On the Nehari univalence criterion and quasicircles.

Comment. Math. Helvetici 59 (1984) 226–242
0010–2571/84/020226–17$01.50 + 0.20/0

On the Nehari univalence criterion and quasicircles

F. W. GEHRING* and CH. POMMERENKE

1. Jordan domains

We assume throughout the paper that the function f is meromorphic and locally univalent in the unit disk $\mathbb{D}$. The Schwarzian derivative

$$S_f(z)=\frac{d}{dz}\frac{f''(z)}{f'(z)}-\frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 \tag{1.1}$$

is analytic in $\mathbb{D}$. It satisfies

$$S_{\varphi\circ f\circ\psi}(z)=S_f(\psi(z))\psi'(z)^2+S_\psi(z) \tag{1.2}$$

for $\varphi\in$ Möb, where Möb denotes the group of Möbius transformations.

Nehari [13] has shown that if

$$(1-|z|^2)^2\,|S_f(z)|\leqq 2 \quad \text{for} \quad z\in\mathbb{D}, \tag{1.3}$$

then f is univalent in $\mathbb{D}$.

The bound 2 cannot be improved because

$$f(z)=[(1+z)/(1-z)]^{i\varepsilon}, \qquad \varepsilon>0, \tag{1.4}$$

satisfies (1.3) with 2 replaced by $2(1+\varepsilon^2)$ but assumes some values infinitely often in $\mathbb{D}$.

The univalent function

$$f^*(z)=\log\frac{1+z}{1-z} \qquad (z\in\mathbb{D}) \tag{1.5}$$

* This research was supported in part by grants from the Humboldt Foundation and the U.S. National Science Foundation.

satisfies $(1-z^2)^2 S_{f^*}(z) \equiv 2$ and maps $\mathbb{D}$ onto the parallel strip

$$T = \left\{ w : -\frac{\pi}{2} < \operatorname{Im} w < \frac{\pi}{2} \right\}. \tag{1.6}$$

Hence $f(\mathbb{D})$ need not be a Jordan domain in $\hat{\mathbb{C}}$ under the assumption (1.3).

Duren and Lehto [5] asked for conditions of the form

$$(1-|z|^2)^2 |S_f(z)| \leqq 2\lambda(|z|) \qquad (r_0 < |z| < 1)$$

that imply that $f(\mathbb{D})$ is a Jordan domain. They proved that $\lambda(r) = 1 + \varepsilon/\log(1-r)$ with $\varepsilon > 0$ is a possible choice, and this was improved by Becker [3] to $\lambda(r) = 1 + 2(1+\varepsilon)(1-r)/\log(1-r)$.

We shall show that the function f^* defined in (1.5) is essentially the only exception.

THEOREM 1. *Let f be meromorphic in $\mathbb{D}$ and let*

$$(1-|z|^2)^2 |S_f(z)| \leqq 2 \quad \textit{for} \quad z \in \mathbb{D}. \tag{1.7}$$

Then f has a spherically continuous extension to $\bar{\mathbb{D}}$ and $f(\mathbb{D})$ is a Jordan domain or the image of the parallel slit T under a Möbius transformation. Moreover if $z_0 \in \partial\mathbb{D}$ and $f(z_0) \neq \infty$, then

$$|f(rz_0) - f(z_0)| = O(\operatorname{dist}(f(rz_0), \partial f(\mathbb{D}))^{1/2}) \quad \textit{as} \quad r \to 1-0. \tag{1.8}$$

The estimate (1.8) means geometrically that the Jordan curve $\partial f(\mathbb{D})$ can at most have first order cusps (like two tangent circles).

In the second (exceptional) case, we can write

$$f = \varphi \circ f^* \circ \psi \quad \text{with} \quad \varphi, \psi \in \text{Möb}, \ \psi(\mathbb{D}) = \mathbb{D}.$$

Thus $(1-|z|^2)^2 |S_f(z)| = 2$ on some hyperbolic geodesic, by (1.2) and (1.5). Hence we conclude from Theorem 1:

COROLLARY 1. *If*

$$(1-|z|^2)^2 |S_f(z)| < 2 \quad \textit{for} \quad z \in \mathbb{D},$$

then $f(\mathbb{D})$ is a Jordan domain.

The following more precise result will be stated under the normalization $f''(0)=0$.

THEOREM 2. *Let the assumptions of Theorem* 1 *be satisfied and let* $f''(0)=0$. *Then either*

$$f(z)=a\log\frac{e^{i\theta}+z}{e^{i\theta}-z}+b, \qquad a, b\in\mathbb{C}, a\neq 0, \qquad 0\leqq\theta<2\pi, \tag{1.9}$$

or f has a homeomorphic extension to $\bar{\mathbb{D}}$ *with*

$$|f(z)-f(z')|\leqq M_1\left(\log\frac{3}{|z-z'|}\right)^{-1} \qquad (z, z'\in\bar{\mathbb{D}}), \tag{1.10}$$

$$|f(re^{i\theta})-f(e^{i\theta})|\leqq M_2[\operatorname{dist}(f(re^{i\theta}), \partial f(\mathbb{D}))]^{1/2} \qquad (0\leqq r<1, 0\leqq\theta<2\pi) \tag{1.11}$$

for some constants M_1 *and* M_2.

