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Commutators of C^∞-diffeomorphisms. Appendix to "A Curious Remark Concerning the Geometric Transfer Map" by John N. Mather

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We use the notation and reference numbering from Mather's paper. The references which are particular to this appendix are marked [A1] etc. Let M be a connected C^∞-manifold of dimension n.

The point of this paper is to present an elementary proof of the perfectness of $D^\infty(M)^0$. This proof follows almost exactly the lines of Mather's original paper [10] for the case $n+1<r<\infty$. We will not keep on reiterating that the proof here is the same as Mather's original proof. For the convenience of readers we will repeat certain portions of Mather's paper and readers are expected to realize that no claim of originality is being made. That paper relies eventually on an application of the Leray–Schauder Fixed Point Theorem to a certain compact convex set B_ε of C^r-diffeomorphisms and to a mapping θ of B_ε into itself. The new element of this paper is that this compact subset is chosen in a different way. The problem is that θ tends to expand the size of higher derivatives more and more, as more derivatives are taken, so that it is hard to find a B_ε which is mapped into itself. The solution to this problem is to define B_ε so that the first few derivatives are near those of the identity mapping, and then to allow the higher derivatives to range over a increasingly large but bounded domain. (We take ε to be very large instead of very small.) This idea was told to me by John Mather and independently at about the same time by Francis Sergeraert by letter. Sergeraert also said that Belickii [A1] had used the same idea in a somewhat different context, and referred me to the thesis of his student A. Masson [A2], where another group of diffeomorphisms is proved to be perfect by this method. I am most grateful to Mather and Sergeraert for this suggestion and also to Mather for a subsequent helpful conversation while I was working out the detailed proof.

THEOREM. *Let M be a connected C^∞-manifold and let $D^\infty(M)^0$ be the group of C^∞-diffeomorphisms which are compactly isotopic to the identity. Then $D^\infty(M)^0$ is a perfect group. In fact, the universal cover (defined using the C^∞-topology) of this group is perfect.*

§1. Norms on function spaces

1.1. *Reduction.* The first step is to reduce to the case where M is equal to $\mathbf{R}^n$, and where a given diffeomorphism, which we want to prove is equal to a product of commutators, is C^∞ near to the identity. This is possible because, using a partition of unity, we can easily factorise a compactly supported isotopy to a product of a finite number of small isotopies each supported on a small coordinate neighbourhood. Let $f:U\to\mathbf{R}^m$ be a C^r-function, where U is an open subset of $\mathbf{R}^n$. We define

$$\|f\|_r=\sup_{x\in U}\|D^rf(x)\|.$$

This "seminorm" may be infinite in general. However, we will make use of it only when it is finite. The norm $\|D^rf(x)\|$ is the usual norm of an r-multilinear map between normed vector spaces.

We will also have occasion to consider maps between open subsets of spaces like $S^1\times\mathbf{R}^{n-1}$. Regarding S^1 as $\mathbf{R}/\mathbf{Z}$, we may regard all our maps as being between open subsets of euclidean spaces, which are well-defined up to addition of an additive constant. The seminorm is therefore well-defined.

If $1\le r$, and if f is a diffeomorphism, we write

$$M_r(f)=\sup\{\|f-\mathrm{id}\|_1,\|f\|_2,\ldots,\|f\|_r\}.$$

If $\mathbf{f}=(f_1,\ldots,f_k$ is a k-tuple of C^r-diffeomorphisms, we define $M_r(\mathbf{f})=\sup_{1\le i\le k}M_r(f_i)$.

We recall the formulas

$$D(f\circ g)=Df\circ g\,.\,Dg \tag{1.2}$$

and

$$\begin{aligned}D^r(f\circ g)&=D^rf\circ g\,.\,Dg\times\cdots\times Dg+Df\circ g\,.\,D^rg\\&\quad+\sum C(i,j,\ldots,j_i)D^if\circ g\,.\,D^{j_1}g\times\cdots\times D^{j_i}g.\end{aligned} \tag{1.3}$$

where $C(i,j_1,\ldots,j_i)$ is an integer which is independent of f and g and even of the dimensions of their domains and ranges, $1<i<r$, $j_1+\cdots+j_i=r$ and each $j_s\ge1$. Note that this implies that at least one $j_s\ge2$. The second formula is proved by induction on r, using the first formula.

