument, which is based on uniform estimates of these solutions on the set $\mathcal{R}$, will be given in Section 7.

Combining these results one sees that the Floquet representations for the solutions of (1.1) are extended from the resolvent set to a closed subset Λ of the spectrum, which is characterized by (1.7). This set is of positive measure and includes the part of the spectrum found by Dinaburg and Sinai.

More importantly, this argument together with Theorem 1.4 shows that for generic potentials in $\mathfrak{Q}^a(\omega)$ all points of the subset of the spectrum provided by the Dinaburg–Sinai theorem are cluster points of open spectral gaps. One may say that generically this subset lies in the *boundary* of the spectrum.

Method of proof

The essential result of this paper is Theorem 1.3, which is proven by a perturbation argument. By an infinite succession of linear transformations, the system

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(x)-\lambda & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}$$

is transformed into one with constant coefficients. To control the effect of the small divisors occurring one uses a rapidly converging iteration scheme, as it was done in [1, 9, 14, 18]. The main difference to the proof of Theorem 1.2 lies in the difference of the null space of the linear operator

$$U = U(\theta) \to AU = D_\omega U + [U, C], \qquad D_\omega = \sum_{i=1}^{d} \omega_i \, \partial_{\theta_i},$$

where U is 2×2-matrix of trace 0 with real coefficients in $C^a(T^n)$, and $C = \bar{C}$ is a 2×2-matrix with constant coefficients and eigenvalues $\pm w$, $\operatorname{Im} w = \mu$. If $\mu \neq \frac{1}{2}(j, \omega)$ for all j, then the null space of A is 1-dimensional, and this 1-dimensional space can be compensated for by adjusting the parameter λ [14, 18]. But in the resonant case, $\mu = \frac{1}{2}(k, \omega)$, the null space has dimension 3 and can be

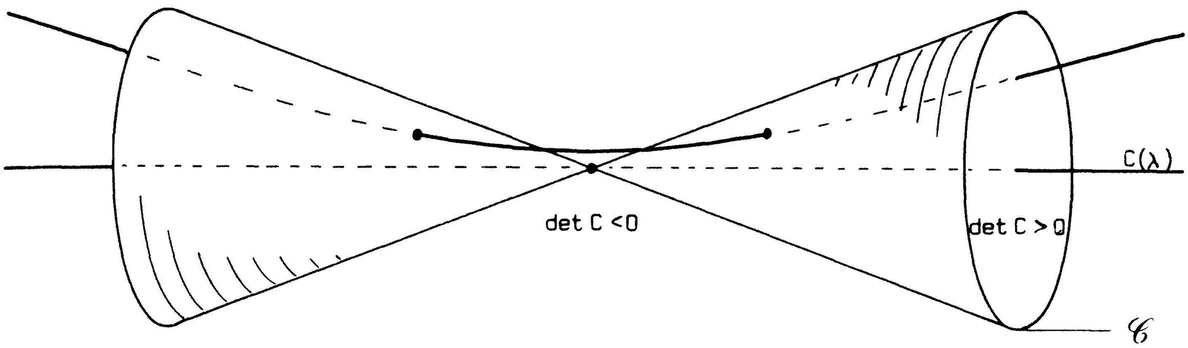

Figure 1

identified with $sl(2, \mathbb{R})$, the Lie algebra of real matrices with trace 0. This space can, of course, not any more be compensated for by a single parameter.

The Lie algebra $sl(2, \mathbb{R})$ contains the cone $\mathscr{C} = \{C : \det C = 0\}$, which separates $sl(2, \mathbb{R})$ into two open regions, the stable ($\det C > 0$) and the unstable ($\det C < 0$) one. The unperturbed system corresponds to a curve $C(\lambda)$ in $sl(2, \mathbb{R})$ for $-\delta < \lambda - \lambda_0 < \delta$ which passes through the vertex of $\mathscr{C}$ for $\lambda = \lambda_0$ and otherwise lies in the stable region. After perturbation, such a curve will generally not any more pass through the vertex of $\mathscr{C}$, but will partially lie in the unstable region. This geometrical fact corresponds to the opening of a gap under perturbation (see Fig. 1). It also shows the exceptional character of a collapsed gap.

The details of the proof of Theorem 1.3 will be carried out in Sections 3–5, after discussing the Floquet representations on the resolvent set in Section 2. In Section 6 we prove Theorem 1.4. In Section 7 we finally extend our results continuously to the Dinaburg–Sinai set.

Open problems

We mention two open problems. Although it is easy to give $q \in \mathfrak{Q}^0(\omega)$ which do not admit a Floquet representation for some λ even if (1.6) holds, such an example in $\mathfrak{Q}^a(\omega)$ is not known to us. In particular, can one have point eigenvalues for $q \in \mathfrak{Q}^a(\omega)$ with ω satisfying (1.6)?

Second, it has to be pointed out that our approach may be excessive for showing that gaps can be opened up by small perturbations, and it is conceivable that this question can be decided by easier means. We do not know whether *generically* for $q \in \mathfrak{Q}^a(\omega)$ *all* gaps are nondegenerate, or at least all gaps with sufficiently large λ.

Acknowledgement

We want to thank Peter Sarnak and Walter Craig for discussions on this topic and, in particular, for supplying their proof of Theorem 7.1.

§2. Floquet representations on the resolvent set

Smoothness of splitting and integration

For λ in the resolvent set, there exist two linearly independent solutions $\psi_+(x, \lambda)$ and $\psi_-(x, \lambda)$ of (1.1) where $\psi_+ \in L^2(0, \infty)$ and $\psi_- \in L^2(-\infty, 0)$. This is well known for $\operatorname{Im} \lambda \neq 0$. For real $\lambda \in \rho(L)$ one can choose, for instance,

$$(L-\lambda)^{-1} f = \begin{cases} \psi_+, & x > x_+ \\ \psi_-, & x < x_- \end{cases},$$

where f is any continuous function with compact support in (x_-, x_+) such that $\int_{-\infty}^{\infty} fy\, dx \neq 0$ for every nontrivial solution y of (1.1).

These solutions ψ_+, ψ_- are unique up to a multiplicative constant. Moreover, for $\operatorname{Im} \lambda \neq 0$, they have no roots, and their logarithmic derivatives

$$m_+(x, \lambda) = \frac{\psi_+'}{\psi_+}, \qquad m_-(x, \lambda) = \frac{\psi_-'}{\psi_-}$$

are uniquely defined. By a theorem of G. Scharf [20], $m_\pm$ are almost periodic functions whose frequency module is contained in $\mathcal{M}(q)$. In particular, if $q \in \mathfrak{Q}(\omega)$, there exists a unique continuous function $M_\pm = M_\pm(\theta, \lambda)$ on T^d, called the extension of $m_\pm$, such that

$$m_\pm(x, \lambda) = M_\pm(\omega x, \lambda). \tag{2.1}$$

For $\operatorname{Im} \lambda = 0$, the roots of $\psi = \psi_\pm$ give rise to singularities of $m = m_\pm$. But for real λ, the function m is also real, and one avoids such singularities by considering, for example,

$$\tilde{m} = \frac{\psi'}{\psi + i\psi'} = \frac{m}{1 + im}$$

instead of m. In other words, we consider the values of m on the projective space

$P^1_{\mathbf{C}}$. We say that

$$m \in \mathcal{F}_{\mathrm{P}},$$

where $\mathcal{F} = \mathfrak{Q}^{\alpha}(\omega)$ with $\alpha = r, \infty, a$, if

$$\frac{am+b}{cm+d} \in \mathcal{F}$$

for some complex constants a, b, c, d with $ad - bc = 1$.

We will prove Theorem 1.1 in two steps. The first step consists in showing that m is of the same regularity as q, in the sense described above, the second in showing that $\psi_\pm$ can be written in the form (1.2). The latter requires just the integration of an almost periodic function. Therefore, the first step is the more important one, and it is worth observing that it does not require any small divisor conditions, but depends only on the fact that the exponent

$$w(\lambda) = [m_+] = -[m_-] \tag{2.2}$$

satisfies

$$\operatorname{Re} w(\lambda) < 0, \qquad \lambda \in \rho(L).$$

For $\operatorname{Im} \lambda \neq 0$ this was established in [7], and the full statement follows from the maximum principle applied to the harmonic function $\operatorname{Re} w(\lambda)$. Since in a spectral gap I,

$$\lim_{\varepsilon \downarrow 0} \operatorname{Im} w(\lambda + i\varepsilon) = \alpha(\lambda) = \mu = \text{const.},$$

the reflection principle shows that $w(\lambda)$ admits an analytic continuation across I to $\operatorname{Im} \lambda < 0$ by setting $w(\bar{\lambda}) - i\mu = \overline{w(\lambda) - i\mu}$. We see that $\operatorname{Re} w(\lambda)$ is one valued across I and thus harmonic, hence negative, in $\rho(L)$.

THEOREM 2.1. *Let $\mathcal{F} = \mathfrak{Q}^{\alpha}(\omega)$ with $\alpha = r, \infty, a$. If $q \in \mathcal{F}$ and $\lambda \in \rho(L)$ then*

$$m = m_\pm \in \mathcal{F}_{\mathrm{P}}.$$

Precisely, $m \in \mathcal{F}$ *and* $\tilde{m} = \dfrac{m}{1+im} \in \mathcal{F}$ *for* $\operatorname{Im} \lambda \neq 0$ *and* $\operatorname{Im} \lambda = 0$, *respectively.*

COROLLARY 2.2. *Under the above assumptions there exists a* 2×2-*matrix* $T=T(x,\lambda)$ *such that the elements of* T, T^{-1} *belong to* $\mathcal{F}$ *and the transformation*

$$\begin{pmatrix} y \\ y' \end{pmatrix} = T(x,\lambda)z$$

takes (1.3) *into*

$$z' = D(x,\lambda)z,$$

where $D(x,\lambda)$ *is a diagonal matrix with elements in* $\mathcal{F}$. *Moreover,*

$$\left[\frac{d}{dx}\log\det T(x,\lambda)\right] = 0. \tag{2.3}$$

Proof of Corollary 2.2. For $\operatorname{Im}\lambda \neq 0$ we choose

$$T(x,\lambda) = \begin{pmatrix} \psi_+ & \psi_- \\ \psi'_+ & \psi'_- \end{pmatrix}\begin{pmatrix} \psi_+ & 0 \\ 0 & \psi_- \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ m_+ & m_- \end{pmatrix}$$

so that $z' = Dz$ possesses $\begin{pmatrix} \psi_+ \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ \psi_- \end{pmatrix}$ as solutions, that is,

$$D = \operatorname{diag}(m_+, m_-), \qquad [D] = \operatorname{diag}(w, -w)$$

by (2.2), hence $\operatorname{tr}[D] = 0$.

