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Autor(en): De La Harpe, Pierre / McDuff, Dusa

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# Acyclic groups of automorphisms

PIERRE DE LA HARPE and DUSA MCDUFF<sup>(1)</sup>

### **1. Introduction**

A discrete group  $\Gamma$  is said to be acyclic if its Eilenberg-MacLane homology groups  $H_i(\Gamma)$  with coefficients in the trivial  $\Gamma$ -module  $\mathbb{Z}$  are zero for all i > 0. In this paper we show that certain groups, such as the group GL(V) of all continuous linear automorphisms of an infinite dimensional Hilbert space V, are acyclic. This is a folk theorem which was surely known long ago to experts in the field such as Quillen and Segal. However it seems worthwhile to publish a proof in view of the recent interest shown in such questions. For example, Herman pointed out in [He] that the group of diffeomorphisms of a compact manifold admits a canonical representation in GL(V). Therefore, if GL(V) had carried non-trivial cohomology, one might have been able to define non-trivial characteristic classes for groups of diffeomorphisms. See also section 2.6 in [Ma] and the concluding remark of [H2].

We will consider the following groups.

1. The group  $\Sigma(X)$  of all permutations of an infinite set X.

2. The group  $\mathscr{A}(\Omega)$  of measure preserving automorphisms of a Lebesgue measure space  $(\Omega, \mathcal{B}, \mu)$  where  $\mu$  is infinite and non-atomic. (As usual one identifies two automorphisms which agree  $\mu$ -a.e.)

3. The group GL(W) of all linear automorphisms of an infinite dimensional vector space W.

4. The group GL(V) of all continuous linear automorphisms of an infinite dimensional Hilbert space V over the real, complexes or quaternions, as well as the group U(V) of invertible isometries of V.

5. The group GL(M) of invertible elements in a properly infinite von Neumann algebra M, and the subgroup U(M) of unitary elements.

THEOREM. The groups defined above are acyclic.

The above list is by no means complete. One could add many "classical

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groups" in the sense of [H3], and also the group of continuous linear automorphisms of an infinite dimensional topological vector space E for suitable E. The Banach spaces  $c_0$  and  $l_p$ ,  $1 \le p < \infty$ , are possible candidates: see proposition 2.a.2 in [LT]. However Douady [D] constructs a Banach space E for which the group of connected components of GL(E) is isomorphic to  $\mathbb{Z}$ . It follows that GL(E) is not perfect and hence not acyclic. Therefore the above theorem does not hold for GL(E) where E is an arbitrary Banach space. See also [St]. For acyclic groups of a quite different nature from those of our list, see [BDH] and [BDM].

Here is one consequence of the theorem.

COROLLARY. Let G be one of the groups above and let A be a finitely generated abelian group. Then any extension

 $0 \rightarrow A \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ 

is trivial.

**Proof.** Any non-trivial normal subgroup of G is of uncountable index. (See Appendix 1.) In particular any homomorphism from G to Aut (A) is trivial and so G acts trivially on A in the above extension. Our main theorem implies that  $H^2(G; A)$  is zero. Hence the extension is a semi-direct product. Again using the fact that the action of G on A is trivial, we see that the product is direct.

A notable feature of the groups in 2, 4 and 5 is that they are contractible when given their natural topologies. (See [Ke] for  $\mathcal{A}(\Omega)$ , [DD] for U(V) and U(M)with the strong topology, [Ku] for GL(V) and U(V) with the uniform topology, and [BW] for GL(M) and U(M) with the uniform topology.) There are many other contractible groups of automorphisms which are acyclic when considered as discrete groups: for example, the group of compactly supported homeomorphisms of  $\mathbb{R}^n$  [M], and the group of diffeomorphisms of  $\mathbb{R}^n$  which are the identity near the origin [Se]. On the other hand, Sah pointed out that the universal cover  $SL(2, \mathbf{R})$  of  $SL(2, \mathbf{R})$  is contractible as a topological group but is not acyclic as a discrete group [SW]. The main tool which we use in proving acyclicity is the infinite repetition argument of Mather [M] and Wagoner [W]. (See also [BDH] §4 and [Be] ch. 3.) There are several contractible groups which are more "flexible" than  $SL(2, \mathbf{R})$ , but are still not large enough for this argument to be used. We have in mind groups such as  $\mathcal{A}(\Omega)$ , where  $\Omega$  has finite measure, or the group of compactly supported homeomorphisms of  $\mathbb{R}^n$  which preserve Lesbegue measure, for n > 2. These groups are known to be perfect [F1], [F2], and it would be interesting to know if they are acyclic. One could also consider much bigger groups such as the group of all homeomorphisms of a Hilbert cube or a Hilbert

space. These are shown to be contractible in [Re]. The groups GL(M) and U(M), where M is a finite continuous von Neumann factor, are not contractible. They are discussed further in section 4.

The theorem is not hard to prove. We first show that the subgroup  $G_F$  of elements in G which are the identity on an appropriately defined "flag" F is acyclic. Then we show, using a technique due to Segal (§2 in [Se]), that this forces the whole group G to be acyclic. The first of these two steps uses the infinite repetition argument of [M] and [W] and, in the general case, an elegant algebraic trick due to Quillen [Q2]. The second step works essentially because the Tits building (or partially ordered set) formed by the flags is contractible. We give the proof for GL(V) in full detail, and in section 4 sketch the modifications needed for the other groups.

We discuss in Appendix 1 the results about normal subgroups of G needed for the corollary above. Though these are old results, we indicate for GL(W) and GL(V) a proof much shorter than the originally published ones. Doing this, we again show that G is perfect, namely that  $H_1(G)$  is trivial. This is what our main result and proof reduce to when cleared from homological machinery.

Finally Appendix 2 describes a result due to Quillen according to which the monoids (or semi-groups) related to our groups are contractible and hence acyclic.