We see that

$$M_1(f \circ g) \leq M_1 f(1+M_1 g)+M_1 g \tag{1.4}$$

by writing

$$f \circ g-\mathrm{id}=(f-\mathrm{id}) \circ g+(g-\mathrm{id}).$$

By an *admissible polynomial* we will mean a polynomial whose coefficients are non-negative integers, and which has no constant or linear term. From (1.3) it follows that for $r \geq 2$ there is an admissible polynomial $F_{1,r}$ of two variables, such that

$$\|g \circ h\|_r \leq \|g\|_r(1+M_1(h))^r+\|h\|_r(1+M_1(g))^r+F_{1,r}(M_{r-1}(g), M_{r-1}(h)). \tag{1.5}$$

$F_{1,2}$ may be taken to be zero.

1.6 PROPOSITION. *For each $r \geq 2$ and each $k \geq 2$, there is an admissible polynomial $F_{2,k,r}$ of one variable with the following property. Let $\mathbf{f}=(f_1, f_2, \ldots, f_k)$, where the domains and ranges of $f_1, \ldots, f_k$ are such that $f_1 \circ \cdots \circ f_k$ makes sense. Then*

$$\|f_1 \circ \cdots \circ f_k\|_r \leq k\,\|\mathbf{f}\|_r(1+M_1(\mathbf{f}))^{r(k-1)}+F_{2,k,r}(M_{r-1}(\mathbf{f})). \tag{1.6.1}$$

Moreover,

$$M_1(f_1 \circ \cdots \circ f_k) \leq k M_1(\mathbf{f})(1+M_1(\mathbf{f}))^{k-1} \tag{1.6.2}$$

Proof. The second formula follows by induction on k from (1.4). The first formula follows by induction on $k+r$ from (1.5).

1.7 LEMMA. *For each $r \geq 2$ and $k \geq 2$, there is an admissible polynomial $F_{3,r}$ of one variable, with the following property. Let g be a diffeomorphism of $\mathbf{R}^n$ and let $M_1(g) \leq 1/2$ and $r \geq 2$. Then*

$$\|g^{-1}\|_r \leq(1+M_1(g))^{3(r+1)}\|g\|_r+F_{3,r}(M_{r-1}(g)). \tag{1.7.1}$$

Also

$$M_1(g^{-1}) \leq M_1(g)(1+M_1(g))^2 \leq 3 M_1(g). \tag{1.7.1}$$

Proof. For any isomorphism A of a Banach space (which is $\mathbf{R}^n$ in this case), we have $\|A^{-1}\| \le (1-\|\mathrm{id}-A\|)^{-1}$ provided that $\|\mathrm{id}-A\|<1$. Hence $\|g^{-1}\|_1 \le (1-M_1(g))^{-1}$, and since $M_1(g)\le 1/2$, $\|g^{-1}\| \le (1+M_1(g))^2 \le 3$. Therefore

$$M_1(g^{-1}) = \|g^{-1}-\mathrm{id}\|_1 = \|(\mathrm{id}-g)\circ g^{-1}\|_1 \le M_1(g)\,\|g^{-1}\|_1 \le 3M_1(g).$$

This proves the second inequality and enables us to estimate first derivatives when proving the first inequality. Next note that by (1.3),

$$\begin{aligned} D^r g^{-1}\circ g = {} & -Dg^{-1}\circ g\,.\,D^r g\,.\,(Dg\times\cdots\times Dg)^{-1} \\ & -\sum C(i, j_1, \ldots, j_i)(D^i g^{-1})\circ g\,.\,(D^{j_1}g\times\cdots\times D^{j_i}g) \\ & \times(Dg\times\cdots\cdot\times Dg)^{-1}. \end{aligned} \tag{1.8}$$

The first inequality follows easily.

§2. A criterion for conjugacy

We now describe Mather's technique, giving a sufficient (but not a necessary) condition for two diffeomorphisms of $\mathbf{R}^n$ to be conjugate.

Let $A>2$ be a fixed large number (exactly how large will emerge in due course). Let $\alpha:\mathbf{R}\to[0,1]$ be a C^∞-function which is equal to 1 on $(-\infty,0)$, is equal to zero on $(1,\infty)$, and has negative derivative on $(-1,1)$. We define $\rho:\mathbf{R}\to[0,1]$ by $\rho=1$ on $[-2A,2A]$, $\rho(t)=\alpha(t-2A)$ for $t\ge 2A$ and $\rho(t)=\alpha(-t+2A)$ for $t\le -2A$. Abusing notation, we define $\rho:\mathbf{R}^n\to\mathbf{R}$ by

$$\rho(x_1,\ldots,x_n)=\rho(x_1)\cdots\rho(x_n).$$

The support of ρ is $[-2A-1,2A+1]^n$.