As the proof will show (see (3.4)), it is sufficient to assume instead of (1.7) that

$$\operatorname{Re}[e^{2i\theta}S_f(re^{i\theta})]\leqq\frac{2}{(1-r)^2)^2} \qquad (0\leqq\theta<2\pi, 0\leqq r<1) \tag{1.12}$$

in order to prove (1.10). This condition was considered by Steinmetz [16] who proved (1.10) with an extra factor $1-2(1-r^2)/\log[8/(1-r^2)]$ in (1.12).

2. Quasidisks

The Jordan curve Γ is called a *quasicircle with constant M* if

$$\min[\operatorname{diam}\Gamma_1, \operatorname{diam}\Gamma_2]\leqq M|w_1-w_2| \quad \text{for} \quad w_1, w_2\in\Gamma \tag{2.1}$$

where Γ_1 and Γ_2 are the components of $\Gamma\backslash\{w_1, w_2\}$. A domain bounded by a quasicircle will be called a quasidisk. If f is univalent in $\mathbb{D}$, the $f(\mathbb{D})$ is a quasidisk if and only if f has a quasiconformal extension to $\hat{\mathbb{C}}$ as Ahlfors [1] has shown.

THEOREM 3. *If f is meromorphic in* $\mathbb{D}$ *and if*

$$(1-|z|^2)^2|S_f(z)|\leqq b<2 \quad \textit{for} \quad z\in\mathbb{D}, \tag{2.2}$$

then $f(\mathbb{D})$ *is a quasidisk with constant*

$$M \leqq 8\left(1-\frac{b}{2}\right)^{-1/2}. \tag{2.3}$$

This result was proved by Ahlfors and Weill [2] except for the above estimate for the constant M. When $b<2$ the function

$$f(z)=\frac{[(1+z)/(1-z)]^a-1}{[(1+z)/(1-z)]^a+1}\ (z\in\mathbb{D}), \qquad a=\left(1-\frac{b}{2}\right)^{1/2},$$

satisfies (2.2) while (2.1) holds for $\Gamma=\partial f(\mathbb{D})$ only if

$$M \geqq \left(2\sin\frac{\pi a}{4}\right)^{-1} \geqq \frac{2}{\pi}\left(1-\frac{b}{2}\right)^{-1/2}.$$

Thus the order of the bound for M in (2.3) is best possible as $b \to 2$.

We give an extension of the Ahlfors–Weill theorem.

THEOREM 4. *Let f be meromorphic in $\mathbb{D}$ and let*

$$\limsup_{|z|\to 1}\,(1-|z|^2)^2\,|S_f(z)|<2. \tag{2.4}$$

Then f has a spherically continuous extension to $\bar{\mathbb{D}}$ and there exists $p<\infty$ such that f assumes every value at most p times in $\bar{\mathbb{D}}$. If $p=1$ then $f(\mathbb{D})$ is a quasidisk.

The number p can be arbitrarily large because every function that is meromorphic and locally univalent in $\bar{\mathbb{D}}$ satisfies (2.4).

The last assertion was conjectured by Becker [4]. He proved it under the additional hypothesis

$$\limsup_{|z|\to 1}\,(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right|<2.$$

If f is not injective on $\partial\mathbb{D}$, then $f(\mathbb{D})$ need not be a quasidisk as the example $f(z)=e^{\pi z}$ shows.

COROLLARY 2. *If the meromorphic function f satisfies* (1.7) *and* (2.4), *then* $f(\mathbb{D})$ *is a quasidisk.*

This follows at once from Theorems 1 and 4; the exceptional case in Theorem 1 cannot occur because of (2.4).

Our next result is a quantitative version of a theorem of Sullivan [17]. It is a consequence of a result of Mañé, Sad, and Sullivan [11] for which we give an invariant version in terms of the cross ratio

$$(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_1 - z_4} \cdot \frac{z_2 - z_4}{z_2 - z_3}. \tag{2.5}$$

The Jordan curve $\Gamma \subset \hat{\mathbb{C}}$ is a quasicircle if and only if [1, p. 295]

$$|(z_1, z_2, z_3, z_4)| \leqq K_0 \tag{2.6}$$

for all ordered quadruples z_1, z_2, z_3, z_4 on Γ and some constant K_0.

THEOREM 5. *Let the domain $G \subset \hat{\mathbb{C}}$ be bounded by a quasicircle Γ satisfying* (2.6). *Let the function*

$$g = g(z, \lambda) : G \times \mathbb{D} \to \hat{\mathbb{C}}$$

be injective in z (for fixed λ) and meromorphic in λ (for fixed z). Let $g(z, 0) \equiv z$. If $\lambda \in \mathbb{D}$, then $g(G, \lambda)$ is bounded by a quasicircle $g(\Gamma, \lambda)$ with

$$|(w_1, w_2, w_3, w_4)| \leqq \frac{1}{16} \exp\left[(\pi + \log K_0) \frac{1 + |\lambda|}{1 - |\lambda|}\right] \tag{2.7}$$

for all ordered quadruples w_1, w_2, w_3, w_4 on $g(\Gamma, \lambda)$.