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### **2.** Subgroups of GL(V)

In this section and the next one, V denotes an infinite dimensional Hilbert space. Let F be a flag in V: we mean by this that F is a nested sequence  $S_1 \supset S_2 \supset S_3 \supset \cdots$  of closed subspaces of  $V = S_0$  such that  $S_{i-1}/S_i$  is isomorphic to V for each  $i \ge 1$ . Define

 $G_i = \{g \in GL(V) \mid g = \text{id on } S_i\}$ and  $G'_i = \{g \in G_i \mid g(S_i^{\perp}) = S_i^{\perp}\}$ 

for each  $i \ge 0$ . Define also  $G_{\infty}$  to be the union of the  $G_i$ 's and  $G'_{\infty}$  that of the  $G'_i$ 's. Then

$$1 = G_0 \subset G_1 \subset \cdots \subset G_i \subset \cdots \subset G_{\infty}$$
$$\parallel \bigcup \qquad \bigcup \qquad \bigcup$$
$$G'_0 \subset G'_1 \subset \cdots \subset G'_i \subset \cdots \subset G'_{\infty}.$$

For  $g \in G_{\infty}$ , observe that g = id on  $S_{\infty} = \bigcap S_i$ . For notational convenience, we assume  $S_{\infty} = \{0\}$ . (But proposition 1 as well as its consequences in section 3 and the variations of section 4 would obviously hold without this assumption.) The result of this section is:

**PROPOSITION** 1. The groups  $G'_{\infty}$  and  $G_{\infty}$  are acyclic.

We shall recall the following facts from §2 in [W]. A *flabby* group is a group  $\Gamma$  such that there exist homomorphisms

 $\boldsymbol{\mu}: \boldsymbol{\Gamma} \times \boldsymbol{\Gamma} \to \boldsymbol{\Gamma} \quad (\text{direct sum})$ 

 $\tau: \Gamma \to \Gamma$  (infinite repetition)

with the following properties: For any finite subset  $\Phi \subset \Gamma$ , there are elements a, b, c in  $\Gamma$  satisfying

(1)  $g\mu 1 = aga^{-1}$ ,  $1\mu g = bgb^{-1}$  where 1 is the identity element in  $\Gamma$ ,

(2)  $g\mu\tau(g) = c\tau(g)c^{-1}$ for all  $g \in \Phi$ .

LEMMA 2 (Wagoner). A flabby group is acyclic.

Sketch of proof. Any inner automorphism of  $\Gamma$  acts trivially on homology. By (1), this implies first that  $\mu$  induces a (non associative) ring structure  $\mu_*: H_*(\Gamma) \otimes H_*(\Gamma) \to H_*(\Gamma)$  on homology, with two-sided unit the number 1 in  $H_0(\Gamma) = \mathbb{Z}$ . By (2), this implies also that  $\mu(\mathrm{id} \times \tau) \Delta$  and  $\tau$  act the same way on homology, where  $\Delta: \Gamma \to \Gamma \times \Gamma$  is the diagonal map.

Let *i* be an integer, i > 0, and assume inductively that  $H_n(\Gamma)$  is trivial for 0 < n < i (this holds trivially if i = 1). Choose  $z \in H_i(\Gamma)$ . By the Künneth formula

$$\Delta_{\mathbf{*}}(z) = z \otimes 1 + 1 \otimes z \in H_{\mathbf{i}}(\Gamma) \otimes H_{\mathbf{0}}(\Gamma) + H_{\mathbf{0}}(\Gamma) \otimes H_{\mathbf{i}}(\Gamma) = H_{\mathbf{i}}(\Gamma \times \Gamma)$$

so that

$$(\boldsymbol{\mu}(\mathrm{id}\times\tau)\boldsymbol{\Delta})_{\boldsymbol{\ast}}(z) = \boldsymbol{\mu}_{\boldsymbol{\ast}}(z\otimes 1+1\otimes\tau_{\boldsymbol{\ast}}(z)) = z+\tau_{\boldsymbol{\ast}}(z)\in H_{i}(\Gamma).$$

As this must coincide with  $\tau_*(z)$  one has z = 0. Hence  $H_i(\Gamma)$  is trivial.

LEMMA 3. The group  $G'_{\infty}$  is flabby.

**Proof.** Let  $T_0^0$  be a Hilbert space isomorphic to V. For any pair (j, k) of positive integers, let  $T_i^k$  be a copy of  $T_0^0$ . We identify V and  $T = \bigoplus_k \bigoplus_j T_j^k$  in such

a way that

$$S_i = \bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$$

(where  $\bigoplus_k \text{ means } \bigoplus_{k=0}^{\infty}$ ). For each  $j \ge 0$  define an isometry  $\rho_j$  from  $\bigoplus_k T_j^k$  onto  $T_j^0$  and an isometry (shift)  $\sigma_j$  from  $\bigoplus_k T_j^k$  onto  $\bigoplus_{k=1}^{\infty} T_j^k$  with  $\sigma_j(T_j^k) = T_j^{k+1}$  for all  $k \ge 0$ . Denote by  $\rho$  the isometry  $\bigoplus_j \rho_j$  from T onto  $\bigoplus_j T_j^0$  and by  $\sigma$  the shift  $\bigoplus_j \sigma_j$ . Define the maps

$$\boldsymbol{\mu}: \begin{cases} GL(T) \times GL(T) \to GL(T) \\ (g, h) \to \rho g \rho^* + \sigma h \sigma^* \end{cases}$$

and

$$\tau: \begin{cases} GL(T) \to GL(T) \\ g \to \sum_{k} \sigma^{k} \rho g \rho^{*} \sigma^{*k} \end{cases}$$

(The series converges strongly, and  $\rho^*$  is the adjoint of  $\rho$ ; in view of section 4, it is appropriate to define  $\rho^*$  by  $\rho^*(\xi) = \eta$  if  $\eta = \rho(\xi) \in \text{Im}(\rho)$  and  $\rho^*(\xi) = 0$  if  $\xi \perp \text{Im}(\rho)$ .)

It is easy to check that  $\mu$  and  $\tau$  are homomorphism because  $\rho$  and  $\sigma$  are isometries with orthogonal complementary ranges. Similarly  $\mu(id \times \tau)\Delta = \tau$ . For each  $i \ge 0$  one has  $\mu(G'_i \times G'_i) \subset G'_i$  and  $\tau(G'_i) \subset G'_i$  because  $\rho_i \rho_i^* + \sigma_i \sigma_i^*$  coincides with the identity on  $\bigoplus_k T_i^k$  for  $j \ge i$ . It follows that  $\mu$  and  $\tau$  induce homomorphisms  $G'_{\infty} \times G'_{\infty} \to G'_{\infty}$  and  $G'_{\infty} \to G'_{\infty}$ , denoted below by  $\mu$  and  $\rho$  again. Requirement (2) in the definition of a flabby group obviously holds (with c = 1).

