

On the homotopy groups of a finite dimensional space.

Autor(en): **McGibbon, C.A. / Neisendorfer, J.A.**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **59 (1984)**

PDF erstellt am: **26.05.2024**

Persistenter Link: <https://doi.org/10.5169/seals-45395>

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

On the homotopy groups of a finite dimensional space

C. A. McGIBBON and J. A. NEISENDORFER

The purpose of this note is to prove the following.

THEOREM 1. *Let X be a 1-connected space and p a prime number such that*

- (i) $H_n(X; \mathbb{Z}/p) \neq 0$ for some $n > 0$, and
- (ii) $H_n(X; \mathbb{Z}/p) = 0$ for all n sufficiently large.

Then for infinitely many n , $\pi_n X$ contains a subgroup of order p . \square

Thirty years ago, J.-P. Serre conjectured such a result for $p = 2$ [3, page 219]. He arrived at this conjecture after having proved the 2-primary part of the following result.

THEOREM 2. *Let X and p be as in Theorem 1. Moreover, assume that $H_*(X; \mathbb{Z})$ is of finite type. Then for infinitely many n , $\pi_n X$ contains an element whose order either equals p or is infinite. \square*

Serre's proof in this case used, among other things, Poincaré series and methods of analytic number theory. Later Y. Umeda, [5], showed that these methods could be modified to work for odd primes as well.

Notice that Theorem 1 represents an improvement over Theorem 2 in two respects. First, of course, it establishes the existence of torsion in $\pi_* X$ in infinitely many dimensions and, second, it does so without the hypothesis of finite type.

The key ingredient in our proof is the following recent result of Haynes Miller, [1].

THEOREM 3. *Let X and p be as in Theorem 1. Let $B = B\mathbb{Z}/p$, the classifying space for the group \mathbb{Z}/p . Then the space of pointed maps from B to X is weakly contractible; that is, $\pi_n(\text{map}_*(B, X)) = 0$ for all $n \geq 0$. \square*

Of course, in this theorem, B may also be regarded as the Eilenberg–MacLane space $K(\mathbb{Z}/p, 1)$ or, in the case when $p = 2$, as the infinite real projective space RP^∞ . We should add that we have not stated Miller's result in its most general form. However, for our purposes the statement above is sufficient.

Theorem 3 indicates a remarkable property of the iterated loop spaces, $\Omega^n X$, of such a space X . In more detail, notice that if $\text{map}_*(B, X)$ is weakly contractible then so is its iterated loop space $\Omega^n(\text{map}_*(B, X))$. This latter space, however, is easily seen to be homeomorphic to $\text{map}_*(B, \Omega^n X)$. Hence Theorem 3 implies that for all $n \geq 0$, the space $\text{map}_*(B, \Omega^n X)$ is weakly contractible, or equivalently, for all $n \geq 0$, every map from B to $\Omega^n X$ is null homotopic.

To begin the proof, let X and p satisfy the hypothesis of Theorem 1. Without loss of generality, we may assume that X has been localized at p . Notice that the conditions on X do not rule out the possibility that some of the groups $\pi_n X$ may contain rational vector spaces.

Our first goal is to establish that for infinitely many n , the mod p homotopy groups $\pi_n(X; \mathbb{Z}/p) \neq 0$. Recall that these groups are defined for $n \geq 2$, as

$$\pi_n(X; \mathbb{Z}/p) = \pi_0(\text{map}_*(S^{n-1} \cup_p e^n, X)).$$

They are related to the ordinary homotopy groups of X by a short exact sequence

$$0 \rightarrow \pi_n X \otimes \mathbb{Z}/p \rightarrow \pi_n(X; \mathbb{Z}/p) \rightarrow \text{Tor}(\pi_{n-1} X, \mathbb{Z}/p) \rightarrow 0.$$

For more details, see [2].

Suppose that at most a finite number of the mod p homotopy groups of X are nontrivial. Then by condition (i) and the mod p Hurewicz theorem we can choose a largest integer, say m , such that $\pi_m(X; \mathbb{Z}/p) \neq 0$.

What does this supposition imply about the ordinary homotopy groups of X ? By the universal coefficient sequence, mentioned earlier, it follows that there are just two possibilities; either

Case 1. $\pi_m X \otimes \mathbb{Z}/p \neq 0$, or

Case 2. $\pi_m X \otimes \mathbb{Z}/p = 0$ and $\text{Tor}(\pi_{m-1} X, \mathbb{Z}/p) \neq 0$

Moreover in both cases, if $\pi = \pi_n X$, then $\pi \otimes \mathbb{Z}/p = 0$ if $n > m$ and $\text{Tor}(\pi, \mathbb{Z}/p) = 0$ if $n \geq m$.

The second case is the easier to handle. In it we see that \mathbb{Z}/p is a subgroup of $\pi_{m-1} X = \pi_1 \Omega^{m-2} X$. Hence there is an essential map

$$f_1: K(\mathbb{Z}/p, 1) \rightarrow K(\pi_{m-1} X, 1)$$

Consider the obstructions to lifting this map up the Postnikov tower of $\Omega^{m-2}X$ to a map

$$f_\infty: K(\mathbb{Z}/p, 1) \rightarrow \Omega^{m-2}X$$

These obstructions take values in $\tilde{H}^*(K(\mathbb{Z}/p, 1); \pi)$ where $\pi = \pi_n X$ and $n > m$. By the universal coefficient theorem for cohomology [4, page 246], these obstruction groups are trivial since $\pi \otimes \mathbb{Z}/p = \text{Tor}(\pi, \mathbb{Z}/p) = 0$.