For real λ in a spectral gap I with $\alpha(\lambda) = \frac{1}{2}(k,\omega)$ we set

$$\begin{aligned} T(x,\lambda) &= \begin{pmatrix} \psi_+ & \psi_- \\ \psi'_+ & \psi'_- \end{pmatrix}\begin{pmatrix} (\psi_+ + i\psi'_+)e^{i(k,\omega)x} & 0 \\ 0 & \psi_- + i\psi'_- \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 1 - i\tilde{m}_+ & 1 - i\tilde{m}_- \\ \tilde{m}_+ & \tilde{m}_- \end{pmatrix}\begin{pmatrix} e^{-i(k,\omega)x} & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Then with $E = \begin{pmatrix} \psi_+ + i\psi'_+ & 0 \\ 0 & \psi_- + i\psi'_- \end{pmatrix}\begin{pmatrix} e^{i(k,\omega)x} & 0 \\ 0 & 1 \end{pmatrix}$ we have

$$D = E'E^{-1} = \operatorname{diag}(p_+ + i(k,\omega), p_-)$$

with

$$p_{\pm} = \tilde{m}_{\pm} + (q-\lambda)(\tilde{m}_{\pm}+i) = \frac{(\psi_{\pm} + i\psi'_{\pm})'}{\psi_{\pm} + i\psi'_{\pm}}. \tag{2.4}$$

It follows from (1.4) and the asymptotic behaviour of ψ_+ and ψ_- that

$$[p_+] = \bar{w}, \qquad [p_-] = -w. \tag{2.5}$$

Hence also in this case

$$[D] = \operatorname{diag}(\bar{w} + i(k,\omega), -w) = \operatorname{diag}(w, -w)$$

and $\operatorname{tr}[D] = 0$, since $\operatorname{Im} w = \alpha(\lambda) = \frac{1}{2}(k,\omega)$.

By Theorem 2.1 the coefficients of T so chosen belong to $\mathcal{F}$. Moreover,

$$\det T = m_- - m_+, \qquad (\tilde{m}_- - \tilde{m}_+)e^{-i(k,\omega)x},$$

respectively, which is bounded away from zero, if $\lambda \in \rho(L)$. Because of

$$D = T^{-1}AT - T^{-1}T',$$

where A is the matrix in (1.3), we conclude from $\operatorname{tr}[D] = 0$, $\operatorname{tr} A = 0$ that

$$0 = [\operatorname{tr} T^{-1}T'] = \lim_{x\to\infty} \frac{1}{x}\int_0^x \frac{(\det T)'}{\det T}\, dt,$$

proving (2.3) and the corollary.

The proof of Theorem 1.1 is now immediate. In order to transform the diagonal system

$$z_j' = d_j(x)z_j, \qquad j = 1, 2, \tag{2.6}$$

into constant coefficients, we merely have to set

$$z_j = e^{g_j(x)}\zeta_j, \qquad g_j = \int_0^x (d_j(t) - [d_j])\, dt.$$

While in general for a quasi-periodic function d_j the integral is not bounded, it is quasi-periodic and of the same class as d_j, if d_j belongs to $\mathfrak{Q}^a(\omega)$ or $\mathfrak{Q}^\infty(\omega)$ and ω

satisfies (1.5) or (1.6), respectively. The above transformation maps (2.6) into

$$\zeta_j' = [d_j]\zeta_j,$$

with $[d_1] = w = -[d_2]$. Therefore, equation (1.1) admits a fundamental system of the form (1.2), and Theorem 1.1 follows.

It remains to prove Theorem 2.1.

Riccati equation

For $\operatorname{Im} \lambda \neq 0$ the proof of Theorem 2.1 will be based on the fact that $m = m_\pm$ is a bounded solution of the Riccati equation

$$m' + m^2 = q - \lambda.$$

In the quasi-periodic case, due to the theorem of Scharf mentioned above, m as well as q admit unique extensions to continuous functions M and Q on T^d. Although M is in general not differentiable, it does admit a directional derivative in the direction ω, which we denote by $D_\omega M$. Then the above equation extends to

$$D_\omega M + M^2 = Q - \lambda \tag{2.7}$$

on the torus T^d.

More generally, we now consider the differential equation

$$D_\omega M + P(A, M) = 0, \qquad P(A, M) = A_0 + A_1 M + \tfrac{1}{2} A_2 M^2, \tag{2.8}$$

where A_0, A_1, A_2 are complex valued functions on T^d. We denote with $C^0_\omega(T^d)$ the space of all those $M \in C^0(T^d)$, which admit a continuous directional derivative in the direction ω, and set $|M|_{0,\omega} = |M|_0 + |D_\omega M|_0$.

LEMMA 2.3. *Suppose* $M \in C^0_\omega(T^d)$ *and* $A = (A_0, A_1, A_2)$ *satisfy* (2.8) *and*

$$\operatorname{Re}[A_1 + A_2 M] \neq 0.$$

Then there exists a neighbourhood U *of* A *in* $C^0 \times C^0 \times C^0$ *and a unique analytic map*

$$\Phi : U \to C^0_\omega(T^d), \qquad \Phi(A) = M$$

such that $N = \Phi(B)$ *satisfies the equation* $D_\omega N + P(B, N) = 0$ *for all* $B \in U$.

Proof. This lemma is a straightforward consequence of the implicit function theorem. Consider the analytic map from $C^0 \times C^0 \times C^0 \times C^0_\omega$ into C^0 given by $(B_0, B_1, B_2, N) \to D_\omega N + P(B, N)$, which vanishes at (A, M). We show that its first partial derivative with respect to N at $N = M$, that is, the linear map taking $X \in C^0_\omega$ into

$$D_\omega X + (A_1 + A_2 M)X = G \tag{2.9}$$

in C^0, has a bounded inverse. Then our claim follows.

We abbreviate $R = A_1 + A_2 M$ which satisfies $\mathrm{Re}\,[R] \neq 0$ by assumption. We assume that

$$\mathrm{Re}\,[R] = -\delta < 0,$$

the other case is handled analogously. Then (2.9) has a unique solution $X \in C^0_\omega$ for $G \in C^0$ which can be written in the form

$$X(\theta) = K_R G(\theta) = \int_0^\infty K_R(\theta, s) G(\theta + s\omega)\, ds,$$

where

$$K_R(\theta, s) = -\exp\left(\int_0^s R(\theta + \sigma\omega)\, d\sigma\right).$$

Since R is continuous and the flow $s \to s\omega$ ergodic on T^d,

$$\frac{1}{s}\int_0^s R(\theta + \sigma\omega)\, d\sigma \to [R] = -\delta,$$

and we can choose s_* so large that

$$\frac{1}{s}\int_0^s R(\theta + \sigma\omega)\, d\sigma < -\tfrac{1}{2}\delta, \qquad s \geq s_*$$

holds uniformly in θ by the compactness of T^d. With $c_0 = |R|_0$ it follows that

$$\begin{aligned} |X|_0 &\leq \left(\int_0^\infty |K_R(\theta, s)|\, ds\right) |G|_0 \\ &\leq \left(\int_0^{s_*} e^{c_0 s}\, ds + \int_{s_*}^\infty e^{-\delta s/2}\, ds\right) |G|_0 = c_1 |G|_0, \end{aligned}$$

and by the differential equation,

$$|X|_{0,\omega} \leq |X|_0 + |G|_0 + |A_1 + A_2 M|_0\, |X|_0 \leq c_2\, |G|_0.$$

Thus, K_R is bounded as a linear map from C^0 into C^0_ω, and the lemma is proven.

COROLLARY 2.4. *If A and M satisfy the hypotheses of Lemma 2.3 and*

$$A \in C^\alpha(T^d), \qquad \alpha = r, \infty, a,$$

then also

$$M \in C^\alpha(T^d).$$

Proof. This regularity result follows easily from Lemma 2.3, since Φ is analytic in a neighbourhood of A. For instance, if $A \in C^1$, and $\hat{\theta}_k$ denotes the kth unit vector, then

$$M(\theta + t\hat{\theta}_k) = \Phi(A(\theta + t\hat{\theta}_k))$$

is well defined for small t and continuously differentiable,

$$\frac{d}{dt} M(\theta + t\hat{\theta}_k)|_{t=0} = \Phi'(A) \frac{\partial}{\partial \theta_k} A(\theta),$$

showing that $M \in C^1$. Similarly, one shows that $M \in C^r$ if $A \in C^r$.

If A is analytic, we can extend M to a complex neighbourhood of T^d by setting

$$M(\theta + i\zeta) = \Phi(A(.+i\zeta))(\theta)$$

for small $|\zeta|$. Then M is continuously differentiable in θ and ζ. With $\xi_k = \theta_k + i\zeta_k$ we find

$$\frac{\partial}{\partial \bar{\xi}_k} M(\theta + i\zeta) = \Phi'(A(.+i\zeta)) \frac{\partial}{\partial \bar{\xi}_k} A(\theta + i\xi) = 0.$$

Hence M satisfies the Cauchy Riemann equations and is an analytic function of $\xi_1, \ldots, \xi_d$.

Proof of Theorem 2.1

For $\operatorname{Im} \lambda \neq 0$ the extension $M = M_\pm$ of $m = m_\pm$ is a solution of the Riccati equation (2.7) which corresponds to the choice

$$A_0 = Q - \lambda, \qquad A_1 = 0, \qquad A_2 = 2$$

in (2.8). By the ergodicity of the flow $s \to s\omega$ on T^d,

$$[A_1 + A_2 M] = 2[M] = 2[m] = \pm 2w$$

has a non-vanishing real part for $\operatorname{Im} \lambda \neq 0$. We conclude from Corollary 2.4 that M is as regular as Q, that is,

$$m_\pm \in \mathfrak{Q}^\alpha(\omega) \quad \text{for} \quad q \in \mathfrak{Q}^\alpha(\omega).$$

This proves Theorem 2.1 for $\operatorname{Im} \lambda \neq 0$.

For $\operatorname{Im} \lambda = 0$ we consider

$$\tilde{m} = \frac{\psi'}{\psi + i\psi'} = \frac{m}{1 + im}.$$

One readily checks that $\tilde{m}$ satisfies the Riccati equation

$$\tilde{m}' + (a_0 + a_1 \tilde{m} + \tfrac{1}{2} a_2 \tilde{m}^2) = 0$$

with

$$a_0 = -(q - \lambda), \qquad a_1 = 2i(q - \lambda), \qquad a_2 = 2(q - \lambda) + 2.$$

Moreover, $\tilde{m} \in \mathfrak{Q}(\omega)$ by the same arguments as for $\operatorname{Im} \lambda \neq 0$. Hence $\tilde{m}$ extends to a function $\tilde{M} \in C^0_\omega(T^d)$ which satisfies

$$D_\omega \tilde{M} + (A_0 + A_1 \tilde{M} + \tfrac{1}{2} A_2 \tilde{M}^2) = 0,$$

where A_ν are the extensions of a_ν, $\nu = 1, 2, 3$.