Consider some integer  $i \ge 0$ . Let  $a_i$  be an invertible isometry of T which acts as  $\bigoplus_{j=0}^{i-1} \rho_j$  on  $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$ , as the identity on  $\bigoplus_k \bigoplus_{j=i+1}^{\infty} T_j^k$ , and (thus) maps in some way  $\bigoplus_k T_i^k$  onto

$$\left(\bigoplus_{k=1}^{\infty}\bigoplus_{j=0}^{i-1}T_{j}^{k}\right)\oplus\left(\bigoplus_{k}T_{i}^{k}\right).$$

One has  $a_i \in G'_{i+1} \subset G'_{\infty}$  and  $a_i g a_i^* = g \mu 1$  for all  $g \in G'_i$ . Similarly, let  $b_i$  be an invertible isometry of T which acts as  $\bigoplus_{j=0}^{i-1} \sigma_j$  on  $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$  and as the identity on  $\bigoplus_k \bigoplus_{j=i+1}^{\infty} T_j^k$ . Then  $b_i \in G'_{i+1}$  and  $b_i g b_i^* = 1 \mu g$  for all  $g \in G'_i$ . It follows that requirement (1) above holds.

We know thus that  $G'_{\infty}$  is acyclic. The reader who is interested in U(V) and not in GL(V) may skip the end of this section since  $G_{\infty} \cap U(V) = G'_{\infty} \cap U(V)$ .

Let us now recall what we need from a result due to Quillen (theorem 1' of [Q2]). Let A be a **Q**-algebra with unit, let  $\Gamma$  be the group of invertible  $(2 \times 2)$ -matrices over A which have the form  $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ , let  $\Gamma'$  be the subgroup of  $\Gamma$  consisting of diagonal matrices and let  $\pi: \Gamma \to \Gamma'$  be the homomorphism defined by

$$\pi \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

If R is a  $\mathbb{Z}[\Gamma]$ -module, we denote by  $H_i(\Gamma, R)$  the *i*<sup>th</sup> Eilenberg-MacLane homology group of  $\Gamma$  with coefficients in R; moreover R is assumed to have the trivial  $\mathbb{Z}[\Gamma]$ -structure if there is no strong reason for any other one (such as  $R = H_t(N; \mathbf{K})$  below).

LEMMA 4 (Quillen). Let **K** be a field which is either finite or the rationals. Then  $\pi$  induces an isomorphism on  $H_*(-; \mathbf{K})$ .

**Proof.** Let N be the subgroup of  $\Gamma$  consisting of matrices of the form  $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ , which is isomorphic to the additive group of the algebra A. As N is torsion-free and abelian,  $H_{\mathbf{x}}(N; \mathbf{Z})$  is isomorphic to the additive group  $\bigwedge_{\mathbf{Z}} N$ . (This holds for finitely generated free abelian groups, as one checks knowing homology of compact tori; this holds in general because N and the inductive limit of finitely generated subgroups of N have the same homology.) It follows that  $H_{\mathbf{x}}(N; \mathbf{K}) \approx (\bigwedge_{\mathbf{Z}} N) \otimes_{\mathbf{Z}} \mathbf{K}$  for any field **K**. In particular  $H_{\mathbf{x}}(N; \mathbf{K}) = H_0(N; \mathbf{K}) = \mathbf{K}$  if **K** is finite (because N is divisible) and  $H_{\mathbf{x}}(N; \mathbf{Q}) = \bigwedge_{\mathbf{Q}} A$ . (This is a highly degenerate form of the results described in §8 of [Q2].)

Consider the Hochschild-Serre spectral sequence

$$E_{s,t}^{2} = H_{s}(\Gamma'; H_{t}(N; \mathbf{K})) \Rightarrow H_{s+t}(\Gamma; \mathbf{K})$$

corresponding to the extension

 $0 \to N \to \Gamma \to \Gamma' \to 1.$ 

If **K** is a finite field, one has  $H_t(N; \mathbf{K}) = 0$  for t > 0 and  $H_0(N; \mathbf{K}) = \mathbf{K}$ . The spectral sequence therefore degenerates, giving the desired result.

Suppose  $\mathbf{K} = \mathbf{Q}$ . Make  $\mathbf{Q}^*$  act on  $\Gamma$  by

$$\lambda \cdot \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a & \lambda b \\ 0 & 1 \end{pmatrix}.$$

Thus  $\lambda \in \mathbf{Q}^*$  acts on the exact sequence

$$0 \to N \to \Gamma \to \Gamma' \to 1$$
$$\downarrow^{\lambda} \qquad \qquad \downarrow^{\lambda} \qquad \qquad \downarrow^{\text{id}}$$
$$0 \to N \to \Gamma \to \Gamma' \to 1$$

and consequently also on the spectral sequence. As  $\lambda \in \mathbf{Q}^*$  acts on  $H_t(N; \mathbf{Q}) = \bigwedge_{\mathbf{Q}}^t (N \otimes_{\mathbf{Z}} \mathbf{Q})$  by multiplying by  $\lambda^t$ , and acts trivially on  $\Gamma'$ , it follows that  $\lambda$  acts on  $E_{s,t}^2$  by multiplying by  $\lambda^t$ . Assume  $\lambda \neq \pm 1$ ; as the differentials commute with the  $\mathbf{Q}^*$ -action and as  $\lambda^t \neq \lambda^{t'}$  for  $t \neq t'$ , all differentials are zero. It follows that

 $E_{s,t}^2 = E_{s,t}^\infty$  for all  $s, t \ge 0$ .

Now  $\bigoplus_{s+t=n} E_{s,t}^{\infty}$  is the graded object associated to the natural filtration of  $H_n(\Gamma; \mathbf{Q})$  for each integer  $n \ge 1$ . Since  $\mathbf{Q}^*$  acts on  $\Gamma$  by inner automorphisms, the induced action on  $H_n(\Gamma; \mathbf{Q})$  is trivial; thus  $\mathbf{Q}^*$  acts trivially on each  $E_{s,t}^{\infty}$ . Hence  $E_{s,t}^{\infty} = 0$  for any (s, t) with  $s \ge 0$  and t > 0. This shows that  $H_s(\Gamma'; \mathbf{Q}) = H_s(\Gamma; \mathbf{Q})$  for any  $s \ge 0$ .