Let

$$\begin{aligned} D_i &= [-2,2]^i\times[-2A,2A]^{n-i} \\ &= \{x\in\mathbf{R}^n : -2\le x_j\le 2 \text{ for } 1\le j\le i \text{ and } -2A\le x_j\le 2A \text{ for } i<j<n\}. \end{aligned}$$

Then

$$[-2,2]^n = D_n \subset D_{n-1} \subset\cdots\subset D_0 = [-2A,2A]^n.$$

The construction we are about to give depends on i, but this will often be suppressed in the notation. Let ∂_i be the unit vector field in the direction of the

i-th coordinate axis. Let $\tau = \exp(\rho\partial_i)$ be the time one integral of the vector field $\rho\partial_i$. Then τ is a diffeomorphism of $\mathbf{R}^n$ with support $[-2A-1, 2A+1]^n$. Let $\varphi: \mathbf{R}^n \to \mathbf{R}^n$ be the function defined by $\varphi_* \partial_i = \rho\partial_i$ and $\varphi(x) = x$ if $x_i = 0$. This means we have n ordinary differential equations, one for each coordinate of φ. Since ρ has compact support, φ is defined on all of $\mathbf{R}^n$. We see that $\varphi(x) = x$ for $x \in [-2A, 2A]^n$ and that φ defines a diffeomorphism from $\{x : |x_j| < 2A+1$ for each $j \neq i\}$ onto $(-2A-1, 2A+1)^n$. An alternative definition of φ is to let ψ_t be the 1-parameter group of diffeomorphisms corresponding to the vector field $\rho\partial_i$. Then

$$\varphi(x_1, \ldots, x_n) = \psi_{x_i}(x_1, \ldots, x_{i-1}, 0, \ldots, x_n).$$

From now on we will change the meaning of φ so that it refers to the diffeomorphism onto $(-2A-1, 2A+1)^n$, and not to the map with domain the whole of $\mathbf{R}^n$.

Let

$$T(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, x_i + 1, \ldots, x_n).$$

Now φ transforms ∂_i into $\rho\partial_i$. Therefore φ transforms $T = \exp\partial_i$ into $\tau = \exp(\rho\partial_i)$. In other words, $\varphi T = \tau\varphi$.

We now describe Mather's process for "rolling up" a diffeomorphism u with support in $[-2A, 2A]^n$. Let $C = \mathbf{R}^{i-1} \times S^1 \times \mathbf{R}^{n-i}$, and let $\pi: \mathbf{R}^n \to C$ be the projection obtained by regarding S^1 as $\mathbf{R}/\mathbf{Z}$. Let $p: \mathbf{R}^n \to \mathbf{R}^{n-1}$ be the projection which omits the i-th factor. Abusing notation, we also write $p: C \to \mathbf{R}^{n-1}$ be the similar projection defined on C.

Let u be supported on $[-2A, 2A]^n$ and suppose that $\|u - \mathrm{id}\|_0 \leq 1/2$. We define $\Gamma(u): C \to C$ as follows. Let $\vartheta \in C$ and let $x \in \mathbf{R}^n$ be such that $\pi x = \vartheta$ and $x_i < -2A$. We choose a positive integer N sufficiently large so that $((Tu)^N x)_i > 2A$. (Explicitly, one could take $-2A - 1 \leq x_i < -2A$ and $N = [8A + 4]$.) We define

$$\Gamma(u)(\vartheta) = \pi(Tu)^N(x).$$

Then $\Gamma(u)$ is a diffeomorphism of C, whose inverse is given by a similar construction using $(Tu)^{-N}$, but with the representative $x \in \mathbf{R}^n$ of $\vartheta \in C$ being chosen so that $x_i > 2A$. Note that if for some $j \neq i$, $|\vartheta_j| > 2A$, then $\Gamma(u)(\vartheta) = \vartheta$. We also note that $\Gamma(\mathrm{id}) = \mathrm{id}_c$.

We claim that Γ is a continuous map from the space of all C^r-diffeomorphisms u supported on $[-2A, 2A]^n$, such that $\|u - \mathrm{id}\|_0 \leq \frac{1}{2}$, to the space of C^r-diffeomorphisms of C with compact support, provided that we give the C^r-topology to both domain and range of Γ. To see this, we first prove the following lemma.