In order to apply Corollary 2.4 we have to check the average

$$[A_1 + A_2 M] = [a_1 + a_2 m].$$

For this purpose we notice that

$$a_1+a_2\tilde{m}=2\frac{(\psi+i\psi')'}{\psi+i\psi'}.$$

It therefore follows from (2.4) and (2.5) that $[a_1+a_2m]=\pm 2[w]$ has a non-vanishing real part. Now again Corollary 2.4 applies, and Theorem 2.1 is proven.

§3. Quasi-periodic solutions at resonances

Preliminary transformations

For a quasi-periodic potential q with basic frequencies ω, the second order equation

$$y''=(q(x)-\lambda)y$$

can be written as the autonomous first order equation

$$\begin{pmatrix} y \\ y' \end{pmatrix}'=\begin{pmatrix} 0 & 1 \\ Q(\theta)-\lambda & 0 \end{pmatrix}\begin{pmatrix} y \\ y' \end{pmatrix}, \qquad \theta'=\omega,$$

where $\theta=(\theta_1,\dots,\theta_d)$ are coordinates on the torus T^d, and Q is the unique extension of q to T^d such that $Q(\omega x)=q(x)$.

In complex coordinates

$$\begin{pmatrix} y \\ y' \end{pmatrix}=\begin{pmatrix} 1 & 1 \\ i\sqrt{\lambda} & -i\sqrt{\lambda} \end{pmatrix}u, \qquad u=\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

this system becomes

$$u'=\begin{pmatrix} i\sqrt{\lambda} & 0 \\ 0 & -i\sqrt{\lambda} \end{pmatrix}u+\frac{Q(\theta)}{2\sqrt{\lambda}}\begin{pmatrix} -i & -i \\ i & i \end{pmatrix}u, \qquad \theta'=\omega. \tag{3.1}$$

For large λ, this can be viewed as a perturbation of a family of rotations with angular velocity $\sqrt{\lambda}$. We consider this family in a neighbourhood of a resonance $\mu=\frac{1}{2}(k,\omega)$.

It is convenient to introduce rotating coordinates v, setting

$$u=M(\theta)v, \qquad M(\theta)=\begin{pmatrix} e^{i(k,\theta)/2} & 0 \\ 0 & e^{-i(k,\theta)/2} \end{pmatrix}. \tag{3.2}$$

We also write $\sqrt{\lambda}=\mu+\sigma$. Then the equations become

$$v'=\begin{pmatrix} i\sigma & 0 \\ 0 & -i\sigma \end{pmatrix}v+R(\theta,\sigma)v, \qquad \theta'=\omega \tag{3.3}$$

with

$$R(\theta,\sigma)=\frac{Q(\theta)}{2(\mu+\sigma)}\begin{pmatrix} -i & -ie^{-i(k,\theta)} \\ ie^{i(k,\theta)} & i \end{pmatrix}. \tag{3.4}$$

Our aim is to construct an interval $[\sigma_-, \sigma_+]$ and a continuous family of coordinate changes

$$v=T(\theta,\sigma)z, \qquad \sigma_-\leqslant\sigma\leqslant\sigma_+,$$

which takes this system into a system

$$z'=C(\sigma)z, \qquad \theta'=\omega$$

with constant coefficients such that

$$\operatorname{tr} C(\sigma)=0, \qquad \det C(\sigma)\begin{cases} =0, & \sigma=\sigma_\pm \\ <0, & \sigma_-<\sigma<\sigma_+ \end{cases} \tag{3.5}$$

but $C=0$ if and only if $\sigma_-=\sigma_+$. This means in particular that C is similar to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ at the endpoints of the interval $[\sigma_-, \sigma_+]$, if $\sigma_-<\sigma_+$. A suitable basis of solutions of $z'=C(\sigma_\pm)z$ will then transform into the desired Floquet solutions for $\lambda_\pm=(\mu+\sigma_\pm)^2$ with rotation number μ.

We note that the above will imply that

$$[\lambda_-, \lambda_+]=\alpha^{-1}(\mu).$$

For the Liapunov number $\sqrt{-\det C}$ has to be strictly positive in the interior of $\alpha^{-1}(\mu)$, so $[\lambda_-, \lambda_+]$ cannot be properly contained in $\alpha^{-1}(\mu)$. On the other hand, the rotation number is clearly constant in $[\lambda_-, \lambda_+]$, so that this interval agrees with $\alpha^{-1}(\mu)$.

The main result

We have to make some definitions. The matrix R is analytic and bounded in some complex domain

$$\mathcal{D}: |\mathrm{Im}\,\theta|<r, \qquad |\sigma|<1.$$

On such a domain, we introduce the weighted matrix norm

$$\|R\|_{\mathcal{D},k} = |MRM^{-1}|_{\mathcal{D}} = \sup_{(\theta,\sigma)\in\mathcal{D}} |MRM^{-1}(\theta,\sigma)|, \tag{3.6}$$

where $M=M(\theta)$ as in (3.2), and $|.|$ denotes the maximum of the moduli of the matrix elements. Thus, $\|.\|_{\mathcal{D},k}$ is the usual sup-norm in non-rotating coordinates. In this norm,

$$\|R\|_{\mathcal{D},k} \leq \frac{1}{\mu}|Q|_{\mathcal{D}} \tag{3.7}$$

is small for large μ *independently* of k.

The non-resonance conditions of Theorem 1.3 can be combined to the assumption

$$0, \tfrac{1}{2}(k,\omega)\in\mathcal{R}(\Omega).$$

Ω is supposed to be a continuous, monotone increasing function $[0,\infty)\to[1,\infty)$ such that

$$\frac{1}{s}\log\Omega(s)\searrow 0, \qquad 0\leq s\to\infty$$

and

$$\int_{\varepsilon}^{\infty}\frac{1}{s^2}\log\Omega(s)\,ds<\infty, \qquad \varepsilon>0.$$

In addition, we impose the growth condition

$$\Omega(s)\geq s^{d-1}, \qquad s\geq 0. \tag{3.8}$$

This is reasonable in view of Dirichlet's theorem which states that

$$\min_{0<|j|\leq m} |(j, \omega)| \leq |\omega| \, m^{1-d}$$

for all ω. Functions Ω with these properties are called approximation functions [18].

It is clear that with Ω also any power of Ω is an approximation function. It therefore follows from [18] that for $\rho > 0$,

$$\Phi(\rho) = \int_0^\infty \Omega^{16}\left(\frac{s}{\rho}\right) e^{-s} \, ds \tag{3.9}$$

and

$$\Psi(\rho) = \inf_{\substack{\rho_0 \geq \rho_1 \geq \cdots > 0 \\ \sum \rho_\nu \leq \rho}} \prod_{\nu=0}^{\infty} \Phi(\rho_\nu)^{2^{-\nu-1}} \tag{3.10}$$

are finite, monotone decreasing functions of ρ. We will see later that Φ measures the influence of the resonances in the linearized problem, while Ψ measures their influence in the nonlinear problem. The exponent of Ω, however, is chosen to obtain convenient estimates – see (4.21) – and is in no way optimal.

For the following to be true, the matrix R need not be of the special form (3.4). It suffices that R has trace 0 and is *real analytic* on $\mathfrak{D}$ in the sense that all coefficients are analytic on $\mathfrak{D}$ and have the form

$$R(\theta, \sigma) = \begin{pmatrix} R_1 & R_2 \\ \overline{R_2} & \overline{R_1} \end{pmatrix}$$

for real (θ, σ). This is precisely the condition in order that the system (3.3) gives rise to a *real* system. We refer to this notion of real analyticity of matrices in the following two sections.

Finally, we may assume $|\omega| \leq 1$ without loss of generality.

THEOREM 3.1. *Suppose* $0, \frac{1}{2}(k, \omega) \in \mathfrak{R}(\Omega)$ *and* $|\omega| \leq 1$. *Suppose the matrix R is real analytic on* $\mathfrak{D}$ *and has trace* 0. *If*

$$\|R\|_{\mathfrak{D},k} = \delta \leq 4^{-d-13} \Psi^{-3}(\rho), \qquad 0 < \rho < 1, \frac{r}{2},$$

then there exists a – possibly collapsed – interval $J = [\sigma_-, \sigma_+]$ *and a coordinate* *transformation*

$$v = T(\theta, \sigma)z, \qquad \det T = 1,$$

which takes for $\sigma \in J$ the system (3.1) *into a system $z' = C(\sigma)z$, $\theta' = \omega$ with constant coefficients such that* (3.5) *holds.*

On the domain $\mathcal{D}_*: |\mathrm{Im}\,\theta| < r - 2\rho$, $\sigma \in J$, the transformation T is analytic in θ, both T and C are continuous in σ, and we have the estimates

$$\|T - I\|_{\mathcal{D}_*,k}, \qquad \left\| C - \begin{pmatrix} i\sigma & 0 \\ 0 & -i\sigma \end{pmatrix} \right\|_{\mathcal{D}_*,k} \leq 4^{d+12}\Psi^3(\rho) \cdot \delta. \tag{3.11}$$

Furthermore,

$$\tfrac{1}{4}|b|_J \leq |J| \leq 2\,|b|_J, \qquad |\sigma_\pm| \leq 5\delta, \tag{3.12}$$

where b is an off diagonal element of C.

We will prove this theorem in Sections 4 and 5. Actually we will show that the dependence on σ is C^1 in J. After a slight modification in the definition of Φ, one can in fact prove that this dependence is C^∞.

Consequences

Theorem 1.3 follows immediately from Theorem 3.1. We already noted that a suitable basis of solutions of $z' = C(\sigma_\pm)z$ transforms into the desired Floquet solutions of (3.1). Also, with $\delta = |Q|_{\mathcal{D}}/\mu$, the estimates (3.11), (3.12) yield

$$|J| \leq \frac{c}{\mu} e^{-|k|(r-2\rho)}, \qquad |\sigma_\pm| \leq \frac{c}{\mu},$$

from which the corresponding estimates for the λ-interval $I(\mu)$ of Theorem 1.3 follow.

In rotating coordinates, we are able to transform into a system with constant coefficients, but have to deal with a weighted norm, which depends on the resonance. It is useful to free ourselves from this weighted norm and return to estimates in a fixed norm by looking at the transformation $S = MTM^{-1}$. Now the final system will no longer have constant coefficients, but will nevertheless be of an equally simple form.