Hence Case 2 implies the existence of the essential map, f_∞ , which in turn contradicts Theorem 3. That leaves us with Case 1. In it, we see that $\mathbb{Z}_{(p)}$ is a subgroup of $\pi_m X = \pi_2 \Omega^{m-2}X$. More precisely we see that there is a monomorphism

$$g: \mathbb{Z}_{(p)} \rightarrow \pi_m X$$

which, when tensored with \mathbb{Z}/p , is still injective. This, in turn, implies that the following composition

$$g_2: K(\mathbb{Z}/p, 1) \rightarrow K(\mathbb{Z}_{(p)}, 2) \rightarrow K(\pi_m X, 2)$$

is essential. Here the first map represents a generator of $H^2(K(\mathbb{Z}/p, 1); \mathbb{Z}_{(p)}) = \mathbb{Z}/p$, and the second map is determined by g .

Let $\Omega^{m-2}X\langle 1 \rangle$ denote the 1-connective cover of $\Omega^{m-2}X$. The map g_2 can be taken to be a map into the first stage of the Postnikov tower for this cover. The obstructions to lifting g_2 up to a map

$$g_\infty: K(\mathbb{Z}/p, 1) \rightarrow \Omega^{m-2}X\langle 1 \rangle,$$

are zero for the same reasons as before. Thus g_∞ exists and is essential. The composition of g_∞ with the covering projection back into $\Omega^{m-2}X$ would likewise be essential. Once again we have reached a contradiction of Theorem 3. We therefore conclude that $\pi_n(X; \mathbb{Z}/p) \neq 0$ for infinitely many n . Notice that Theorem 2 is an immediate consequence of this fact.

To complete the proof of Theorem 1, suppose that $\text{Tor}(\pi_n X, \mathbb{Z}/p) \neq 0$ for at most a finite number of n . Then we may choose $m > 0$ large enough so that

- (i) $\text{Tor}(\pi_q \Omega^m X, \mathbb{Z}/p) = 0$ for all $q > 0$, and
- (ii) $\pi_2 \Omega^m X \otimes \mathbb{Z}/p \neq 0$.

These conditions on $\pi_2\Omega^m X$, in particular, are the same as those in the case just considered. Hence, as before there is a commutative diagram of essential maps

$$\begin{array}{ccc} K(\mathbb{Z}_{(p)}, 2) & \xrightarrow{h} & K(\pi_2\Omega^m X, 2) \\ & \nwarrow \quad \nearrow j & \\ & K(\mathbb{Z}/p, 1) & \end{array}$$

This time, however, we will consider the lifting problem for h , rather than working directly with map j .

We want to lift h up through the Postnikov tower for $\Omega^m X\langle 1 \rangle$. At the n -th stage this involves the diagram

$$\begin{array}{ccccc} & & E_{n+1} & & \\ & \nearrow h_{n+1} & \downarrow & & \\ K(\mathbb{Z}_{(p)}, 2) & \xrightarrow{h_n} & E_n & \xrightarrow{k} & K(\pi', q) \end{array}$$

where h_n is some lift of h . As usual, the next lift, h_{n+1} , exists if and only if the composition kh_n is trivial. With this in mind, note that under rationalization the k -invariant (and hence kh_n) is taken to zero. This follows because $\Omega^m X\langle 1 \rangle$ is an H -space. On the other hand, since π' is torsion-free, a simple calculation shows that

$$H^*(K(\mathbb{Z}_{(p)}, 2), \pi') \rightarrow H^*(K(\mathbb{Z}_{(p)}, 2), \pi' \otimes Q)$$

is injective. We conclude that kh_n must therefore represent the zero class in the first group. Thus kh_n is null homotopic and there is a solution, h_{n+1} , to the lifting problem.

In summary, the map h has been shown to lift to a map into $\Omega^m X\langle 1 \rangle$. Composing this lift with maps previously considered we obtain an essential map $K(\mathbb{Z}/p, 1) \rightarrow \Omega^m X$. This third and final contradiction of Theorem 3, completes the proof of Theorem 1.

We thank Dan Burghilea and Clarence Wilkerson for useful comments on this paper.

REFERENCES

- [1] H. MILLER, *The Sullivan conjecture*, Bull. AMS 9, 1983, 75–78.
- [2] J. A. NEISENDORFER, *Primary Homotopy Theory*, Memoirs AMS 232, 1980.
- [3] J.-P. SERRE, *Cohomologie modulo 2 des complexes d'Eilenberg–MacLane*, Comm. Math. Helv. 27 (1953), 198–232.

- [4] E. SPANIER, *Algebraic Topology*, McGraw-Hill, 1966.
- [5] Y. UMEDA, *A remark on a theorem of J.-P. Serre*, Proc. Japan Acad. 35 (1959), 563–566.

Wayne State University
Detroit, Michigan

Ohio State University
Columbus, Ohio

Received September 8, 1983