Let now G be a simply connected domain and let ρ_G denote the hyperbolic (Poincaré) metric of G. Let the functions f be meromorphic and locally univalent in G. Ahlfors [1] and Gehring [8] have proved that, if and only if G is a quasidisk, there is a constant $a > 0$ such that

$$|S_f(z)| \leqq a\rho_G(z)^2 (z \in G) \qquad \text{implies } f \text{ univalent in } G.$$

It follows from the argument given in [8] that also the image $f(G)$ is a quasidisk if a is replaced by a smaller number.

We show now that the last fact holds in a much more general context.

THEOREM 6. *Let G be bounded by a quasicircle Γ satisfying* (2.6) *and let ρ*

be any positive function. Suppose that

$$|S_f(z)| \leqq a\rho(z)^2 (z \in G) \qquad \text{implies } f \text{ is univalent in } G. \tag{2.8}$$

If $0 \leqq b < a$ *and*

$$|S_f(z)| \leqq b\rho(z)^2 \qquad (z \in G), \tag{2.9}$$

then $f(G)$ *is bounded by a quasicircle* $f(\Gamma)$ *with*

$$|(w_1, w_2, w_3, w_4)| \leqq \frac{1}{16} \exp\left[(\pi + \log K_0)\frac{a+b}{a-b}\right] \tag{2.10}$$

for all ordered quadruples w_1, w_2, w_3, w_4 *on* $g(\Gamma)$.

In we choose $G = \mathbb{D}$, $\rho(z) = (1-|z|^2)^{-1}$ and $a = 2$, then (2.8) becomes the Nehari criterion. Hence we obtain a new proof of the Ahlfors–Weill theorem. It turns out however that, for b close to 2, the bound is substantially larger than the one obtained in Theorem 3.

Remark. A similar argument can be used to prove the following analogue of Theorem 6. Let the functions f be analytic and locally univalent in the simply connected domain $G \subset \mathbb{C}$. If there is a constant $a > 0$ such that

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq a\rho(z) (z \in G) \qquad \text{implies } f \text{ univalent in } G \tag{2.11}$$

and if $0 \leqq b < a$, then

$$\left|\frac{f''(z)}{f'(z)}\right| \leqq b\rho(z) (z \in G) \qquad \text{implies } f(G) \text{ is a quasidisk.} \tag{2.12}$$

Martio and Sarvas [12, Theorem 4.9] have shown that (2.11) holds for some $a > 0$ and $\rho = \rho_G$ if G is a quasidisk. Astala and Gehring have just established the converse of this result, namely that (2.11) holds for some $a > 0$ and $\rho = \rho_G$ only if G is a quasidisk.

3. Proof of Theorem 2

(a) Let $0 \leqq \theta < 2\pi$. The function

$$h(t) = e^{i\theta}\frac{e^t - 1}{e^t + 1} \qquad (t \in T) \tag{3.1}$$

maps the strip T conformally onto $\mathbb{D}$ and

$$g = f \circ h \tag{3.2}$$

is meromorphic and (at least) locally univalent in T. Computation shows that

$$|g'(t)| = \tfrac{1}{2}(1-r^2)\,|f'(re^{i\theta})| \quad \text{for} \quad t \in \mathbb{R},\, h(t) = re^{i\theta}. \tag{3.3}$$

Since $S_h(t) = -\frac{1}{2}$, it follows from (1.2) and (1.12) that

$$\operatorname{Re} S_g(t) = -\tfrac{1}{2} + \tfrac{1}{4}(1-r^2)^2 \operatorname{Re}[e^{2i\theta} S_f(re^{i\theta})] \leqq 0 \tag{3.4}$$

for $t \in \mathbb{R}$ and $h(t) = re^{i\theta}$.

We define

$$v(t) = |g'(t)|^{-1/2} \quad \text{for} \quad t \in \mathbb{R}; \tag{3.5}$$

this function is zero at a possible pole of g. We see that

$$\frac{v'}{v} = -\tfrac{1}{2} \operatorname{Re} \frac{g''}{g'},\ \frac{v''}{v} - \left(\frac{v'}{v}\right)^2 = -\tfrac{1}{2} \operatorname{Re}\left[\frac{d}{dt}\frac{g''}{g'}\right] \tag{3.6}$$

and therefore

$$v''(t) = p(t)v(t) \quad \text{for} \quad t \in \mathbb{R} \tag{3.7}$$

(except where g has a pole) where

$$p(t) = -\tfrac{1}{2} \operatorname{Re} S_g(t) + \left(\tfrac{1}{2} \operatorname{Im} \frac{g''(t)}{g'(t)}\right)^2 \geqq 0 \tag{3.8}$$

by (3.4). Hence v is non-negative and convex in $\mathbb{R}$; this is also true if g has a pole at $t_0 \in \mathbb{R}$ in which case $v(t_0) = 0$.