COROLLARY 5 (a universal coefficient argument). The homomorphism  $\pi: \Gamma \to \Gamma'$  induces an isomorphism on  $H_*(-) = H_*(-; \mathbb{Z})$ .

**Proof.** We know that  $\pi$  induces an isomorphism for  $H_*(-; R)$  if R is the additive group of a finite field. Using direct products and extensions of the coefficients, one checks the same holds for R a finite abelian group. As homology commutes with inductive limits of coefficients, this holds also when R = Q/Z. Using the sequence

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0$$

and the fact that  $\pi_*$  is an isomorphism for R = Q and R = Q/Z, one proves the claim.

The proof of Proposition 1. We use again the notations defined earlier in this section, and we denote by L(V) the algebra of all bounded operators on V. For each i > 0 the spaces  $S_i^{\perp}$  and  $S_i$  are both isomorphic to V. It follows that  $G_i$  is isomorphic to

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a, b \in L(V) \text{ with } a \text{ invertible} \right\}$$

and that  $G'_i$  consists of matrices in  $G_i$  with b = 0. Quillen's argument shows that the inclusion of  $G'_i$  in  $G_i$  induces an isomorphism  $H_*(G'_i) \approx H_*(G_i)$ . It follows that the inclusion of  $G'_{\infty}$  in  $G_{\infty}$  induces also an isomorphism  $H_*(G') \approx H_*(G)$ , so that the proof of proposition 1 is complete.

Let us end this section by two observations. First the groups of our main theorem are not flabby. Consider for example G = U(V) with V an infinite dimensional separable complex Hilbert space, and suppose there exists a "direct sum" homomorphism  $\mu: G \times G \to G$  with property (1) preceding lemma 2; we shall reach a contradiction.

Choose an orthonormal basis  $(e_j)_{j \in N}$  of V and a sequence  $(\lambda_j)_{j \in N}$  of pairwise distinct numbers in the interval  $]-\pi, \pi[$ . Define  $r \in G$  by  $r(e_j) = \exp(i\lambda_j)e_j$  for  $j \in N$ . The centralizer of r in G is the abelian group T of unitary operators which are diagonal with respect to the chosen basis.

Consider the homomorphism  $\mu_1: G \to G$  given by  $g \mapsto \mu(g, 1)$ . By hypothesis  $\mu_1(g)$  is conjugate to g. Therefore,  $\mu_1$  is injective and, because its image commutes with  $\mu(1, r)$ , the centralizer of  $\mu(1, r)$  is not abelian. But there exists  $b \in G$  with  $\mu(1, r) = brb^{-1}$ . Therefore the centralizer of  $\mu(1, r)$  is the abelian group  $bTb^{-1}$ . This contradiction shows that G is not flabby.

The second observation is that there are plenty of (non trivial) G-modules R with non trivial  $H_*(G, R)$  or  $H^*(G, R)$ . Consider for example a subgroup  $G_1$  of G and a  $G_1$ -module  $R_1$ . Let  $R = \text{Hom}_{\mathbb{Z}G_1}(\mathbb{Z}G, R_1)$ , where  $\mathbb{Z}G$  is considered as a left  $\mathbb{Z}G_1$ -module and as a right  $\mathbb{Z}G$ -module; then R is naturally a G-module (namely a left  $\mathbb{Z}G$ -module). A standard result known as Shapiro's lemma states that  $H^n(G_1, R_1)$  is naturally isomorphic with  $H^n(G, R)$  for all  $n \ge 0$ ; see for example §34.2 in [Bab]. Choose in particular a finite cyclic subgroup  $G_1$  of G and let  $R_1$  be a trivial  $G_1$ -module isomorphic to  $G_1$  as abelian group. Then  $H^n(G, R) \ne 0$  for all  $n \ge 0$ .

This is quite a general construction. Indeed, let  $\Gamma$  be any group with more than one element. One shows by induction from a (possibly infinite) cyclic subgroup of  $\Gamma$  that there exists a  $\Gamma$ -module M and an integer n > 0 with  $H^n(\Gamma; M) \neq 0$ .

#### 3. The set of flags

Let Gr be the set of those closed subspaces S of V which are isomorphic to V/S. (Thus Gr is the set of points in a Grassmannian space.)

LEMMA 6. Let  $\{S_1, \ldots, S_p\}$  be a finite subset of Gr. There exist  $S'_1, \ldots, S'_p \in Gr$ with  $S'_m \subset S_m$   $(1 \leq m \leq p)$  and  $S'_m \perp S'_n$   $(1 \leq m < n \leq p)$ .

**Proof.** Any subspace of V whose codimension is strictly smaller than the dimension of V intersects non trivially any element of Gr. One may thus choose unit vectors as follows

$$v_{1,1} \in S_1, v_{2,1} \in S_2 \cap \{v_{1,1}\}^{\perp}, \ldots, v_{p,1} \in S_p \cap \{v_{1,1}, \ldots, v_{p-1,1}\}^{\perp}$$

and in general

$$v_{1,i} \in S_1 \cap \{v_{1,1}, \ldots, v_{p,1}, \ldots, v_{1,i-1}, \ldots, v_{p,i-1}\}^{\perp}, \\ \dots, \\ v_{p,i} \in S_p \cap \{v_{1,1}, \ldots, v_{p,1}, \ldots, v_{1,i}, \ldots, v_{p-1,i}\}^{\perp}.$$

(The index *i* runs over  $\mathbb{N}^*$  if *V* is separable and over some suitable infinite set if *V* is "larger".) Define  $S'_m$  to be the closed linear span of the  $v_{m,i}$ 's. Then  $S'_1, \ldots, S'_p$  have the desired properties.

LEMMA 7. Let  $S_1, \ldots, S_p \in Gr$  and let  $h_1, \ldots, h_p \in GL(V)$ . There exist  $S'_1, \ldots, S'_p \in Gr$  with  $S'_m \subset S_m$   $(1 \leq m \leq p)$ ,  $S'_m \perp S'_n$  and  $h_m(S'_m) \perp h_n(S'_n)$   $(1 \leq m < n \leq p)$ .