The following corollary will be used in Section 7 to recover the results of Dinaburg and Sinai.

COROLLARY 3.2. *For* $0, \mu = \frac{1}{2}(k, \omega) \in \mathcal{R}(\Omega)$ *and* μ *sufficiently large, there exists on the interval*

$$I(\mu) = [\lambda_-, \lambda_+] = \alpha^{-1}(\mu)$$

a continuous family of real analytic coordinate transformations $S = S(\theta, \lambda)$ *with determinant* 1, *which takes* (3.1) *into the differential equation*

$$\zeta' = i(\beta + \mu)\zeta + be^{i(k,\theta)}\bar{\zeta}, \qquad \theta' = \omega$$

and its complex conjugate. Here $\beta \in \mathbb{R}$, $b \in \mathbb{C}$ *depend continuously on* λ *and satisfy* $|\beta| \leq |b|$. *Moreover, for any* $\varepsilon > 0$, *we have for* $\mu > \mu_*(\varepsilon)$ *the uniform estimates*

$$\sup_{|\mathrm{Im}\,\theta| < \gamma} |S - I| < \varepsilon, \qquad |b| < \varepsilon e^{-\gamma |k|}, \qquad |\beta + \mu - \sqrt{\lambda}| < \varepsilon \tag{3.13}$$

with some constant $\gamma > 0$, *which is independent of* ε.

The proof is immediate. The transformation $S = MTM^{-1}$ takes (3.1) into a system with coefficient matrix

$$MCM^{-1} - M(M^{-1})' = MCM^{-1} + i\mu\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

With $C = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ and $a = i\beta$ purely imaginary, equation (3.13) follows. The inequality $|\beta| \leq |b|$ is equivalent with $\det C \leq 0$ and the estimates follow from (3.6) and (3.11) for sufficiently small δ.

§4. Proof of Theorem 3.1

Theorem 3.1 is proven by a rapidly converging iteration process. The transformation T is obtained as an infinite product of transformations T_n, where each T_n provides a better approximation to some hyperbolic or parabolic system

$$z' = C_n(\sigma)z, \qquad \sigma \in J_n.$$

The θ- and σ-variables remain unchanged during this process, but have to be restricted to shrinking domains. In our estimates we use the approximation function technique introduced by Rüssmann [18].

Set up

To define the iteration process we need to introduce some sequences of real numbers. The function $\Psi(\rho)$ defined in (3.10) is lower semicontinuous on the space of all non-increasing, positive sequences ρ_ν with sum not greater than ρ, endowed with the topology of pointwise convergence. Therefore, there exists a sequence $\rho_0 \geqslant \rho_1 \geqslant \cdots > 0$ such that

$$\Psi(\rho) = \prod_{\nu=0}^{\infty} \Phi_\nu^{2^{-\nu-1}}, \qquad \Phi_\nu = \Phi(\rho_\nu).$$

Clearly, $\sum \rho_\nu = \rho$, for otherwise $\Psi(\rho)$ is not minimal. We fix such a sequence and set

$$r_n = r - 2\sum_{\nu=0}^{n-1} \rho_\nu, \qquad n \geqslant 0.$$

If we assume $\rho < r/2$, then $r = r_0 > r_1 > \cdots > r_n \to r_\infty = r - 2\rho > 0$.

Next, we set for $n \geqslant 0$

$$\delta_n = c^{1-2^{-n}} \prod_{\nu=0}^{n-1} \Phi_\nu^{3 \cdot 2^{-\nu-1}} \cdot \delta, \qquad c = 4^{d+11}. \tag{4.1}$$

Then $\delta = \delta_0 < \delta_1 < \cdots < \delta_n \to \delta_\infty = c\Psi^3(\rho) \cdot \delta$, and we require that δ is chosen so small that

$$\delta_n < \delta_\infty = c\Psi^3(\rho) \cdot \delta < 4^{-2}. \tag{4.2}$$

This agrees with the smallness condition in Theorem 3.1. The product in the definition of δ_n will turn out to be the accumulated effect of the small divisors during the first $n-1$ steps of the iteration. In particular, the factor 3 appears because the third power of the small divisors enters at each step – see (4.30).

Finally, we define ε_n by

$$\varepsilon_n = \delta_n^{2^n}, \qquad n \geqslant 0, \tag{4.3}$$

and m_n, s_n by

$$e^{-m_n \rho_n} = \varepsilon_n^2, \qquad n \geqslant 0, \tag{4.4}$$

and

$$s_n = 4^{-n-2}\Omega^{-2}(m_n), \qquad n \geq 0. \tag{4.5}$$

Then $\varepsilon_n \downarrow 0$ and $m_n \uparrow \infty$, $s_n \downarrow 0$.

Induction step

Suppose we are given a closed interval $J_n = [\xi_n, \eta_n]$ and a system of differential equations

$$z' = C_n(\sigma)z + R_n(\theta, \sigma)z, \qquad \theta' = \omega, \tag{4.6}$$

where the 2×2-matrices C_n, R_n have trace 0 and are real analytic on the complex domain

$$\mathcal{D}_n : |\mathrm{Im}\, \theta| < r_n, \qquad |\sigma - J_n| < s_n,$$

such that for real σ,

$$\det C_n(\sigma) \begin{cases} = 0, & \sigma = \xi_n, \eta_n \\ < 0, & \xi_n < \sigma < \eta_n \end{cases}. \tag{4.7}$$

Suppose that

$$\|R_n\|_{\mathcal{D}_n} \leq \varepsilon_n \tag{4.8}$$

and

$$\|C_n - C_0\|_{\mathcal{D}_n}, \sqrt{\delta_\infty}\, \|\dot{C}_n - \dot{C}_0\|_{\mathcal{D}_n} \leq \delta_\infty(2 - 2^{1-n}) \leq \tfrac{1}{8}, \tag{4.9}$$

where $\|.\|_{\mathcal{D}_n} = \|.\|_{\mathcal{D}_n,k}$ and $C_0 = \begin{pmatrix} i\sigma & 0 \\ 0 & -i\sigma \end{pmatrix}$. The dot indicates differentiation with respect to σ.

Our aim is to transform this system into a new system

$$\zeta' = C_{n+1}(\sigma)\zeta + R_{n+1}(\theta, \sigma)\zeta, \qquad \theta' = \omega, \tag{4.10}$$

such that the new error term R_{n+1} is roughly of order ε_n^2. Precisely, we will construct below a new closed interval $J_{n+1} = [\xi_{n+1}, \eta_{n+1}]$ and a transformation

$$z = T_n(\theta, \sigma)\zeta, \qquad \det T_n = 1,$$

which is real analytic on the smaller domain

$$\mathcal{D}_{n+1}: |\mathrm{Im}\,\theta| < r_{n+1}, \qquad |\sigma - J_{n+1}| < s_{n+1}$$

and takes (4.6) into (4.10) such that the same properties as before hold, when n is replaced by $n+1$.

In this section, we will obtain the estimates

$$\|T_n - I\|_{\mathcal{D}_{n+1}} \leqslant 2^{d+11} \Phi_n \varepsilon_n \leqslant \delta_{\chi}^{2^n}, \tag{4.11}$$

$$\|C_{n+1} - C_n\|_{\mathcal{D}_{n+1}}, \qquad \frac{s_n}{2} \|\dot{C}_{n+1} - \dot{C}_n\|_{\mathcal{D}_{n+1}} \leqslant \varepsilon_n \tag{4.12}$$

and

$$|\xi_{n+1} - \xi_n|, |\eta_{n+1} - \eta_n| \leqslant 4\varepsilon_n, \tag{4.13}$$

$$\tfrac{1}{4} |J_{n+1}| \leqslant |b_{n+1}|_{J_{n+1}} \leqslant 2\, |J_{n+1}|, \tag{4.14}$$

where b_{n+1} is any off diagonal element of C_{n+1}.

Theorem 3.1 follows from this construction. With

$$R_0 = R, \qquad J_0 = \{0\}$$

and $s_0 \leqslant 1$ the system considered in Theorem 3.1 satisfies our set up for $n = 0$. In particular,

$$\|R_0\|_{\mathcal{D}_0} \leqslant \|R\|_{\mathcal{D}} = \delta \leqslant 4^{-d-13} \Psi^{-3}(\rho)$$

is sufficiently small. So the iteration scheme applies.

The intervals converge to some interval $J = [\xi, \eta] = [\sigma_-, \sigma_+]$, and

$$T_0 \circ \cdots \circ T_n \to T, \qquad C_n \to C, \qquad R_n \to 0$$

uniformly on

$$\mathcal{D}_* = \bigcap_{\nu \geqslant 0} \mathcal{D}_\nu : |\mathrm{Im}\,\theta| < r - 2\rho, \qquad \sigma \in J.$$

It follows that T takes the initial system into

$$z' = C(\sigma) z, \qquad \theta' = \omega\ .$$

for $\sigma \in J$. The remaining statements of Theorem 3.1 follow easily.

The iterative construction

We describe the construction of T_n and C_{n+1}, R_{n+1}. Suppose we are at the nth step of the iteration. Then set

$$C_{n+1}=C_n+[R_n], \qquad [R_n]=(2\pi)^{-d}\int_{T^d} R_n(\theta,.)\,d\theta. \tag{4.15}$$

C_{n+1} is clearly real analytic and has trace 0. The new interval J_{n+1} will be uniquely determined by C_{n+1}, and we will see that $\mathcal{D}_{n+1}\subset\mathcal{D}_n$.

To define the transformation T_n we will solve the linearized matrix equation

$$LU_n=D_\omega U_n+[U_n, C_n]=R_n^t-[R_n^t], \tag{4.16}$$

where $D_\omega=\sum \omega_i\,\partial_{\theta_i}$ and R_n^t is a suitable truncation of the Fourier series of R_n. We pick a solution U_n with trace and mean value 0; it is real analytic. We then set

$$T_n=I+U_n+u_nI, \qquad u_n=1-\sqrt{(1-\det U_n)}, \tag{4.17}$$

where the small correction term u_nI is added to achieve $\det T_n=1$.

The change of coordinates $z=T_n\zeta$ takes (4.6) into (4.10), where

$$R_{n+1}=T_n^{-1}(-D_\omega T_n-T_nC_{n+1}+(C_n+R_n)T_n).$$

R_{n+1} is real analytic and has trace 0, since $\operatorname{tr} C_{n+1}=0$, $\operatorname{tr}(C_n+R_n)=0$, and

$$\operatorname{tr} T_n^{-1}D_\omega T_n=D_\omega(\det T_n)=0.$$

Inserting the definition of C_{n+1} we obtain

$$R_{n+1}=T_n^{-1}(-D_\omega T_n-[T_n, C_n]-T_n[R_n]+R_nT_n),$$

and using (4.16), (4.17),

$$R_{n+1}=T_n^{-1}(-D_\omega u_nI+((R_n-[R_n])-(R_n^t-[R_n^t]))+(R_n\tilde{T}_n-\tilde{T}_n[R_n])), \tag{4.18}$$

where $\tilde{T}_n=T_n-I=U_n+u_nI$. Below we will use this form to estimate R_{n+1}.