2.1 LEMMA. *Let M be a smooth manifold. Then the group of C^r-difleomorphisms with compact support forms a topological group under composition provided one takes the C^r-topology on the group.*

Proof. Using uniform continuity of the drivatives, (1.3) implies that composition is continuous, and 1.8 implies that taking the inverse is continuous.

Since A is fixed, we can take N to be fixed and the result follows since the composite of $2N$ diffeomorphisms depends continuously on each of the diffeomorphisms. Therefore $\Gamma(u)$ depends continuously on u. Since we may take N in the definition of Γ to be $[8A+4]$, it follows from 1.6.1 that there is a universal constant K_1, such that

$$\|\Gamma(u)\|_r \leq K_1 A \, \|u\|_r (1+M_1(u))^{rK_1A} + F_{4,A,r}(M_{r-1}(u)), \tag{2.2}$$

where $F_{4,A,r}$ is an admissible polynomial of one variable. Also from 1.6.2,

$$M_1(\Gamma(u)) \leq K_1 A M_1(u)(1+M_1(u))^{K_1A}. \tag{2.3}$$

The manifold C has an obvious S^1-action. Let G be the group of S^1-diffeomorphisms of C. The following proposition is proved in Mather [10] pp. 521–523.

2.4 PROPOSITION. *Let u and v be C^r-diffeomorphisms $(1 \leq r \leq \infty)$ of $\mathbf{R}^n$ with support in $[-A, A]^n$. If u and v are sufficiently C^1-close to the identity, and if $\Gamma(v_i)\Gamma(u_i^{-1}) \in G$, then τu and τv are conjugate elements of D_n^∞. Moreover the conjugating diffeomorphism depends continuously on u and v, if C^r-topologies are used throughout.*

3. Construction of the mappings Ψ_i.

We define, for each diffeomorphism u supported in $(-2,2)^{i-1} \times (-2A, 2A)^{n-i+1}$, such that u is sufficiently C^1-close to the identity, a diffeomorphism $v = \Psi_i(u)$, supported in $(-2,2)^i \times (-2A, 2A)^{n-i}$. Ψ_i is continuous in the C^r-topology for each $r \geq 1$, and therefore v is also C^1-close to the identity. Ψ_i has the property that $\tau_i u$ and $\tau_i v$ are conjugate. Here $\tau_i = \tau$, the diffeomorphism described in the previous section, and Proposition 2.4, the main result of that section, will be used to guide us in the construction of v.

Let $C_i = C = \mathbf{R}^{i-1} \times S^1 \times \mathbf{R}^{n-i}$ as in the preceding section. We define g to be the unique S^1-diffeomorphism of C^i such that

$$g\,|\{\vartheta_i=0\}=\Gamma(u)|\,\{\vartheta_i=0\}.$$

Explicitly,

$$g(x_1,\ldots,x_{i-1},\vartheta_i,x_{i+1},\ldots,x_n)$$
$$=\Gamma(u)(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_n)+(0,\ldots,0,\vartheta_i,0,\ldots,0).$$

To see that g is a diffeomorphism, note that $\Gamma(u)$ is C^1 near the identity, and so the same is true for g. Clearly, $\|g\|_r\le\|\Gamma(u)\|_r$, so that 2.2 implies

$$\|g\|_r\le K_1A\ \|u\|_r(1+M_1(u))^{rK_1A}+F_{4,A,r}(M_{r-1}(u)),\tag{3.1}$$

and similarly

$$M_1(g)\le K_1AM_1(u)(1+M_1(u))^{K_1A}.\tag{3.2}$$

If $M_1(u)\le 1/A$, then for some universal constant K_2

$$M_1(g)\le AM_1(u)K_2,\tag{3.3}$$

and for some admissible polynomial $F_{5,A,r}$ of one variable

$$\|g\|_r\le A\ \|u\|_rK_2r+F_{5,A,r}(M(u,r-1))\tag{3.4}$$

Now take $M_1(u)\le 1/2K_2A$. Then $M_1(g)\le\frac{1}{2}$. We can therefore apply (1.7.1), (1.7.2), (3.3) and (3.4) to deduce