(b) We use now the hypothesis that $f''(0) = 0$. It follows from (3.2) that $g''(0) = 0$. Hence (3.6) shows that $v'(0) = 0$. Therefore v has its minimum at 0 where $v(0) > 0$, and we conclude that $g(t) \neq \infty$ for $t \in \mathbb{R}$.

Let first $v'(t_0) = 0$ for some $t_0 \neq 0$, say $t_0 > 0$. Since v is convex, we conclude that $v'(t) = 0$ for $0 \leqq t \leqq t_0$ and thus $v''(t) = 0$. Hence $\operatorname{Re}[g''/g'] = 0$ by (3.6) and $\operatorname{Im}[g''/g'] = 0$ by (3.4) and (3.8). We conclude that $g''(t) = 0$ for $0 \leqq t \leqq t_0$ and thus for $t \in T$ by the identity theorem. It therefore follows from (3.1) and (3.2) that f has the form (1.9).

Suppose next that f is not of the form (1.9). Then the above argument shows that $v'(1) > 0$ for each choice of the constant θ in (3.1). It follows by continuity that

$$v'(t) \geqq \alpha > 0 \quad \text{for} \quad 1 \leqq t < \infty$$

for some constant α and therefore

$$v(t) \geqq v(t_0) + \alpha(t - t_0) \quad \text{for} \quad 1 \leqq t_0 \leqq t < \infty. \tag{3.9}$$

In view of (3.5) this means that

$$|g'(t)| \leqq \frac{1}{[v(t_0) + \alpha(t - t_0)]^2} \quad \text{for} \quad 1 \leqq t_0 \leqq t < \infty. \tag{3.10}$$

(c) We obtain from (3.1), (3.3), and (3.10) that

$$|f'(z)| \leqq 2\alpha^{-2}(1 - |z|^2)^{-1}\left(\log\frac{1+|z|}{1-|z|} - 1\right)^{-2} \quad \text{for} \quad |z| \geqq \frac{e-1}{e+1}.$$

Hence there are constants a and b such that

$$|f'(z)| < \frac{a}{1-|z|}\left(\log\frac{8}{1-|z|}\right)^{-2} + b \quad \text{for} \quad z \in \mathbb{D}. \tag{3.11}$$

We apply now a standard method (see for instance [15]) to derive (1.10) from (3.11). It is sufficient to consider $z, z' \in \mathbb{D}$ because then (1.10) shows that f is uniformly continuous in $\mathbb{D}$ and hence has a continuous extension to $\bar{\mathbb{D}}$. Let Γ be the hyperbolic segment joining z and z' in $\mathbb{D}$. Then Γ has length $l \leqq \pi|z - z'|/2$ and

$$\min(s, l - s) \leqq \frac{\pi}{2}(1 - |\zeta|) \tag{3.12}$$

for each $\zeta \in \Gamma$, where s is the length of the part of Γ between z and ζ. We see from (3.11) and (3.12) that

$$
\begin{aligned}
|f(z)-f(z')| &\leqq \int_\Gamma |f'(\zeta)|\,|d\zeta| \\
&\leqq \int_\Gamma \frac{a}{1-|\zeta|}\left(\log\frac{8}{1-|\zeta|}\right)^{-2}|d\zeta| + bl \\
&\leqq 2a\int_0^{l/2} \frac{\pi}{2s}\left(\log\frac{4\pi}{s}\right)^{-2} ds + bl \\
&\leqq \pi a\left(\log\frac{16}{|z-z'|}\right)^{-1} + \frac{\pi b}{2}|z-z'| \leqq M_1\left(\log\frac{3}{|z-z'|}\right)^{-1}
\end{aligned}
$$

because $\frac{1}{x}\left(\log\frac{8}{x}\right)^{-2}$ is decreasing in (0, 1).

(d) We also obtain from (3.5) and (3.10) that

$$
\int_{t_0}^{\infty} |g'(t)|\,dt \leqq \int_{t_0}^{\infty} \frac{dt}{[v(t_0)+\alpha(t-t_0)]^2} = \frac{1}{\alpha v(t_0)} = \frac{1}{\alpha}|g'(t_0)|^{1/2}
$$

for $1 \leqq t_0 < \infty$. Hence we see from (3.1), (3.2), and (3.3) that

$$
|f(e^{i\theta}) - f(re^{i\theta})| \leq \frac{1}{\alpha}\left[\frac{1}{2}(1-r^2)\,|f'(re^{i\theta})|\right]^{1/2}, \tag{3.13}
$$

and (1.11) follows from a consequence of the Koebe distortion theorem [14, p. 22]. This completes the proof of Theorem 2 except for the statement that f is injective on $\partial\mathbb{D}$.