We now provide the necessary estimates.

Auxiliary inequalities

We first establish some inequalities among the sequences defined at the beginning, which will be used repeatedly.

Since Φ_ν increases with ν,

$$\begin{aligned}\Phi_n\left(\prod_{\nu=0}^{n-1}\Phi_\nu^{2^{-\nu-1}}\right)^{2^n} &= \prod_{\nu=n}^{\infty}\Phi_n^{2^{n-\nu-1}}\cdot\prod_{\nu=0}^{n-1}\Phi_\nu^{2^{n-\nu-1}}\\ &\leq \prod_{\nu=0}^{\infty}\Phi_\nu^{2^{n-\nu-1}}\\ &= \Psi^{2^n}(\rho),\end{aligned}$$

which leads together with (4.1)–(4.3) to

$$c\Phi_n^3\varepsilon_n = c\Phi_n^3\delta_n^{2^n} \leq (c\delta)^{2^n}(\Psi(\rho))^{3\cdot 2^n} = \delta_\infty^{2^n}. \tag{4.19}$$

Furthermore, a straightforward calculation shows that

$$c\Phi_n^3\varepsilon_n^2 = \varepsilon_{n+1}. \tag{4.20}$$

To estimate derivatives with respect to σ using the Cauchy inequality, we need an estimate for ε_n/s_n. This requires a bound on $\Omega(m_n)$ in terms of Φ_n and ε_n. By the definition of Φ in (3.9) and the monotonicity of Ω,

$$\begin{aligned}\Phi_n \geq \int_{m_n\rho_n}^{\infty}\Omega^{16}\left(\frac{s}{\rho_n}\right)e^{-s}\,ds &\geq \Omega^{16}(m_n)\int_{m_n\rho_n}^{\infty}e^{-s}\,ds\\ &= \Omega^{16}(m_n)e^{-m_n\rho_n},\end{aligned}$$

or with (4.4),

$$\Omega^{16}(m_n) \leq \Phi_n e^{m_n\rho_n} = \Phi_n\varepsilon_n^{-2}. \tag{4.21}$$

By the choice of the exponent 16 this yields an estimate of $\Omega(m_n)$ in terms of a sufficiently small power of ε_n^{-1}, as is needed in the following.

Using again (4.4) and $\rho_n \leq 1$ we see that $e^{-m_n} \leq \varepsilon_n^2 = \delta_n^{2^{n+1}}$ which leads to the first half of

$$4^{n+2} \leq m_n^d \leq \Omega^2(m_n),$$

while the second half follows from (3.8). Therefore we obtain

$$\frac{1}{s_n}=4^{n+2}\Omega^2(m_n)\leq\Omega^4(m_n)\leq\Phi_n\varepsilon_n^{-1/2}$$

and

$$2\frac{\varepsilon_n}{s_n}\leq(c\Phi_n^2\varepsilon_n)^{1/2}\leq\delta_\infty^{2^{n-1}}$$

by (4.19). Finally, $\varepsilon_n\leq\delta_\infty^{2^n}\leq\delta_\infty\cdot 4^{-n}$ by (4.2), so we get

$$\varepsilon_n\leq\delta_\infty\cdot 2^{-n},\qquad 2\frac{\varepsilon_n}{s_n}\leq\sqrt{\delta_\infty}\cdot 2^{-n}. \tag{4.22}$$

Proof of the estimates (4.8–11) *for* $n+1$

The difference $C_{n+1}-C_n$ is easy to bound. On

$$\mathscr{D}_n':|\mathrm{Im}\,\theta|<r_n,\qquad|\sigma-J_n|<s_n/2$$

we have by (4.15), (4.8) and the Cauchy inequality

$$\|C_{n+1}-C_n\|_{\mathscr{D}_n'},\qquad\frac{s_n}{2}\|\dot{C}_{n+1}-\dot{C}_n\|_{\mathscr{D}_n'}\leq\varepsilon_n. \tag{4.23}$$

It follows from (4.22) and assumption (4.9) that

$$\|C_{n+1}-C_0\|_{\mathscr{D}_n'},\ \sqrt{\delta_\infty}\,\|\dot{C}_{n+1}-\dot{C}_0\|_{\mathscr{D}_n'}\leq\delta_\infty(2-2^{-n})\leq\tfrac{1}{8}. \tag{4.24}$$

These estimates suffice to show – see the end of this section – that $\mathscr{D}_n\supset\mathscr{D}_n'\supset\mathscr{D}_{n+1}$, so that (4.12) and (4.9) for $n+1$ will follow from them.

The truncation R_n^t of R_n and the solution U_n of the linearized equation (4.16) will be constructed in the next section. On the domain

$$\mathscr{D}_n'':|\mathrm{Im}\,\theta|<r-\rho_n,\qquad|\sigma-J_n|<s_n,$$

we will obtain the estimates (Section 5)

$$\|U_n\|_{\mathscr{D}_n''}\leq 2^{d+8}\Phi_n\varepsilon_n, \tag{4.25}$$

$$\|R_n-R_n^t\|_{\mathscr{D}_n''}\leq 2^d\Phi_n\varepsilon_n^2. \tag{4.26}$$

Accepting these estimates for now, the remaining estimates are straightforward. Then

$$|\det U_n|_{\mathscr{D}''_n} \leq 2\,\|U_n\|^2_{\mathscr{D}''_n} \leq 4^{d+9}\Phi_n^2\varepsilon_n^2 < 1, \tag{4.27}$$

hence $u_n = 1 - \sqrt{(1-\det U_n)}$ is analytic on $\mathscr{D}''_n$ with

$$\|u_n I\|_{\mathscr{D}''_n} = |u_n|_{\mathscr{D}''_n} \leq |\det U_n|_{\mathscr{D}''_n}. \tag{4.28}$$

It follows that

$$\|T_n - I\|_{\mathscr{D}''_n} \leq \|U_n\|_{\mathscr{D}''_n} + \|u_n I\|_{\mathscr{D}''_n} \leq 2^{d+10}\Phi_n\varepsilon_n \tag{4.29}$$

by (4.19) and (4.27). This gives (4.11) once we have $\mathscr{D}''_n \supset \mathscr{D}_{n+1}$.

The estimate of R_{n+1} consists of three pieces – see (4.18). First,

$$\|R_n\tilde{T}_n - \tilde{T}_n[R_n]\|_{\mathscr{D}_{n+1}} \leq 2^{d+12}\Phi_n\varepsilon_n^2$$

by (4.29) and (4.8). Second,

$$\|(R_n - [R_n]) - (R_n^t - [R_n^t])\|_{\mathscr{D}_{n+1}} \leq 2^{d+1}\Phi_n\varepsilon_n^2$$

by (4.26). And third, with $\mathscr{D}_{n+1} \subset \mathscr{D}''_n$,

$$\|D_\omega u_n I\|_{\mathscr{D}_{n+1}} \leq \frac{|\omega|}{\rho_n}\|u_n I\|_{\mathscr{D}''_n} \leq 4^{d+9}\Phi_n^3\varepsilon_n^2 \tag{4.30}$$

by (4.28), (4.27) and

$$\Phi(\rho) \geq \int_0^\infty \frac{s}{\rho}\, e^{-s}\, ds = \frac{1}{\rho},$$

which follows from (3.8) and (3.9). Finally, $\|T_n^{-1}\|_{\mathscr{D}_{n+1}} \leq 2$ by (4.29). It thus follows from (4.18), (4.20) and $\Phi_n \geq 1$ that

$$\|R_{n+1}\|_{\mathscr{D}_{n+1}} \leq 4^{d+11}\Phi_n^3\varepsilon_n^2 = c\Phi_n^3\varepsilon_n^2 = \varepsilon_{n+1}.$$

This proves (4.8) for $n+1$ in place of n.

Position of the new interval J_{n+1}

We define $J_{n+1}=[\xi_+, \eta_+]$ as the interval where $\det C_{n+1}(\sigma)\leqslant 0$, and estimate its position as given in (4.13), (4.14).

For real σ,

$$C_{n+1}(\sigma)=\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},$$

where a, $\bar{a}$ are purely imaginary, since C_{n+1} has trace 0. We write its determinant in the form

$$\det C_{n+1}(\sigma)=fg, \qquad f=-ia+|b|, \qquad g=-ia-|b|,$$

so that ξ_+, η_+ appear as roots of f and g, respectively.

We determine these roots. On the real interval $J_n': |\sigma-J_n|<s_n/2$, $\sigma\in R$, we have $|\dot{a}-i|, |\dot{b}|\leqslant\sqrt{\delta_\infty}\leqslant\frac{1}{4}$ by (4.24), hence the functions f, g are strictly increasing with

$$|\dot{f}-1|, |\dot{g}-1|\leqslant\tfrac{1}{2} \tag{4.31}$$

on J_n'. The determinant of C_n vanishes at ξ_n and η_n, which implies

$$|f(\xi_n)|, |g(\eta_n)|\leqslant 2\,\|C_{n+1}-C_n\|_{\mathscr{D}_n'}\leqslant 2\varepsilon_n$$

by (4.23). Since $2\varepsilon_n\leqslant s_n/8$ by (4.22), we see that f and g each have a unique root $\xi_+\leqslant\eta_+$ in J_n', with

$$|\xi_+-\xi_n|, |\eta_+-\eta_n|\leqslant 4\varepsilon_n\leqslant s_n/4.$$

This defines J_{n+1} uniquely and proves (4.13). Moreover, since $s_{n+1}\leqslant s_n/4$, the interval $|\sigma-J_{n+1}|<s_{n+1}$ is contained in the interval $|\sigma-J_n|<s_n/2$, hence $\mathscr{D}_{n+1}\subset\mathscr{D}_n'$, $\mathscr{D}_n''\subset\mathscr{D}_n$.

We estimate the length of J_{n+1}. Since $f(\xi_+)=0=g(\eta_+)$,

$$|f(\xi_+)-f(\eta_+)|=|f(\eta_+)-g(\eta_+)|=2\,|b(\eta_+)|,$$

which with (4.31) gives

$$\tfrac{1}{2}\,|\xi_+-\eta_+|\leqslant 2\,|b(\eta_+)|\leqslant 2\,|\xi_+-\eta_+|.$$

Also,

$$|b(\sigma)-b(\eta_+)|\leq|\dot{b}|\,|\sigma-\eta_+|\leq|J_{n+1}|,\qquad \sigma\in J_{n+1}$$

by (4.24), so we have $|b|_J\leq 2\,|J|$. From this, (4.14) follows.