**Proof.** By Lemma 6 there exist  $S''_1, \ldots, S''_p \in Gr$  with  $S''_m \subset S_m$   $(1 \le m \le p)$  and  $S''_m \perp S''_n$   $(1 \le m < n \le p)$ . Define  $T_m = h_m(S''_m)$   $(1 \le m \le p)$ . There exist also  $T'_1, \ldots, T'_p \in Gr$  with  $T'_m \subset T_m$   $(1 \le m \le p)$  and  $T'_m \perp T'_n$   $(1 \le m \le p)$ . Define  $S'_m = h_m^{-1}(T'_m)$   $(1 \le m \le p)$ .

Now consider the set  $\mathfrak{F}$  of flags  $F = \{S_1 \supset S_2 \supset \cdots\}$  with  $\bigcap S_i = \{0\}$  as defined in section 2. Let  $F = \{S_1 \supset S_2 \supset \cdots\}$ ,  $F' = \{S'_1 \supset S'_2 \supset \cdots\}$  and  $h \in GL(V)$ . We write  $F' \leq F$  if  $S'_i \subset S_i$  for all *i*. If  $S'_i \perp S_i$ , we write  $F' \perp F$ . If in addition  $S_1 \oplus S'_1 \in Gr$ , the spaces  $S_1 \oplus S'_1 \supset S_2 \oplus S'_2 \supset \cdots$  form a flag which we call  $F' \oplus F$ . Finally the flag  $\{h(S_1) \supset h(S_2) \supset \cdots\}$  is called h(F).

We may reformulate lemma 7 for flags.

LEMMA 8. Let  $F_1, \ldots, F_p \in \mathfrak{F}$  and let  $h_1, \ldots, h_p \in GL(V)$ . There exist  $F'_1, \ldots, F'_p \in \mathfrak{F}$  with  $F'_m \leq F_m$   $(1 \leq m \leq p)$ ,  $F'_m \perp F'_n$  and  $h_m(F'_m) \perp h_n(F'_n)$   $(1 \leq m < n \leq p)$ .

**Proof.** Let  $F_m = \{S_{m,1} \supset S_{m,2} \supset \cdots \}$  and write  $T_{m,i} = S_{m,i}^{\perp} \cap S_{m,i-1}$  where  $S_{m,0} = V$  $(1 \le m \le p \text{ and } i \ge 1)$ . Then  $S_{m,i} = \bigoplus_{j=i+1}^{\infty} T_{m,j}$ . The result now follows by applying lemma<sup>7</sup> 7 to the spaces  $T_{1,j}, \ldots, T_{p,j}$  for each  $j \ge 1$ .

We review now the Milnor construction for classifying space (see e.g. [Hu], chap. 4, §11). Given any (discrete) group  $\Gamma$ , let  $E\Gamma$  be the simplicial complex whose *p*-simplices are the ordered subsets  $(\gamma_0, \ldots, \gamma_p)$  of  $\Gamma$ . We denote by  $|E\Gamma|$  the topological space obtained by realizing  $E\Gamma$ . It is well-known and easy to see that  $|E\Gamma|$  is contractible (compare the proof of lemma 10 below). Moreover the group  $\Gamma$  acts freely on  $|E\Gamma|$  by multiplication on the left. Thus the quotient space  $B\Gamma = \Gamma \setminus |E\Gamma|$  is a model (the "infinite join" model) for the classifying space of the group  $\Gamma$ . In particular this means that the groups  $H_i(\Gamma)$  ( $i \in \mathbb{N}$ ) are just the integral homology groups of the space  $B\Gamma$ .

For the rest of this section, we will write G for GL(V), E for EGL(V) and B for BGL(V). For each flag  $F = \{S_1 \supset S_2 \supset \cdots\}$  in  $\mathfrak{F}$ , let  $G_F$  be the subgroup of G containing those operators which agree with the identity on  $S_i$  for *i* large enough, and let  $E_F$  be the subcomplex of E defined as follows: a k-simplex  $(g_0, \ldots, g_k)$  of E is in  $E_F$  if  $g_0, \ldots, g_k$  agree on  $S_i$  for *i* large enough. (For short, we will say that  $g_0, \ldots, g_k$  agree on F.) Let  $F, F' \in \mathfrak{F}$ . If  $F' \leq F$ , observe that  $G_F \subset G_{F'}$  and that  $E_F$  is a subcomplex of  $E_{F'}$ . If  $F \perp F'$  and if  $F \oplus F' \in \mathfrak{F}$ , then  $G_{F \oplus F'} = G_F \cap G_{F'}$ .

LEMMA 9. For any  $F \in \mathfrak{F}$ , the complex  $E_F$  is G-invariant and the quotient  $G \setminus |E_F|$  is naturally isomorphic to  $BG_F$ .

*Proof.* "Naturally" means that, if  $F, F' \in \mathfrak{F}$  with  $F' \leq F$ , then the map  $BG_F \rightarrow BG_{F'}$  induced by  $G_F \hookrightarrow G_{F'}$  is just the inclusion of  $BG_F$  in  $BG_{F'}$  (both are subspaces of B).

The space  $|E_F|$  is not connected. Indeed two 0-simplices (g) and (g') define points lying in the same connected component if and only if there is a sequence of 1-simplices in  $E_F$  of the form

$$(g, g_1), (g_1, g_2), \ldots, (g_m, g').$$

This holds if and only if g and g' agree on F, namely if and only if g and g' belong to the same right coset of  $G_F$  in G. It follows that connected components of  $|E_F|$  are parametrized by  $G/G_F$ . The coset  $G_F$  corresponds to  $|E'_F|$ , where  $E'_F$  is the subcomplex of  $E_F$  consisting of simplices  $(g_0, \ldots, g_k)$  where  $g_0, \ldots, g_k$  agree with the identity on F.

It is clear that  $E_F$  is G-invariant. It follows from the discussion above that  $G \setminus |E_F|$  may be identified with  $G_F \setminus |E'_F|$ , which is nothing but the infinite join model  $BG_F$  for the classifying space of  $G_F$ .