4. Proof of Theorem 1

There exists $\varphi \in$ Möb such that $(\varphi\circ f)''(0) = 0$. Hence it follows from Theorem 2 that $\varphi\circ f$ and therefore f has a spherically continuous extension to $\bar{\mathbb{D}}$.

Suppose now that f is not injective on $\partial\mathbb{D}$. Since S_f is invariant under Möbius transformations, we may assume that

$$
f(z_1) = f(z_2) = \infty, \qquad z_1, z_2 \in \partial\mathbb{D}, \qquad z_1 \neq z_2. \tag{4.1}
$$

Let Γ be the hyperbolic geodesic joining z_1 and z_2 in $\mathbb{D}$ and let h map the strip T conformally onto $\mathbb{D}$ such that $h(\mathbb{R}) = \Gamma$.

We set $g=f\circ h$. Then g is analytic in T and we see as in part (a) of the proof of Theorem 2 that

$$v(t)=|g'(t)|^{-1/2} \qquad (t\in\mathbb{R})$$

is convex and positive. Suppose that $v'(t_0)\neq 0$ for some $t_0\in\mathbb{R}$. If $v'(t_0)=\alpha>0$ then we obtain (3.10) as in part (b) of the proof of Theorem 2. This implies $g(+\infty)\neq\infty$ in contradiction to (4.1). Similarly $v'(t_0)<0$ leads to $g(-\infty)\neq\infty$ contradicting (4.1). Thus $v'(t)\equiv 0$, $g''(t)\equiv 0$ and $g\in$ Möb. Hence $f(\mathbb{D})$ is the image of T under the Möbius transformation g.

5. Proofs of Theorems 3 and 4

We need the following characterization of quasidisks. We say that the domain $G\subset\mathbb{C}$ has a *c-accessible boundary* if each z_1, $z_2\in\partial G$ can be joined by an open arc $A\subset G$ such that

$$\min_{j=1,2}|z-z_j|\leqq c\operatorname{dist}(z,\partial G) \quad \text{for} \quad z\in A. \tag{5.1}$$

It follows from (5.1) that $c\geqq 1$.

LEMMA 1. *Let G be a Jordan domain in $\mathbb{C}$. Suppose that there is a constant c such that, for all $\varphi\in$ Möb with $\varphi(G)\subset\mathbb{C}$, the domains $\varphi(G)$ have c-accessible boundaries. Then ∂G is a quasi-circle with constant $M\leqq 2c$.*

It easily follows from [9, Theorem III.2.3] that the converse holds except for the constants.

Proof. We show first that each w_1, $w_2\in\partial G$ can be joined by an open arc $B\subset G$ such that

$$|w-w_1|\leqq c\,|w_1-w_2| \quad \text{for} \quad w\in B. \tag{5.2}$$

We may assume that w_1, w_2 are finite and set

$$\varphi(w)=(w-w_1)/(w-w_2).$$

Then $\varphi(G)\subset\mathbb{C}$ with 0, $\infty\in\partial\varphi(G)$. By hypothesis there is an open arc A joining 0 and ∞ in $\varphi(G)$ such that

$$|w| \leqq c \operatorname{dist}(w, \partial\varphi(G)) \leqq c\,|w-1| \quad \text{for} \quad w \in A$$

because $1 \notin \varphi(G)$. If $w \in B = \varphi^{-1}(A)$ we deduce that

$$|w-w_1| = \frac{|\varphi(w)|}{|\varphi(w)-1|}\,|w_1-w_2| \leqq c\,|w_1-w_2|.$$

Now fix w_1, $w_2 \in \partial G$ and suppose that

$$\min(\operatorname{diam}\Gamma_1, \operatorname{diam}\Gamma_2) > 2c\,|w_1-w_2|$$

where Γ_1 and Γ_2 are the components of $\partial G \backslash \{w_1, w_2\}$. Then we can choose $z_j \in \Gamma_j$ with

$$\min_{j,k=1,2} |z_j - w_k| > c\,|w_1-w_2|. \tag{5.3}$$

Let C be the open segment (w_1, w_2) and suppose first that $C \cap \partial G = \varnothing$.

If $C \subset G$ then we join z_1, z_2 by an open arc $A \subset G$ satisfying (5.1). Since C separates z_1 and z_2 in G we can choose $z \in A \cap C$ in which case

$$\operatorname{dist}(z, \partial G) \leqq \tfrac{1}{2}\,|w_1-w_2|.$$

Thus, by (5.1),

$$\min_{j=1,2} |z_j - w_k| \leqq \frac{c}{2}\,|w_1-w_2| + |z-w_k| \leqq c\,|w_1-w_2| \tag{5.4}$$

where w_k is the endpoint of C nearest to z.

If $C \subset \mathbb{C} \backslash \bar{G}$ let B be an open arc joining w_1, w_2 in G for which (5.2) holds. Then $B \cup \bar{C}$ is a Jordan curve which separates z_1 and z_2, and hence

$$\min_{i=1,2} |z_j - w_1| \leqq \max_{w \in B \cup \bar{C}} |w-w_1| \leqq c\,|w_1-w_2|$$

by (5.2). Together with (5.4) this shows that

$$\min_{j,k=1,2} |z_j - w_k| \leqq c\,|w_1-w_2| \tag{5.5}$$

whenever $C \cap \partial G = \varnothing$.