Infinite differentiability in σ

We finally indicate how to obtain infinite differentiability in σ. Define the function Φ by

$$\Phi(\rho)=\int_0^\infty \Omega^{\alpha(s)}\left(\frac{s}{\rho}\right)e^{-s}\,ds,\qquad \rho>0,$$

where $\alpha\geq 16$ is a monotone increasing, *unbounded* function, chosen so that $\Omega^{\alpha(s)}(s)$ is still an approximation function. An example is given below.

All the preceding estimates remain valid. In addition,

$$\Omega^{\alpha(m_n)}(m_n)\leq\Phi_n e^{m_n\rho_n}=\Phi_n\varepsilon_n^{-2}$$

replaces (4.21). Given an integer $l\geq 1$ and an arbitrary constant $c_l>0$, this subsequently leads, for sufficiently large n, to

$$c_l s_n^{-l}\leq\Omega^{4l+1}(m_n)\leq\Phi_n\varepsilon_n^{10l/\alpha(m_n)}\leq\Phi_n\varepsilon_n^{-1/2}$$

and

$$c_l\frac{\varepsilon_n}{s_n^l}\leq(c\Phi_n^2\varepsilon_n)^{1/2}\leq\delta_\infty^{2^{n-1}}.$$

This and the Cauchy inequalities imply that all derivatives of T_n, C_n, R_n with respect to σ converge on $\mathscr{D}_*$ as n tends to infinity.

The function α can be obtained as follows. For $s\geq s_*$ sufficiently large,

$$\alpha_1(s)=-\log\left(\frac{1}{s}\log\Omega(s)\right),\qquad \alpha_2(s)=-\log\int_s^\infty\frac{1}{t^2}\log\Omega(t)\,dt$$

define monotone increasing, unbounded functions, such that Ω^{α_1} and Ω^{α_2} satisfy the monotonicity and integrability condition for approximation functions, respectively. We obtain α from α_1 by inserting intervals of constancy into α_1, if necessary, such that $\alpha\leq\alpha_2$. Then Ω^α is an approximation function.

§5. Solution of linearized equation

To complete the proof of Theorem 3.1, we have to construct a truncation R_n^t of the Fourier series of R_n and a solution U_n of the linearized equation

$$LU_n = D_\omega U_n + [U_n, C_n] = R_n^t - [R_n^t],$$

such that the estimates (4.25), (4.26) hold.

In the following, we usually drop the index n and write R for R_n, $\mathcal{D}$ for $\mathcal{D}_n$, and so on.

Truncation of Fourier series

We first derive an estimate for the Fourier coefficients of

$$R = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}.$$

The weighted norm $\|R\|_{\mathcal{D}} = \|R\|_{\mathcal{D},k}$ can be written

$$\|R\|_{\mathcal{D}} = \max_{1 \leq \lambda \leq 4} |R_\lambda e^{-i(k_\lambda, \theta)}|_{\mathcal{D}}, \qquad k_1 = 0 = k_4, \qquad k_2 = -k = -k_3. \tag{5.1}$$

It follows from the Bessel inequality (see [16]) that the Fourier coefficients of $R_\lambda = \sum \hat{R}_{\lambda j} e^{i(j,\theta)}$ satisfy

$$\sum_{j \in \mathbb{Z}^d} |\hat{R}_{\lambda j}|_{\mathcal{J}}^2 \, e^{2|j-k_\lambda| r} \leq 2^d \, \|R\|_{\mathcal{D}}^2, \tag{5.2}$$

where $\mathcal{J} : |\sigma - J| < s$.

R^t is obtained by truncating the Fourier series of $R_\lambda e^{-i(k_\lambda,\theta)}$ at order $m = m_n$, where m_n was defined in (4.4):

$$R_\lambda^t = \sum_{|j-k_\lambda| \leq m} \hat{R}_{\lambda j} e^{i(j,\theta)}.$$

R^t is real analytic and has trace 0.

We estimate the cut off error $\|R - R^t\|$ on $\mathcal{D}'' : |\mathrm{Im}\,\theta| < r - \rho$, $|\sigma - J| < s$. By the

Schwarz inequality and (5.2),

$$|(R_\lambda - R_\lambda^t)e^{-i(k_\lambda,\theta)}|^2_{\mathscr{D}''} \leq \Big(\sum_{|j-k_\lambda|>m} |\hat{R}_{\lambda j}|\, e^{|j-k_\lambda|(r-\rho)}\Big)^2$$
$$\leq 2^d \|R\|^2_{\mathscr{D}''} \sum_{|j|>m} e^{-2|j|\rho}.$$

Using that $m\rho \geq d$ by (4.4), the sum can be bounded by 2^d times

$$\sum_{l>m} l^{d-1}e^{-2l\rho} \leq m^d e^{-2m\rho} \leq \Omega^2(m)\varepsilon^4 \leq \Phi\varepsilon^2$$

by (3.8), (4.4) and (4.21). Since $\|R\|_{\mathscr{D}} \leq \varepsilon$ it follows that

$$\|R - R^t\|_{\mathscr{D}''} \leq 2^d \Phi \varepsilon^2.$$

This proves (4.26).

Construction of U

We solve $D_\omega U + [U, C] = R^t - [R^t]$. The right hand side has trace and mean value 0. So we can normalize the solution by

$$\operatorname{tr} U = 0, \qquad [U] = 0,$$

and write

$$U = \begin{pmatrix} U_1 & U_2 \\ U_3 & -U_1 \end{pmatrix}, \qquad U_\lambda = \sum_{j\neq 0} \hat{U}_{\lambda j} e^{i(j,\theta)}.$$

If $C = \begin{pmatrix} a/2 & b \\ c & -a/2 \end{pmatrix}$, we then have to solve the system of linear equations

$$\begin{pmatrix} i(j,\omega) & c & -b \\ 2b & i(j,\omega)-a & 0 \\ -2c & 0 & i(j,\omega)+a \end{pmatrix} \begin{pmatrix} \hat{U}_{1j} \\ \hat{U}_{2j} \\ \hat{U}_{3j} \end{pmatrix} = \begin{pmatrix} \hat{R}_{1j} \\ \hat{R}_{2j} \\ \hat{R}_{3j} \end{pmatrix} \tag{5.3}$$

for the Fourier coefficients of U. Here, j can be restricted to the set

$$\mathscr{X} = \bigcup_{\lambda=1}^{3} \{j \neq 0 : |j - k_\lambda| \leq m\},$$

since the other coefficients of R^t vanish by truncation.

The determinant of the 3×3-matrix is

$$\Delta_j = -i(j,\omega)((j,\omega)^2 - 4\det C).$$

This determinant does not vanish for $j\neq 0$ and $\sigma\in J$, for then $\det C\leqslant 0$. We show that $\Delta_j\neq 0$ also for $\sigma\in\mathcal{J}$, if we restrict ourselves to $j\in\mathcal{Z}$.

On one hand, our assumption $0, \frac{1}{2}(k,\omega)\in\mathcal{R}(\Omega)$ implies

$$|(j,\omega)|\geqslant \Omega^{-1}(|j|),\ \Omega^{-1}(|j\pm k|), \qquad j\neq 0, \tag{5.4}$$

hence

$$(j,\omega)^2\geqslant \Omega^{-2}(m)$$

for all $j\in\mathcal{Z}$. On the other hand, (4.9) implies $|(\det C)^{\cdot}|\leqslant 1$ on $\mathcal{J}$, and since $\det C\leqslant 0$ on J, we must have

$$\operatorname{Re}\det C(\sigma)\leqslant|\sigma - J|<s$$

on $\mathcal{J}$. By definition and the above estimate, $s\leqslant 4^{-2}\Omega^{-2}(m)\leqslant 4^{-2}(j,\omega)^2$, and we see that

$$\begin{aligned} |(j,\omega)^2 - 4\det C| &\geqslant |(j,\omega)^2 - 4\operatorname{Re}\det C(\sigma)| \\ &\geqslant (j,\omega)^2 - 4s \\ &\geqslant \tfrac{3}{4}(j,\omega)^2, \end{aligned}$$

hence

$$|\Delta_j|\geqslant\tfrac{3}{4}|(j,\omega)|^3>0 \tag{5.5}$$

on $\mathcal{J}$ for all $j\in\mathcal{Z}$.

Thus, we can solve (5.3) and find

$$\hat{U}_j = A_j\hat{R}_j$$

where $\hat{U}_j$, $\hat{R}_j$ stand for the column vectors in (5.3), and

$$A_j = \Delta_j^{-1}\begin{pmatrix} \xi_j - a^2 & -c(\xi_j + a) & b(\xi_j - a) \\ -2b(\xi_j + a) & \xi_j(\xi_j + a) - 2bc & -2b^2 \\ 2c(\xi_j - a) & -2c^2 & \xi_j(\xi_j - a) - 2bc \end{pmatrix}, \qquad \xi_j = i(j,\omega)$$

is the inverse matrix to that of (5.3).

Estimate in the weighted norm

It remains to estimate U in the weighted norm. By comparison with (5.2), this requires that we estimate $|\hat{U}_{\lambda j}|\, e^{|j-k_\lambda|r}$, or, equivalently,

$$|M_j\hat{U}_j|, \qquad M_j = \operatorname{diag}(e^{|j-k_\lambda|r})_{\lambda=1,2,3}.$$

We recall that $k_1 = 0$ and $k_2 = -k = -k_3$.

The crucial observation is that we have a bound on $|M_jA_jM_j^{-1}|$ which only depends on the small divisors $\xi_j = i(j, \omega)$. Indeed, we have

$$|a| \leqslant 1, \qquad |b|, |c| \leqslant e^{-|k|r}$$

uniformly on $\mathcal{J}$ by (4.9), hence the $\lambda\mu$th coefficient of the matrix A_j is bounded by

$$\frac{8}{|\xi_j|^3} e^{-|k_\lambda - k_\mu|}$$

by (5.5) if $|\xi_j| \leqslant 1$. For $|\xi_j| > 1$ we can replace $|\xi_j|^3$ by $|\xi_j|$. Since

$$e^{|j-k_\lambda|} e^{-|k_\lambda - k_\mu|} e^{-|j-k_\mu|} \leqslant 1$$

by the triangle inequality, we conclude that

$$|M_jA_jM_j^{-1}| \leqslant \frac{8}{D_j}, \qquad D_j = \min(|\xi_j|^3, |\xi_j|)$$

uniformly on $\mathcal{J}$. Consequently

$$|M_j\hat{U}_j| \leqslant 3\,|M_jA_jM_j^{-1}|\,|M_j\hat{R}_j| \leqslant \frac{2^5}{D_j}|M_j\hat{R}_j|$$

on $\mathcal{J}$.