$$M_1(g^{-1})\le 4K_2AM_1(u)\tag{3.5}$$

and for some universal constant K_3 and some admissible polynomial $F_{6,A,r}$

$$\|g^{-1}\|_r\le A\ \|u\|_rK_3^r+F_{6,A,r}(M_{r-1}u).$$

Now let $h=g^{-1}\Gamma(u)$. Then h depends continuously on U by Lemma 2.4, and $h=\mathrm{id}_c$ if $u=\mathrm{id}_{\mathbf{R}^n}$. On $\{\vartheta\in C:\vartheta_i=0\}$, h is the identity. From (1.4), (1.5), (2.2), (2.3), (3.4), (3.5), we see that for any $r\ge 2$ there is a universal constant K_4 and an admissible polynomial $F_{7,A,r}$ such that

$$\|h\|_r\le A\ \|u\|_rK_4^r+F_{7,A,r}(M_{r-1}u)\tag{3.6}$$

and

$$M_1(h) \le K_4 A M_1(u), \tag{3.7}$$

provided $u \in N_A$, where N_A is a certain neighbourhood of the identity in the space of C^1-diffeomorphisms which are supported in $[-2A, 2A]^n$. Note that this neighbourhood is independent of r, though it does depend on A and n.

Provided that N_A is small enough, we can lift $h - \mathrm{id}: C \to C$ to a mapping $\gamma: C \to \mathbf{R}^n$, such that $\pi\gamma = h - \mathrm{id}$ and $\|\gamma\|_0 < \frac{1}{2}$. Here $\pi: \mathbf{R}^n \to C$ is the obvious projection. Let ζ be a bump function which is equal to 1 in a neighbourhood of 0 and to 0 in a neighbourhood of $\frac{1}{2}$. We define

$$h_0(x_1, \ldots, \vartheta_i, \ldots, x_n) = \pi(\zeta(\vartheta_i)\gamma(x_1, \ldots, \vartheta_i, \ldots, x_n)) + (x_1, \ldots, \vartheta_i, \ldots, x_n).$$

Then γ and h_0 depend continuously on u, and, if $u = \mathrm{id}$, then $h_0 = \mathrm{id}$. Therefore, reducing the size of N_A if necessary, we see that h_0 is a diffeomorphism. Moreover h_0 is the identity on a neighbourhood of $\{\vartheta \in C: \vartheta_i = \frac{1}{2}\}$ and also on

$$\{\vartheta = (x_1, \ldots, \vartheta_i, \ldots, x_n) \in C: |x_j| > 2A\}$$

for each $j \neq i$. We define $h_1 = h_0^{-1}h$. Then h_1 depends continuously on u, and is the identity if u is the identity.

By the Leibniz formula for the derivative of a product, we have for some universal constant K_5 and for some admissible polynomial $F_{8,A,r}$,

$$\sup(\|h_0\|_r, \|h_1\|_r) \le AK_5^r \|u\|_r + F_{8,A,r}(M_{r-1}u).$$

The actual value of γ is used in the Leibniz formula, but this can be estimated from the first derivative, since h is equal to the identity when $\vartheta_i = 0$. Moreover,

$$\sup(M_1(h_0), M_1(h_1)) \le AK_5 M_1(u)$$

provided $u \in N_A$, where N_A is some small C^1-neighbourhood, independent of r, of the identity in the space of C^1-diffeomorphisms of $\mathbf{R}^n$ supported on $[-2A, 2A]^n$.

Let

$$E_- = \{x \in \mathbf{R}^n: ; -1 < x_i < 0\}$$

and let

$$E_+ = \{x \in \mathbf{R}^n: \tfrac{1}{2} < x_i < \tfrac{3}{2}\}.$$

We define $v:\mathbf{R}^n \to \mathbf{R}^n$ by the following equations:

$$v \mid \mathbf{R}^n \backslash (E_+ \cup E_-) = \mathrm{id},$$
$$\pi v \mid E_+ = h_0 \pi \mid E_+, \qquad v(E_+) = E_+,$$
$$\pi v \mid E_- = h_1 \pi \mid E_-, \qquad v(E_-) = E_-,$$