Thus we see from (5.3) that $C\cap\partial G\neq\emptyset$. Let C_1 and C_2 denote the components of $\partial G\backslash\{z_1, z_2\}$. For $j=1,2$ we choose $w'_j\in\bar{C}\cap C_j$ such that

$$|w'_1-w'_2|=\operatorname{dist}(\bar{C}\cap C_1, \bar{C}\cap C_2)$$

and let $C'=(w'_1, w'_2)$. Then z_1 and z_2 lie in different components of $\partial G\backslash\{w'_1, w'_2\}$. Since $C'\cap\partial G=\emptyset$ it follows from (5.5) that

$$\min_{j,k=1,2}|z_j-w'_k|\leqq c\,|w'_1-w'_2|.$$

It is easy to see that this is a contradiction to (5.3). Thus ∂G is a quasicircle with constant $M\leqq 2c$.

Proof of Theorem 3. We show first that G is c-accessible. We verify (5.1) where it is sufficient to consider $z_1=f(-1)$, $z_2=f(1)$ because of (1.2).

We employ the notation of Section 3 with $\theta=0$. It follows from (2.2) and from (3.4) through (3.8) that

$$v''(t)\geqq a^2v(t)\quad\text{for}\quad -\infty<t<\infty \tag{5.6}$$

where $a^2=(2-b)/8$. For given t_0 we may assume that $v'(t_0)\geqq 0$; otherwise we replace $g(t)$ by $g(-t)$.

We compare the differential inequality (5.6) with the initial value problem

$$u''(t)=a^2u(t)(t\geqq t_0),\qquad u(t_0)=v(t_0),\qquad u'(t_0)=0$$

which is solved by

$$u(t)=v(t_0)\cosh a(t-t_0).$$

From a well-known comparison theorem, or directly from

$$\begin{aligned}\frac{d}{dt}\frac{v(t)}{u(t)}&=\frac{v'(t)u(t)-v(t)u'(t)}{u(t)^2}\\ &=u(t)^{-2}\int_{t_0}^{t}(v''u-vu'')\,ds+v'(t_0)v(t_0)\geqq 0\end{aligned}$$

for $t\geqq t_0$, we deduce that $v(t)\geqq u(t)$ for $t\geqq t_0$. Thus, by (3.5),

$$\int_{t_0}^{\infty}|g'(t)|\,dt\leqq|g'(t_0)|\int_{t_0}^{\infty}[\cosh a(t-t_0)]^{-2}\,dt=\frac{1}{a}|g'(t_0)|.$$

If $z_0 \in (-1, +1)$ is given, we choose t_0 such that $z_0 = h(t_0)$ and obtain

$$\min_{j=1,2} |z_j - f(z_0)| \leqq \frac{1}{a} |g'(t_0)| \leqq \frac{2}{a} \operatorname{dist}(f(z_0), \partial G)$$

by (3.3) and the Koebe distortion theorem. Thus (5.1) holds with

$$c = 4\left(1 - \frac{b}{2}\right)^{-1/2}. \tag{5.7}$$

Since the Schwarzian derivative is Möbius invariant, we therefore conclude that the assumption of Lemma 1 is satisfied with (5.7) and $G = f(\mathbb{D})$. Thus $f(\mathbb{D})$ is a quasidisk with constant

$$M \leqq 2c = 8\left(1 - \frac{b}{2}\right)^{-1/2}.$$

Proof of Theorem 4. By (2.4) there exist $\delta > 0$ and $r_1 < 1$ such that

$$(1 - |z|^2)^2 |S_f(z)| < 2 - 5\delta \quad \text{for} \quad r_1 \leqq |z| < 1. \tag{5.8}$$

Let $\alpha > 0$. The function

$$\varphi(\zeta) = e^{-i\pi\delta/2}\left(\frac{1+\zeta}{1-\zeta}\right)^{1-\delta} - i\alpha \qquad (\zeta \in \mathbb{D}) \tag{5.9}$$

maps $\mathbb{D}$ conformally onto a wedge of vertex $-i\alpha$ and angle $\pi(1-\delta)$ that lies in the right-hand halfplane and has $[-i\alpha, -i\infty]$ as one boundary line. Hence

$$\psi(\zeta) = e^{i\theta} \frac{\varphi(\zeta) - 1}{\varphi(\zeta) + 1}, \qquad 0 \leqq \theta \leqq 2\pi, \tag{5.10}$$

maps $\mathbb{D}$ conformally onto a domain H in $\mathbb{D}$ bounded by an arc of $\partial\mathbb{D}$ together with a circle through $e^{i\theta}$ and $e^{i\theta}(\alpha - i)/(\alpha + i)$ that forms the angle $\pi(1-\delta)$ with $\partial\mathbb{D}$. Hence we can choose α so large that $H \subset \{r_1 < |z| < 1\}$. We see that, for some fixed $\beta > 0$ independent of θ,

$$\{e^{it} : \theta - \beta \leqq t \leqq \theta\} \subset \partial H. \tag{5.11}$$