The remaining estimates follow the usual lines. We have

$$\begin{aligned} |U_\lambda e^{-i(k_\lambda,\theta)}|_{\mathcal{D}''} &\leqslant \sum_{j\in\mathfrak{X}} |M_j\hat{U}_j|_{\mathcal{J}}\, e^{-|j-k_\lambda|\rho} \\ &\leqslant \sum_{j\in\mathfrak{X}} \frac{2^5}{D_j} |M_j\hat{R}_j|_{\mathcal{J}}\, e^{-|j-k_\lambda|\rho} \\ &= 2^5 \sum_{l\in Z} V_l e^{-l\rho}, \qquad V_l = \sum_{\substack{|j-k_\lambda|=l \\ j\in\mathfrak{X}}} \frac{|M_j\hat{R}_j|_{\mathcal{J}}}{D_j}. \end{aligned} \tag{5.6}$$

By the Schwarz inequality,

$$\left(\sum_{l=0}^{p} V_l\right)^2 \leq \sum_{j\in\mathbb{Z}^d} |M_j\hat{R}_j|^2_{\mathcal{J}} \cdot \sum_{\substack{|j-k_\lambda|\leq p \\ j\in\mathfrak{X}}} D_j^{-2}.$$

The first factor is bounded by $2^{d+2}\|R\|^2_{\mathcal{D}}$ by (5.2), while the second factor is bounded by

$$2^{d+3}\left(\min_{\substack{|j-k_\lambda|\leq p \\ j\neq 0}} |(j,\omega)|^6\right)^{-1} \leq 2^{d+3}\Omega^6(p)$$

by [17] and (5.4). It follows that

$$\sum_{l=0}^{p} V_l \leq 2^{d+3}\Omega^3(p)\,\|R\|_{\mathcal{D}}. \tag{5.7}$$

Applying Abel's partial summation formula to (5.6) and (5.7) (see [16]) we finally obtain

$$\begin{aligned}
\|U\|_{\mathcal{D}''} &= \max_{1\leq\lambda\leq 3} |U_\lambda e^{-i(k_\lambda,\theta)}|_{\mathcal{D}''} \\
&\leq 2^{d+8}\|R\|_{\mathcal{D}} \int_0^\infty \Omega^3\left(\frac{s}{\rho}\right) e^{-s}\,ds \\
&\leq 2^{d+8}\Phi(\rho)\,\|R\|_{\mathcal{D}}.
\end{aligned}$$

Since $\|R\|_{\mathcal{D}} \leq \varepsilon$ we have proven the desired estimate (4.25).

§6. Coexistence of almost periodic solutions

The proof of Theorem 1.4 is based on a simple perturbation argument. We replace q by $q+\varepsilon\hat{q}$ and study the effect on the solutions by a "variations of constants" formula – see (6.1). Applying the averaging method we will show that $\hat{q}$ can be chosen as to make the resulting system hyperbolic. This is the adaption of an argument given in [15], where it was used to construct limit periodic potentials with a Cantor set spectrum.

Suppose q is almost periodic with an arbitrary frequency module, and the squares of all solutions of

$$y'' = (q(x) - \lambda_0)y$$

are in $\mathscr{A}=\mathscr{A}(\mathscr{M})$. Let $F(t)=\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$, $\det F(t)=1$, be a fundamental solution of

$$\begin{pmatrix} y \\ y' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ q(x)-\lambda_0 & 0 \end{pmatrix} \begin{pmatrix} y \\ y' \end{pmatrix}.$$

Replacing q by $q+\varepsilon\hat{q}$ and introducing the new vector ϕ by

$$\begin{pmatrix} y \\ y' \end{pmatrix} = F(t)\phi, \qquad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$

we obtain the differential equation

$$\phi' = \varepsilon\hat{q} \begin{pmatrix} -y_1y_2 & -y_2^2 \\ y_1^2 & y_1y_2 \end{pmatrix} \phi. \tag{6.1}$$

Our assumption means that each element of the two by two matrix is in $\mathscr{A}$. We are now going to choose $\hat{q}$ in $\mathscr{A}$ in such a way that this system becomes hyperbolic.

We apply the method of averaging. Taking mean values in the coefficient matrix in (6.1) and disregarding ε, we obtain a constant matrix with trace 0 and determinant

$$\begin{aligned} D(\hat{q}) &= [y_1^2\hat{q}][y_1^2\hat{q}] - [y_1y_2\hat{q}]^2 \\ &= [Y_1\hat{q}]^2 - [Y_2\hat{q}]^2 - [Y_3\hat{q}]^2, \end{aligned}$$

where

$$Y_1 = \tfrac{1}{2}(y_1^2+y_2^2), \qquad Y_2 = \tfrac{1}{2}(y_1^2-y_2^2), \qquad Y_3 = y_1y_2.$$

The functions Y_1, Y_2, Y_3 are in $\mathscr{A}$, so $D(\hat{q})$ is a quadratic form of type $(1,2)$ on $\mathscr{A}$. We can choose $\hat{q}$ in $\mathscr{A}$ such that

$$D(\hat{q}) < 0,$$

which makes the averaged system hyperbolic. We may even choose $\hat{q}$ to be a trigonometric polynomial and normalize it by

$$D(\hat{q}) = -1.$$

Calling the averaged matrix J, we thus obtain a system

$$\phi' = \varepsilon(J + V(t))\phi, \qquad [V] = 0,$$

where J is similar to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $V \in \mathscr{A}$.

The above first order system is not yet in a suitable form to prove it to be hyperbolic. To improve on this we write

$$\phi = T(t)\psi, \qquad T(t) = I + \varepsilon\tilde{T}(t).$$

Then

$$\psi' = (\varepsilon T^{-1}JT + \varepsilon T^{-1}(V - \tilde{T}' + \varepsilon V\tilde{T}))\psi.$$

Given any $\delta > 0$ we can approximate V by a trigonometric (matrix valued) polynomial V_p in $\mathscr{A}$ such that

$$\sup_x |(V - V_p)(x)| < \delta, \qquad [V_p] = 0.$$

Then we define $\tilde{T}$ by the conditions

$$\tilde{T}' = V_p(t), \qquad [\tilde{T}] = 0,$$

which have a unique *bounded* solution. Choosing ε sufficiently small, the transformation T is invertible, and we have

$$\psi' = \varepsilon(J + W(t))\psi, \qquad \sup_x |W(x)| < 2\delta.$$

It is well known – see [4, 19] – that this system admits an exponential dichotomy for $\varepsilon \neq 0$, if δ is sufficiently small. That is, there are two linearly independent solutions ψ_+ and ψ_- which decay exponentially as $t \to +\infty$ and $t \to -\infty$, respectively. Thus for the potential $q + \varepsilon\hat{q}$ the value λ_0 belongs to the resolvent set, that is, we have an open spectral gap near λ_0 for all sufficiently small $\varepsilon \neq 0$. Its rotation number μ is independent of ε since μ is a continuous function of ε and takes the values in a countable set.

It remains to prove the last statement of Theorem 1.4. The conclusion of the theorem is not affected if a constant c is added to $\hat{q}$, since this amounts only to a shift of the spectral gap on the λ-axis. We choose c so that

$$[Y_1(\hat{q}+c)]=[Y_1\hat{q}]+c[Y_1]=0.$$

This is possible, since Y_1 is positive. It follows that $D(\hat{q})<0$ unless $\hat{q}+c$ lies in the orthogonal complement N of Y_1, Y_2, Y_3. Equivalently, $\hat{q}$ must not lie in the span $\{1, N\}$, a subspace of $\mathscr{A}$ of codimension 2. This completes the proof of Theorem 1.4.

We conclude with the remark that the fundamental solution F can be chosen so that

$$[Y_iY_j]=\delta_{ij}.$$

This was proven in [15] for the periodic case, but the proof applies to the almost periodic case as well. In this representation we may choose

$$\hat{q}=\alpha_2 Y_2+\alpha_3 Y_3 \neq 0.$$

§7. The set $\mathscr{R}(\Omega)$

Distribution of $\mathscr{R}(\Omega)$

We show that for suitable ω and Ω the set $\mathscr{R}(\Omega)$ not only is not empty, but that its set of cluster points

$$\mathscr{R}'(\Omega)=(\operatorname{clos}\mathscr{R}(\Omega))\backslash\mathscr{R}(\Omega)$$

contains the set $\mathscr{N}(\Omega/3)$. For the following arguments we are indebted to Peter Sarnak and Walter Craig.

We recall that

$$\mathscr{N}(\Omega)=\{\xi\in\mathbb{R}:|\xi-\tfrac{1}{2}(j,\omega)|^{-1}\leq\Omega(|j|),\ j\in\mathbb{Z}^d\}$$

and

$$\mathscr{R}(\Omega)=\{\mu=\tfrac{1}{2}(k,\omega):|\mu-\tfrac{1}{2}(j,\omega)|^{-1}\leq\Omega(|j|),\ k\neq j\in\mathbb{Z}^d\}.$$

$\mathscr{N}(\Omega)$ is a Cantor set, while every point in $\mathscr{R}(\Omega)$ is isolated.

Suppose ω satisfies

$$|(j, \omega)|^{-1} \leq \Omega_0(|j|), \qquad 0 \neq j \in \mathbb{Z}^d, \tag{7.1}$$

and also $|\omega| \leq 1$. Let Ω be another approximation function somewhat larger than Ω_0, namely

$$\Omega(s) \geq 2\Omega_0(3s), \qquad s \geq 0. \tag{7.2}$$

In view of the natural growth condition (3.5) it is no restriction to require that Ω satisfies

$$\Omega(2s) \geq 2\Omega(s), \qquad s \geq s_* > 0. \tag{7.3}$$

For example, if $\Omega_0(s) = c_0(1+s)^{\beta_0}$ and $\beta_0 > d-1 \geq 1$, then

$$\Omega(s) = c(1+s)^{\beta}, \qquad c \geq 2 \cdot 3^{\beta_0} c_0, \qquad \beta \geq \beta_0$$

satisfies our assumptions with $s_* = (2^{1/\beta} - 1)/(2 - 2^{1/\beta})$.