Let  $E_*$  be the union of the  $E_F$ 's over  $F \in \mathfrak{F}$ ; it is a subcomplex of E which is invariant by G. Let  $B_* = G \setminus |E_*|$ ; it is a subspace of B which is the union of the  $G \setminus |E_F|$ 's over F in \mathfrak{F}.

LEMMA 10. The space  $E_*$  is contractible.

**Proof.** Let  $\sigma_1, \ldots, \sigma_p$  be simplices in  $E_*$ . Choose

$$F_1 = \{S_{1,1} \supset S_{1,2} \supset \cdots \}, \ldots, F_p = \{S_{p,1} \supset S_{p,2} \supset \cdots \}$$

in  $\mathfrak{F}$  with  $\sigma_m \in E_{F_m}$ . There is an integer k such that the vertices in  $\sigma_m$  agree on  $S_{m,k}$ ; denote by  $h_m$  their common restriction on  $S_{m,k}$   $(1 \le m \le p)$ . Let  $F'_1, \ldots, F'_p$  be as in lemma 8: one has  $\sigma_m \in E_{F'_m} (1 \le m \le p)$ . Then the cone on  $\sigma_1 \cup \cdots \cup \sigma_p$  with vertex  $h_0$  is in  $E_*$ .

It follows that, for any finite subcomplex K of  $E_*$ , there exists a subcomplex L of  $E_*$  containing K and contracting to a point. Hence  $|E_*|$  itself is c  $\cdot$  tractible (see e.g. corollary 7.6.24 in [Sp]).

LEMMA 11. The inclusion  $B_* = \bigcup_{F \in \mathfrak{F}} BG_F \to B = BG$  is a homotopy equivalence.

*Proof.* Since the quotient maps  $|E| \rightarrow B$  and  $|E_*| \rightarrow B_*$  are covering maps, this follows immediately from the two previous lemmas.

The following lemma holds for p = 1 by section 2.

LEMMA 12. Let  $F_1, \ldots, F_p \in \mathfrak{F}$ . Then  $BG_{F_1} \cup \cdots \cup BG_{F_p}$  is contained in an acyclic subspace of  $B_*$ .

**Proof.** Choose any flag  $F_0 \in \mathfrak{F}$ . By Lemma 8 there exist  $F'_0, F'_1, \ldots, F'_p \in \mathfrak{F}$  with  $F'_m \leq F_m$   $(0 \leq m \leq p)$  and  $F'_m \perp F'_n$   $(0 \leq m < n \leq p)$ ; in particular  $F'_1 \oplus \cdots \oplus F'_p$  is a flag in  $\mathfrak{F}$ . As  $BG_{F_m} \subset BG_{F'_m}$   $(1 \leq m \leq p)$ , it suffices to check that  $BG_{F'_1} \cup \cdots \cup BG_{F'_p}$  is acyclic. Hence we may assume without loss of generality that  $F_m \perp F_n$   $(1 \leq m < n \leq p)$  and that  $F_1 \oplus \cdots \oplus F_p \in \mathfrak{F}$ .

Let us assume as induction hypothesis that, in this situation, both

$$BG_{F_1} \cup \cdots \cup BG_{F_{p-1}}$$
 and  $BG_{F_1 \oplus F_{p-1}} \cup \cdots \cup BG_{F_{p-2} \oplus F_{p-1}}$ 

are acyclic. (When p = 2, the former works by proposition 1 and the latter is vacuous.)

Consider first the Mayer-Vietoris homology sequence of the subcomplexes

$$BG_{F_1 \oplus F_p} \cup \cdots \cup BG_{F_{p-2} \oplus F_p}$$
 and  $BG_{F_{p-1} \oplus F_p}$ 

of  $B_*$  with intersection

$$BG_{F_1\oplus(F_{p-1}\oplus F_p)}\cup\cdots\cup BG_{F_{p-2}\oplus(F_{p-1}\oplus F_p)}$$

By the induction hypothesis, two of any three consecutive terms in this sequence vanish. Hence all terms vanish and

 $BG_{F_1\oplus F_p}\cup\cdots\cup BG_{F_{p-1}\oplus F_p}$ 

is acyclic.

Consider now the Mayer-Vietoris sequence of the subcomplexes

 $BG_{F_1} \cup \cdots \cup BG_{F_{n-1}}$  and  $BG_{F_n}$ 

of  $B_*$  with intersection

 $BG_{F_1\oplus F_p}\cup\cdots\cup BG_{F_{p-1}\oplus F_p}$ 

From the previous step and from the induction hypothesis it follows that

 $BG_{F_1} \cup \cdots \cup BG_{F_n}$ 

is acyclic.

THEOREM 13. The group G is acyclic.

**Proof.** The homology of a complex is generated by that of its finite subcomplexes. Thus lemma 12 implies that  $B_*$  is an acyclic space, and lemma 11 that G is acyclic.

#### 4. Variations

Unitary group U(V) of an infinite dimensional Hilbert space V.

The proof that U(V) is acyclic is much simpler than for GL(V) since section 2 may be reduced to Lemmas 2 and 3. Section 3 is unchanged.

#### Symmetric group $\Sigma(X)$ of an infinite set X

Here a flag is a nested sequence  $\{S_1 \supset S_2 \supset \cdots\}$  of subsets of  $X = S_0$  such that  $S_{i-1} - S_i$  is equipotent with X for each  $i \ge 1$  and such that  $\bigcap S_i = \emptyset$ . Define

 $\Sigma_i = \{g \in \Sigma(X) \mid g = \text{id on } S_i\}$ 

for each  $i \ge 0$  (no distinction here between  $\Sigma'_i$  and  $\Sigma_i$ ) and  $\Sigma_{\infty} = \bigcup_{i=0}^{\infty} \Sigma_i$ . The argument of Lemma 3 shows that  $\Sigma_{\infty}$  is a flabby group. Read "disjoint union" instead of "direct sum", "injection" instead of "isometry". The adjoint  $\rho^*$  of an injection  $\rho$  is defined only on the image of  $\rho$  by  $\rho^*\rho = id$ ; then a formula like  $\rho g \rho^* + \sigma h \sigma^*$  is clear because  $\rho g \rho^*$  is a permutation of some subset of X and  $\sigma h \sigma^*$  is a permutation of its complement. The group  $\Sigma_{\infty}$  is consequently acyclic.