We set $\Psi_i(u) = v$. Then v is a diffeomorphism of $\mathbf{R}^n$ with support in $[-2, 2]^i \times [-2A, 2A]^{n-i}$. It is easy to see that $\Gamma(v) = h_0 h_1 = h$. Hence $\Gamma(u)\Gamma(v)^{-1} = g \in G$, where g is the diffeomorphism defined above. By Proposition 2.4, $\tau_i u$ is conjugate to $\tau_i v$, where τ_i is defined as τ in §2. Let $\tau_i v = \lambda_i \tau_i u \lambda_i^{-1}$, where λ_i depends continuously on u (see §4 of the paper to which this is an appendix), and is equal to the identity when $u = \mathrm{id}$. In fact, if we put in the parameters, λ_i continues to be smooth. We have

$$\|v\|_r \leq AK_5^r + F_{8,A,r}(M_{r-1}u)$$

and

$$M_1(v) \leq AK_5 M_1(u).$$

(4.1) The domain of Ψ_i is a certain C^1-neighbourhood $U_{i,A}$ of the identity in the space of C^1-diffeomorphisms of $\mathbf{R}^n$ with support in $[-2, 2]^{i-1} \times [-2A, 2A]^{n-i+1}$. If u is in the domain of Ψ_i, then u is linearly isotopic to the identity. The range of Ψ_i is the space of C^1-diffeomorphism with support in $[-2, 2]^i \times [-2A, 2A]^{n-i}$.

(4.2) $\Psi_i(\mathrm{id}) = \mathrm{id}$.

(4.3) If u is C^r, then so is $\Psi_i(u)$.

(4.4) The restriction of Ψ_i to the set of C^r-diffeomorphisms in its domain is continuous with respect to C^r-topologies on both the domain and range of Ψ_i.

(4.5) If u is in the domain of Ψ_i and if u is C^r, then $[u] = [\Psi_i u]$ in the commutator quotient group of D_n^r.

(4.6) There is a universal constant $K > 1$ and an admissible polynomial F_r for each $r \geq 2$, such that for u in a certain C^1-neighbourhood U_A of the identity, which depends on A and on n, but not on r and not on i,

$$\|\Psi_i u\|_r \leq AK^r \|u\|_r + F_r(M_{r-1}u)$$

and

$$M_1(\Psi_i u) \leq AKM_1 u.$$

Given a C^∞-diffeomorphism f, supported in $[-2,2]^n$, and C^1 near to the identity, we want to write f as a product of commutators. We do this as follows. If u is near to the identity and has support in $[-2,2]^n$, we write, for a suitable large A,

$$u_0 = g_A \circ f \circ u \circ g_A^{-1},$$

where g_A has compact support and is equal to scalar multiplication by A on $[-2A, 2A]^n$. We choose g_A so that it is compactly isotopic to the identity. For example, when $n=1$, we can take the linear isotopy, and when $n>1$, we can take an n-fold product of the 1-dimensional situation. The support of u_0 is contained in $[-2A, 2A]^n$. We regard u_0 as a function of u. We define

$$u_1 = \Psi_1(u_0), \ldots, u_n = \Psi_n(u_{n-1}),$$

so that $\operatorname{supp} u_i \subset [-2,2]^i \times [-2A, 2A]^{n-i}$. The above construction is possible if $u_i \in U_A$ for $1 \le i \le n$. Now by 1.4

$$M_1(u_0) = \|u_0 - \mathrm{id}\|_1 = M_1(f \circ u) \le M_1 f + M_1 g + M_1 f.M_1 g. \tag{5.1}$$

Since each Ψ_i is continuous with respect to the C^1-topology, there is a neighbourhood V_A of the identity in the space of C^1-diffeomorphisms with support in $[-2,2]^n$, such that if $f, u \in V_A$, then $u_0, u_1, \ldots, u_n$ are all defined (i.e. $u_0, u_1, \ldots, u_{n-1} \in U_A$).

We have

$$\|u_0\|_r = A^{1-r}\,\|fu\|_r$$

$$\le A^{1-r}((\|f\|_r + \|u\|_r)(1 + M_1 f + M_1 u)^r + F_{1,r}(M_{r-1}u)). \tag{5.2}$$

where $F_{1,r}$ is the admissible polynomial defined in 1.5. From Property 4.6), we see by induction on i that

$$M_1(u_i) \le A^i K^i M_1(u_0) \tag{5.3}$$

and

$$\|u_i\|_r \le A^i K^{ri}\,\|u_0\|_r + F_{i,A,r}(M_1 u_0), \tag{5.4}$$

where K is a universal constant and $F_{i,A,r}$ is an admissible polynomial.