We obtain from (1.2), (5.10), and (5.9) that

$$S_\psi(\zeta)=S_\varphi(\zeta)=\frac{2\delta(2-\delta)}{(1-\zeta^2)^2}\qquad(\zeta\in\mathbb{D}).\tag{5.10}$$

Since $\psi(\mathbb{D})=H\subset\{r_1<|z|<1\}$, it follows from (1.2), (5.8), and (5.12) that the function $h=f\circ\psi$ satisfies

$$\begin{aligned}|S_h(z)|&\leqq|S_f(\psi(z))|\left(\frac{1-|\psi(z)|^2}{1-|z|^2}\right)^2+|S_\psi(z)|\\&\leqq\frac{(2-5\delta)+4\delta}{(1-|z|^2)^2}=\frac{2-\delta}{(1-|z|^2)^2}\end{aligned}$$

for $z\in\mathbb{D}$. Hence we see from Theorem 3 that h maps $\bar{\mathbb{D}}$ topologically onto a closed quasidisk with constant $M=8(2/\delta)^{1/2}$.

Since the domains H are congruent for all θ, it follows from (5.11) that some annulus $\{r_2<|z|<1\}$ can be covered by finitely many domains H. Hence we obtain from the last paragraph that f has a continuous extension to $\bar{\mathbb{D}}$ and assumes every value at most p times in $\bar{\mathbb{D}}$ for some $p<\infty$.

Assume now that $p=1$. Then $\Gamma=f(\partial\mathbb{D})$ is a Jordan curve. We may assume that diam $\Gamma\leqq 1$ because the Schwarzian is Möbius invariant. Then there exists $d>0$ such that

$$|f^{-1}(w)-f^{-1}(w')|\leqq\frac{\beta}{\pi}\quad\text{if}\quad w,w'\in\Gamma,\ |w-w'|\geqq d.$$

Choose w_1, $w_2\in\Gamma$ and let Γ_1, Γ_2 denote the components of $\Gamma\backslash\{w_1,w_2\}$.

Let first $|w_1-w_2|\leqq d/(2M)$. We show that

$$\min(\operatorname{diam}\Gamma_1,\operatorname{diam}\Gamma_2)\leqq 4M\,|w_1-w_2|.\tag{5.13}$$

Otherwise we could find points $z_1\in\Gamma_1$, $z_2\in\Gamma_2$ with

$$|z_j-w_1|=2M\,|w_1-w_2|\leqq d\tag{5.14}$$

and a domain H such that z_1, z_2, w_1, $w_2\in\partial f(H)$. Then z_1, z_2 would lie in different components of $\partial f(H)\backslash\{w_1,w_2\}$ and (5.14) would contradict the fact that $\partial f(H)$ is a quasicircle with constant M.

If $|w_1-w_2|\geqq d/(2M)$ then

$$\operatorname{diam}\Gamma_1\leqq 1\leqq\frac{2M}{d}|w_1-w_2|.$$

Hence we see from (5.13) that Γ is a quasicircle with constant $M_1 \leqq \max(2M/d, 4M)$.

6. Proofs of Theorems 5 and 6

Theorem 5 is an immediate consequence (with $A = G$) of the following lemma which is a quantitative and Möbius-invariant version of the surprising "λ-lemma" of Mañé, Sad and Sullivan [11].

LEMMA 2. *Let A be any set in $\hat{\mathbb{C}}$ and let the function $g = g(z, \lambda): A \times \mathbb{D} \to \hat{\mathbb{C}}$ be injective in z (for fixed λ) and meromorphic in λ (for fixed z). Let $g(z, 0) \equiv z$. Then $g(z, \lambda)$ has a spherically continuous extension to $\bar{A} \times \mathbb{D}$ that is meromorphic in $\lambda \in \mathbb{D}$ and satisfies*

$$|(w_1, w_2, w_3, w_4)| \leqq \frac{1}{16} \exp\left[(\pi + \log^+ |(z_1, z_2, z_3, z_4)|) \frac{1+|\lambda|}{1-|\lambda|}\right] \tag{6.1}$$

for every quadruple z_1, z_2, z_3, z_4 in $\bar{A}$ where $w_j = g(z_j, \lambda)$.

Proof. Fix distinct points $z_j \in A$ $(j = 1, 2, 3, 4)$. The function

$$h(\lambda) = (g(z_1, \lambda), g(z_2, \lambda), g(z_3, \lambda), g_4(z, \lambda)) \qquad (\lambda \in \mathbb{D}) \tag{6.2}$$

is meromorphic and omits the values 0, 1 and ∞ because the points $g(z_j, \lambda)$ are distinct. Hence we obtain

$$|h(\lambda)| \leqq \tfrac{1}{16} \exp\left[(\pi + \log^+ |h(0)|) \frac{1+|\lambda|}{1-|\lambda|}\right] \tag{6.3}$$

from the precise form of Schottky's Theorem proved by Hempel [7] (see also [6]). Since $h(0) = (z_1, z_2, z_3, z_4)$ this is our assertion (6.1) for the case $z_j \in A$. The general case will follow from the next paragraph by continuity.