Let Gr be the set of those subsets S of X equipotent with their complements  $S^{\perp} = X - S$ . For two subsets  $S_1$ ,  $S_2$  of X, read  $S_1 \cap S_2 = \emptyset$  for  $S_1 \perp S_2$ . Lemmas 7 and 8 may then be repeated without change and all of section 3 with minor changes only. It follows that  $\Sigma(X)$  is acyclic.

#### Automorphism group $\mathcal{A}(\Omega)$ of a Lebesgue space $(\Omega, \mathcal{B}, \mu)$

Let  $(\Omega, \mathcal{B}, \mu)$  be a Lebesgue space where the measure  $\mu$  is infinite and non atomic. A flag is now a nested sequence  $F = \{S_1 \supset S_2 \supset \cdots\}$  of measurable subsets of  $\Omega = S_0$  such that  $S_{i-1} - S_i$  has infinite measure for each  $i \ge 1$  and such that  $\bigcap S_i$  has measure zero. Comments for  $\sum (X)$  above apply to  $\mathcal{A}(\Omega)$ , with the understanding that everything in view is now measurable. Therefore  $\mathcal{A}(\Omega)$  is also acyclic.

Let  $(\tilde{\Omega}, \mathcal{B}, \mu)$  be a Lebesgue measure space. Let X be the set of atoms in  $\tilde{\Omega}$ , let  $X = \coprod_j X_j$  be the partition of X according to the masses of the atoms, and let  $\Omega = \tilde{\Omega} - X$ . Then the sequence

$$1 \to \mathscr{A}(\Omega) \to \mathscr{A}(\tilde{\Omega}) \to \prod_{j} \sum (X_{j}) \to 1$$

is exact (and splits). Suppose  $\mu(\Omega) = \infty$ , and suppose that X is not empty. Then  $\mathcal{A}(\tilde{\Omega})$  is clearly acyclic if and only if each  $X_i$  is either one point or an infinite set.

Automorphisms of an infinite dimensional vector space W over a (possibly skew) field  $\mathbf{F}$ 

Case (i): Char  $\mathbf{F} = 0$ .

A flag is in this case a nested sequence  $\{S_1 \supset S_2 \supset \cdots\}$  of subspaces of  $W = S_0$ such that  $S_{i-1}/S_i$  is isomorphic to W for each  $i \ge 1$  and such that  $\bigcap S_i = \{0\}$ . As in Lemma 3 we may identify W with  $\bigoplus_k \bigoplus_j T_j^k$ , where each  $T_j^k \cong W$ , in such a way that  $S_i = \bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$  for all *i*. Then the subspace  $R_i = \bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$  complements  $S_i$ .

Define

 $G_i^{\mathbf{W}} = \{ g \in GL(W) \mid g = \text{id on } S_i \},$  $G_i^{\mathbf{W}'} = \{ g \in G_i^{\mathbf{W}} \mid g(R_i) = \mathbf{R}_i^{\mathbf{Y}} \}.$ 

One checks as in Lemma 3 that  $G_{\infty}^{W'}$  is flabby. When Char  $\mathbf{F} = 0$ , Lemma 4 and 5 show that  $G_{\infty}^{W}$  is acyclic.

In Lemmas 6 to 8, understand  $S'_m \perp S'_n$  as  $S'_m \cap S'_n = \{0\}$ , and  $v \in S \cap \{v_1, \ldots, v_m\}^{\perp}$  as  $v \in S$  with v not in the linear span of  $\{v_1, \ldots, v_m\}$ . Then section 3 holds for GL(W), which is consequently an acyclic group. All our arguments allow the field **F** to be non-commutative.

Case (ii): Char  $\mathbf{F} = p > 0$ .

The arguments of section 2 show that  $\tilde{H}_*(G^W_\infty; \mathbf{K}) = 0$  if  $\operatorname{Char} \mathbf{K} \neq \operatorname{Char} \mathbf{F}$ (where  $\tilde{H}_*$  denotes reduced homology). It follows that  $\tilde{H}_*(GL(W); \mathbf{K}) = 0$  when  $\operatorname{Char} \mathbf{K} \neq \operatorname{Char} \mathbf{F}$ . Therefore, in order to show that GL(W) is acyclic, it will suffice to prove that  $\tilde{H}_*(GL(W); \mathbf{K}) = 0$  when **K** is the algebraic closure  $\mathbf{k}$  of the finite field  $\mathbf{k}$  with p elements. To do this we need

LEMMA 14. For each flag F and integer d > 0 there is a subgroup  $G_F^d$  of GL(W) which contains  $G_F$  and is such that  $H_i(G_F^d; \bar{\mathbf{k}}) = 0$  for 0 < j < d.

Proof. Quillen proves the following lemma in [Q2] §9.

LEMMA. Let  $\bar{\mathbf{k}}$  be an algebraically closed field and d an integer >0. Then there exists an order D in a number field of degree d over  $\mathbf{Q}$  with the following properties: Given any D-module N, let the group of units D<sup>\*</sup> act on it by multiplication, and let the group homology  $H_*(N, \bar{\mathbf{k}})$  be endowed with the induced action of D<sup>\*</sup>. Then for each t,  $H_t(N, \bar{\mathbf{k}})$  is a direct sum of one-dimensional representations of D<sup>\*</sup> over  $\bar{\mathbf{k}}$ . Furthermore,  $H_t(N, \bar{\mathbf{k}})$  does not contain the trivial representation for 0 < t < d.