THEOREM 7.1. *Under the preceding assumptions,*

$$\mathcal{N}(\Omega/3) \subset \mathfrak{R}'(\Omega) \subset \mathcal{N}(\Omega).$$

Moreover, for all large T,

$$m([T, \infty) \setminus \mathcal{N}(\Omega/3)) \leq 2^{d+3} \int_T^\infty t^{d-1} \Omega^{-1}(t)\, dt.$$

The second statement is straightforward. For $\frac{1}{2}(j, \omega)$ to lie in $[T, \infty)$ it is necessary that

$$2T \leq |(j, \omega)| \leq |j|\, |\omega| \leq |j|.$$

Therefore,

$$[T, \infty) \setminus \mathcal{N}(\Omega/3) \subset \bigcup_{|j| \geq 2T} \{x : |x - \tfrac{1}{2}(j, \omega)| < 3\Omega^{-1}(|j|)\},$$

and the measure of the union is bounded by

$$\sum_{|j|\geq 2T} 6\Omega^{-1}(|j|) \leq 2^{d+3} \sum_{k\geq 2T} k^{d-1}\Omega^{-1}(k)$$
$$\leq 2^{d+3} \int_T^\infty t^{d-1}\Omega^{-1}(t)\,dt$$

for all sufficiently large T.

The inclusion $\mathcal{R}' \subset \mathcal{N}$ is also easy to see. For, if ξ is a cluster point of $\mathcal{R}$, then in the defining inequality of $\mathcal{R}$ we can go to the limit to obtain

$$|\xi - \tfrac{1}{2}(j, \omega)|^{-1} \leq \Omega(|j|), \qquad j \in \mathbb{Z}^d,$$

hence $\xi \in \mathcal{N}$. Since every point in $\mathcal{R}$ is isolated, $\xi \notin \mathcal{R}$. This shows that $\mathcal{R}' \subset \mathcal{N}$.

To prove the inclusion $\mathcal{N}(\Omega/3) \subset \mathcal{R}'(\Omega)$, we need the following lemma. With Δ_j, $j \in \mathbb{Z}^d$, we denote the closed intervals

$$\Delta_j = \{x : |x - \tfrac{1}{2}(j, \omega)| \leq r_j\}, \qquad r_j = 2\Omega^{-1}(|j|).$$

LEMMA 7.2. *If* $\mu = \frac{1}{2}(k, \omega) \notin \mathcal{R}(\Omega)$, *then there exists an integer vector* $l \in \mathbb{Z}^d$ *such that*

$$|l| < \tfrac{1}{2}|k|, \qquad |\mu - \tfrac{1}{2}(l, \omega)|^{-1} > \Omega(|l|).$$

Moreover, if $\frac{1}{2}|k| \geq s_*$, *then* $\Delta_k \subset \Delta_l$.

Proof. By the definition of $\mathcal{R}(\Omega)$ there exists $l \in \mathbb{Z}^d$ such that

$$|\mu - \tfrac{1}{2}(l, \omega)|^{-1} > \Omega(|l|). \tag{7.4}$$

However, if $|l| \geq \frac{1}{2}|k|$, then

$$|k - l| \leq |k| + |l| \leq 3\,|l|,$$

hence, by (7.1),

$$|(k - l, \omega)|^{-1} \leq \Omega_0(|k - l|) \leq \Omega_0(3\,|l|),$$

and by (7.2),

$$|\mu-\tfrac{1}{2}(l,\omega)|^{-1}\le 2\Omega_0(3|l|)\le\Omega(|l|).$$

Hence we must have $|l|<\frac{1}{2}|k|$.

To prove the second statement of the lemma, we have to show that

$$|\tfrac{1}{2}(k-l,\omega)|+r_k<r_l.$$

By (7.4), it suffices to show

$$\frac{1}{\Omega(|l|)}+\frac{2}{\Omega(|k|)}<\frac{2}{\Omega(|l|)},$$

or

$$2\Omega(|l|)<\Omega(|k|).$$

But this clearly holds, since Ω is monotone increasing, $|l|<\frac{1}{2}|k|$, and (7.3) applies for $\frac{1}{2}|k|\ge s_*$.

The above shows that a resonance $\mu=\frac{1}{2}(k,\omega)$ fails to be in $\mathcal{R}$ only because it is too close to a resonance $\frac{1}{2}(l,\omega)$ of *lower* order $|l|<|k|$ and therefore contained in Δ_l.

We can now finish the proof of Theorem 7.1. Clearly, the set

$$\mathcal{K}(\Omega/2)=\mathbb{R}\setminus\bigcup_{j\in\mathbb{Z}^d}\Delta_j=\bigcap_{j\in\mathbb{Z}^d}\left\{x:|x-\tfrac{1}{2}(j,\omega)|^{-1}<\frac{\Omega(|j|)}{2}\right\}$$

contains $\mathcal{N}(\Omega/3)$. We will show that any point $\xi\in\mathcal{K}(\Omega/2)$ is the limit of a sequence μ_n in $\mathcal{R}(\Omega)$. Since $\mathcal{R}(\Omega)$ and $\mathcal{N}(\Omega/3)$ are disjoint, this will prove that $\mathcal{N}(\Omega/3)\subset\mathcal{R}'(\Omega)$.

We construct the μ_n as follows. For $N\ge 0$,

$$\mathcal{U}_N=\bigcup_{|j|\le N}\Delta_j$$

is a closed set not containing ξ. Hence

$$\delta_N=\operatorname{dist}(\xi,\mathcal{U}_N)>0.$$

On the other hand, $\{\frac{1}{2}(k, \omega)\}$ is dense in $\mathbb{R}$, so $\delta_N \to 0$ monotonically. We define the sequence $N_n \to \infty$ by requiring that

$$\delta_{N_n} < \delta_{N_n - 1} = \cdots = \delta_{N_{n-1}}$$

for $n > 0$, and $N_0 = 0$. Then there exists $\xi_n \in \mathcal{U}_{N_n}$ with

$$\operatorname{dist}(\xi, \xi_n) = \delta_{N_n} < \operatorname{dist}(\xi, \mathcal{U}_{N_n - 1}). \tag{7.5}$$

Hence ξ_n belongs to some interval Δ_{k_n} with $|k_n| = N_n$. We set

$$\mu_n = \tfrac{1}{2}(k_n, \omega),$$

and show that μ_n belongs to $\mathcal{R}(\Omega)$, if $N_n \geqslant 2s_*$. Indeed, if we had $\mu_n \notin \mathcal{R}$, then by Lemma 7.2 there would exist an l_n with $|l_n| < \frac{1}{2}|k_n| < N_n$ so that

$$\xi_n \in \Delta_{k_n} \subset \Delta_{l_n} \subset \mathcal{U}_{N_n - 1},$$

which contradicts (7.5). Hence $\mu_n \in \mathcal{R}$. The μ_n converge to ξ, since

$$|\xi - \mu_n| \leqslant |\xi - \xi_n| + r_{|k_n|} = \delta_{N_n} + r_{N_n} \to 0,$$

and Theorem 7.1 is proven.

Continuous extension to the Dinaburgh–Sinai set

Finally, we want to show that Theorem 1.2 of Dinaburg and Sinai is obtained as a limit case of Theorem 1.3. For this purpose, we recall Corollary 3.2, according to which there exists a real analytic matrix $S = S(\theta, \lambda)$ for

$$\lambda \in \Lambda = \alpha^{-1}(\mathcal{R}) \cap (\lambda_*, \infty),$$

such that the transformation $u = S\begin{pmatrix}\zeta \\ \bar{\zeta}\end{pmatrix}$ takes the system (3.1) into

$$\zeta' = i(\beta + \mu)\zeta + b e^{i(k,\theta)}\bar{\zeta}, \qquad \theta' = \omega. \tag{7.6}$$

Moreover, $S - I$ is uniformly bounded in a fixed strip $|\operatorname{Im} \theta| < \gamma$.

In order to extend S and the system (7.6) to the closure $\bar{\Lambda}$ or Λ, it is useful to normalize $S=(s_{ij})$ by the condition

$$[s_{11}]>0. \tag{7.7}$$

This is obviously possible by replacing ζ by $e^{i\varphi}\zeta$ with a real constant φ.

THEOREM 7.3. *If the transformation*

$$u=S(\theta,\lambda)\binom{\zeta}{\bar{\zeta}}$$

satisfies (7.7), *then it extends continuously to* $\bar{\Lambda}$, *remaining analytic in* $|\mathrm{Im}\,\theta|<\gamma$. *For* $\lambda\in\bar{\Lambda}\backslash\Lambda$, *one has* $\beta=b=0$ *in* (7.6), *so that the transformed system becomes*

$$\zeta'=i\mu\zeta. \tag{7.8}$$

This theorem clearly implies Theorem 1.2, since $\bar{\Lambda}\supset\mathcal{N}(\Omega/3)$.

For the proof, we consider any converging sequence $\lambda_\nu\in\Lambda$ with limit $\lambda_*\notin\Lambda$,

$$\alpha(\lambda_\nu)=\mu_\nu=\tfrac{1}{2}(k_\nu,\omega)\in\mathcal{R},$$

and note that $|k_\nu|\to\infty$. On account of the uniform estimate in (3.13), there exists a subsequence λ'_ν such that the matrix functions

$$S_\nu(\theta)=S(\theta,\lambda'_\nu)$$

converge uniformly to a real analytic $S_\infty(\theta)$ in a substrip $|\mathrm{Im}\,\theta|<\gamma'$, $0<\gamma'<\gamma$. Since

$$|\beta_\nu|\leq|b_\nu|\leq e^{-\gamma|k_\nu|},\qquad \beta_\nu=\beta(\lambda_\nu),\qquad b_\nu=b(\lambda_\nu)$$

by (3.13) and $|k_\nu|\to\infty$, we conclude that $\beta_\nu, b_\nu\to 0$. Hence in the limit the transformed system has the form (7.8).

It remains to show that the selection of a subsequence is unnecessary, if we impose the condition (7.7). Indeed, otherwise there exists another subsequence $\tilde{\lambda}_\nu$ with a different limit $\tilde{S}_\infty\neq S_\infty$. But it is easily seen that the most general mapping in our class taking (7.8) into itself is a rotation $\zeta\to e^{i\varphi}\zeta$, φ real constant, from which it follows that

$$\tilde{S}_\infty = S_\infty \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}.$$

The normalization (7.7) forces $e^{i\varphi} = 1$, hence $\tilde{S}_\infty = S_\infty$, a contradiction.

We see that the limit S_∞ is unique. This implies that the extension of $S(\theta, \lambda)$ to $\bar{\Lambda}$ is continuous.

Combining this result with Corollary 1.5 we see that any sufficiently large point in the spectrum, which corresponds to a rotation number in $\mathcal{N}(\Omega/3)$, is the cluster point of spectral gaps, which, in the sense of Corollary 1.5, are generically open. In other words, the subset of the spectrum constructed by Dinaburg and Sinai generically lies in the boundary of the spectrum.

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ETH-Zentrum
Dept. of Mathematics
CH-8092 Zürich

Received July 29, 1983