Let now $z_0 \in \bar{A}$ and let ζ_n, ζ_n' be distinct points in $A \backslash \{z_2, z_4\}$ with $\zeta_n \to z_0$, $\zeta_n' \to z_0$ as $n \to \infty$. The meromorphic functions

$$h_n(\lambda) = (g(\zeta_n, \lambda), g(z_2, \lambda), g(\zeta_n', \lambda), g(z_4, \lambda)) \qquad (\lambda \in \mathbb{D})$$

omit $0, 1, \infty$ and therefore form a normal sequence. Since $h_n(0) = (\zeta_n, z_2, \zeta_n', z_4) \to 0$ as $n \to \infty$, it follows that $h_n(\lambda) \to 0$ locally uniformly in $\lambda \in \mathbb{D}$. Hence $g(\zeta, \lambda)$ has a limit as $\zeta \to z_0$, $\zeta \in A$, and it follows that g has a continuous extension to $\bar{A} \times \mathbb{D}$ which is meromorphic in λ.

Proof of Theorem 6. Choose a point $z_0 \in G$ with $z_0 \neq \infty$. Since the Schwarzian is Möbius invariant we may assume that $f(z_0) = z_0$, $f'(z_0) = 1$, $f''(z_0) = 0$. Let $\lambda \in \mathbb{D}$. Since G is simply connected, it follows from the theory of linear differential equations [10] that the initial value problem

$$S_g(z) = \lambda \frac{a}{b} S_f(z), \qquad g(z_0) = z_0,\ g'(z_0) = 1,\ g''(z_0) = 0$$

has a unique solution $g = g(z, \lambda)$ which is meromorphic in λ. Note that

$$g(z, 0) = z,\ g\left(z, \frac{b}{a}\right) = f(z). \tag{6.4}$$

We see from (2.9) that $|S_g(z)| \leqq a\rho(z)^2$ for $z \in G$ so that $g(z, \lambda)$ is univalent in G by condition (2.8). Hence our assertion follows from (6.4) and Theorem 5.

REFERENCES

[1] AHLFORS, L. V., *Quasiconformal reflections*, Acta Math. *109* (1963), 291–301.
[2] AHLFORS, L. V. and WEILL, G., A uniqueness theorem for Beltrami equations, Proc. Amer. Math. Soc. *13* (1962), 975–978.
[3] BECKER, J., *Über homöomorphe Fortsetzung schlichter Funktionen*, Ann. Acad. Sci. Fenn. Ser. AI, *538* (1973), 1–11.
[4] BECKER, J., *Conformal mappings with quasiconformal extensions*, *Aspects of Contemporary Complex Analysis*, ed. by D. A. Brannan and J. G. Clunie, Acad. Press 1980, 37–77.
[5] DUREN, P. L. and LEHTO, O., *Schwarzian derivatives and homeomorphic extensions*, Ann. Acad. Sci. Fenn. Ser. AI *477* (1970), 1–11.
[6] HAYMAN, W. K., *Some remarks on Schottky's Theorem*, Proc. Cambridge Philos. Soc. *43* (1947), 442–454.
[7] HEMPEL, J. A., *Precise bounds in the theorems of Schottky and Picard*, J. London Math. Soc. *21* (1980), 279–286.
[8] GEHRING, F. W., *Univalent functions and the Schwarzian derivative*, Comment. Math. Helv. *52* (1977), 561–572.
[9] GEHRING, F. W., *Characteristic properties of quasidisks*, Séminaire de Math. Sup., Université de Montréal 1982.
[10] HILLE, E., *Lectures on ordinary differential equations*, Addison-Wesley Publ. Co., Reading and London 1969.
[11] MAÑÉ, R. SAD, P. and SULLIVAN, D., *On the dynamics of rational maps*, preprint 1982.
[12] MARTIO, O. and SARVAS, J., *Injectivity theorems in plane and space*, Ann. Acad. Sci. Fenn. Ser. AI Math. *4* (1978/79), 383–401.
[13] NEHARI, Z., *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. *55* (1949), 545–551.

[14] POMMERENKE, CH., *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen 1975.
[15] STEGBUCHNER, H., *On some extensions of a theorem of Hardy and Littlewood*, Ann. Acad. Sci. Fenn. Ser. AI Math. 7 (1982), 113–117.
[16] STEINMETZ, N., *Locally univalent functions in the unit disk*, Preprint, Universität Karlsruhe, 1983.
[17] SULLIVAN, D., *Quasiconformal homeomorphisms and dynamics II*, preprint 1982.

Dept. of Mathematics
University of Michigan
Ann Arbor, Michigan 48109
USA

Fachbereich Mathematik
Technische Universität
1000 Berlin 12
W. Germany

Received June 10, 1983