Let u and f be two diffeomorphisms in a suitable neighbourhood V_A of the identity supported on $[-2,2]^n$. We have

$$M_1(u_n) \leq A^n K^n (M_1 f + M_1 u) \tag{5.5}$$

and

$$\|u_n\|_r \leq A^{n+1-r} K^m (\|f\|_r + \|u\|_r) + F_{r,A}(M_{r-1}u + M_{r-1}f). \tag{5.6}$$

where K is a universal constant and $F_{r,A}$ is an admissible polynomial.

By choosing A large enough, we can ensure that $A^{n+1-r}K^m \leq \frac{1}{4}$ for $r \geq n+2$. We then choose ε_{n+2} small enough so that if $\sup(\|u\|_{n+2}, \|f\|_{n+2}) \leq \varepsilon_{n+2}$, then $u, f \in V_A$ and $\|u_n\|_{n+2} \leq \varepsilon_{n+2}$. To see that it is possible to choose ε_{n+2} in this way, note that we can estimate $\|u\|_i$ and $\|f\|_i$ for $1 \leq i \leq n+1$ by repeated integration over an interval of length at most $4\sqrt{n}$. In fact, for each $i \geq 2$, $\|u\|_i \leq (4\sqrt{n})^{n+2}\|u\|_{n+2}$ and $M_1 u \leq (4\sqrt{n})^{n+2}\|u\|_{n+2}$ and similarly for f. Note that A is now fixed, and so in 5.6, $F_{n+2,A}(M_{n+1}u + M_{n+1}f)$ can be estimated by a definite polynomial in ε_{n+2}. This polynomial has no linear or constant terms.

We now choose $\varepsilon_{n+3}, \varepsilon_{n+4}, \ldots$ inductively, so that if $\|u\|_i \leq \varepsilon_i$ for $n+2 \leq i \leq r$, then $\|u_n\|_i \leq \varepsilon_i$ for $n+2 \leq i \leq r$. Although ε_{n+2} is small, each ε_i is chosen large compared with ε_{i-1}. To see that this is possible, recall that f and A are now fixed and that $A^{n+1-r}K^m \leq \frac{1}{4}$ for $r \geq n+2$. Suppose $\varepsilon_{n+2}, \ldots, \varepsilon_{r-1}$ are all chosen, and that $\|u\|_i \leq \varepsilon_i$ for $n+2 \leq i < r$. Then

$$\|u_n\|_r \leq (\tfrac{1}{4})(\|f\|_r + \|u\|_r) + a_r \tag{5.7}$$

for some constant $a_r > 1$. So we define $\varepsilon_r = \|f\|_r + 2a_r$, and the induction can continue.

Let L be the following set of diffeomorphisms

$$L = \{u : \|u\|_i \leq \varepsilon_i \text{ for } n+2 \leq i < \infty \text{ and } \operatorname{supp} u \subset [-2,2]^n\}.$$

This is a convex subspace of the Frechet space of C^∞-maps from $\mathbf{R}^n \times I$ to $\mathbf{R}^n$. By Ascoli's Theorem and the Cantor diagonalization process, L is compact in the C^∞-topology.

The map $\vartheta : L \to L$, which sends u to u_n, is a continuous map to which the Schauder Fixed Point Theorem can be applied. Since u_n is equal to $f \circ u$ modulo the commutator subgroup, the existence of a fixed point shows that f is in the commutator subgroup.

Finally, we have to prove that the universal cover of $D^\infty(M)^0$ is perfect. We note that in the above proof, τ_i and g_A are fixed diffeomorphisms, and we for the moment we do not take isotopies of these diffeomorphisms to the identity. We take u and f so close to the identity, that linear isotopes of u and f to the identity are mapped to paths connecting $u_0, \ldots, u_n$ to the identity, which are homotopic to the linear paths. The conjugating diffeomorphisms λ_i, referred to in Proposition 2.4, also depend continuously on u and f. The isotopies of u and f to the identity therefore give rise to isotopies of the λ_i to the identity. We take u and f so close to the identity, that the linear isotopy of λ_i to the identity is homotopic to the isotopy produced by the functional dependence on u and f. Now arrange for the isotopies of u and f to the identity to take place over $\frac{1}{2} \leq t \leq 1$, and the isotopies of g_A and τ_i to the identity to take place over $o \leq t \leq \frac{1}{2}$. In this way we see that f, together with the linear isotopy to the identity, is in the commutator subgroup. Since any connected topological group is generated by any neighbourhood of the identity, we see that the universal cover must be perfect.

References

See the main paper by Mather.
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