Let D be as in this lemma. The choice of a basis over  $\mathbb{Z}$  for D gives rise to a ring homomorphism

$$\rho_0: D \to M_d(\mathbf{Z}) \to M_d(\mathbf{F})$$

where  $M_d(A)$  is the ring of *d*-by-*d* matrices over A and where  $M_d(\mathbb{Z}) \to M_d(\mathbb{F})$  is reduction mod *p*. Let *F* be the flag  $\{S_1 \supset S_2 \supset \cdots\}$ . For each pair (j, k) of positive integers, let now  $T_j^k$  be a copy of  $\mathbb{F}^d$ . We identify *W* and  $T = \bigoplus_k \bigoplus_j T_j^k$  in such a way that  $S_i = \bigoplus_k \bigoplus_{j=i}^{\infty} T_j^k$ , and we denote by  $R_i$  "the" complement  $\bigoplus_k \bigoplus_{j=0}^{i-1} T_j^k$ of  $S_i$ . Define a ring homomorphism  $\rho_i : D \to GL(W)$  by setting

$$\rho_i(\lambda) = \begin{cases} \rho_0(\lambda) & \text{in } T_j^k \text{ for } j \ge i, \text{ all } k \\ \text{id} & \text{in the other } T_j^k. \end{cases}$$

Now put

$$G_i^d = \{g \in GL(W) \mid g = \rho_i(\lambda) \text{ in } S_i \text{ for some } \lambda \in D^*\}$$

and let  $G_F^d = \bigcup_{i \ge 1} G_i^d$ . Clearly  $G_F \subset G_F^d$ . We must show that  $H_j(G_F^d; \mathbf{\bar{k}}) = 0$  for 0 < j < d.

Let

$$G_i^{d'} = \{g \in G_i^d \mid g(R_i) = R_i\}.$$

and consider the induced  $D^*$ -action on the spectral sequence of the extension  $0 \rightarrow N \rightarrow G_i^d \rightarrow G_i^{d'} \rightarrow 1$ . It follows from the lemma that each  $E_{st}^r$ ,  $2 \le r \le \infty$ , breaks up into a sum of one dimensional representations preserved by the differentials. Since  $D^*$  acts trivially on the abutment, the subspaces on which  $D^*$  acts trivially form a spectral sequence which converges to  $H_*(G_i^d; \mathbf{\bar{k}})$ . By the lemma, the terms  $E_{st}^2$  of this sequence vanish when 0 < t < d. Hence  $H_j(G_i^d; \mathbf{\bar{k}}) \cong H_j(G_i^d; \mathbf{\bar{k}})$  for 0 < j < d.

Now note that  $G_i^{d'}$  is the product of  $G_i'$  with  $\rho_i(D^*)$ . But  $\rho_i(D^*)$  is isomorphic to a subgroup of the group of units of  $D/pD \cong \mathbf{k}_d$ , where  $\mathbf{k}_d$  is the field of order  $p^d$ . Hence  $\rho_i(D^*)$  has order prime to p. Therefore  $\tilde{H}_*(\rho_i(D^*); \mathbf{k}) = 0$  which implies that  $H_*(G_i^{d'}; \mathbf{k}) \cong H_*(G_i'; \mathbf{k})$ . Now consider the diagram



We have seen that the inclusions  $\alpha_1$  and  $\alpha_2$  induce an isomorphism on  $H_j(-; \bar{\mathbf{k}})$ , 0 < j < d. Since  $\alpha_4$  factors through a group isomorphic to  $G'_{\infty}$ , it induces the zero map on  $\tilde{H}_j(-; \bar{\mathbf{k}})$ . Hence  $\alpha_3$  must induce the zero map on  $H_j(-; \bar{\mathbf{k}})$ , 0 < j < d. This implies that

$$H_j(G_F^d; \mathbf{\bar{k}}) = \lim_i H_j(G_i^d; \mathbf{\bar{k}}) = 0, \qquad 0 < j < d. \blacksquare$$

To finish the proof of the theorem we must find an appropriate substitute for Lemma 12. If  $F_1, \ldots, F_n$  are disjoint flags such that  $F_1 \oplus \cdots \oplus F_n$  is also a flag, choose groups  $G_{F_1}^d$  as above and, for each subset  $\{i_1, \ldots, i_k\}$  of  $\{1, \ldots, n\}$ , set

$$G^d_{F_{\iota_1}\oplus\cdots\oplus F_{\iota_k}}=G^d_{F_{\iota_1}}\cap\cdots\cap G^d_{F_{\iota_k}}$$

The proof of Lemma 14 shows that these groups  $G_F^d$ , for  $F = F_{i_1} \oplus \cdots \oplus F_{i_k}$ , are acyclic. The inductive argument of Lemma 12 then readily shows that

$$H_{j}(BG_{F_{1}}^{d}\cup\cdots\cup BG_{F_{n}}^{d};\bar{\mathbf{k}})=0 \qquad 0 < j < d-2n.$$

Clearly, this suffices to show that the inclusion  $B_* \hookrightarrow B$  annihilates  $\tilde{H}_*(-; \bar{k})$ .

#### Properly infinite von Neumann algebras

Let *M* be a properly infinite von Neumann algebra, faithfully represented in L(V) for some complex Hilbert space *V*. A flag is a nested sequence  $\{S_1 \supset S_2 \supset \cdots\}$  of closed subspaces of  $V = S_0$  with  $\bigcap S_i = \{0\}$  such that the orthogonal projection  $P_i$  from *V* onto  $S_i$  is in *M* and such that  $P_{i-1} - P_i$  is equivalent to the identity for each  $i \ge 1$ . It is easy to choose every operator appearing in sections 2 and 3 in the algebra *M*. Therefore the appropriately defined groups  $G'_{\infty}$  and  $G_{\infty}$  are acyclic, as well as U(M) and GL(M).

It is likely that the argument applies to a large class of infinite  $C^*$ -algebras. Let B be such an algebra, let M(B) be its multiplier algebra, let U(B) be the subgroup of the unitary group U(M(B)) consisting of those elements g for which  $g-1 \in B$ , and let  $U(B)_0$  be the connected component of U(B) with respect to the norm topology. There are many cases in which  $U(B)_0$  is known to be contractible for the norm topology [Mi]; in these cases,  $U(B)_0$  and the similarly defined "general linear group"  $GL(B)_0$  should "often" be acyclic.

#### Finite von Neumann algebras

Let M be a finite continuous factor, and let U(M) be the group of unitaries in M. When given the norm topology, U(M) has a fundamental group isomorphic to

the additive group of the real numbers: this was first proved in [AS], but it follows also essentially from Bott periodicity as formulated in theorem 1.11 of chapter III of [Ka]. Indeed

$$\pi_i(U(M)_{\text{norm}}) \approx \begin{cases} \mathbf{R} \text{ if } i \text{ is odd, } i \ge 0\\ 0 \text{ if } i \text{ is even, } i > 0 \end{cases}$$

(See III.7.7 in [Ka], or theorem 5 in [Br]; both state the analogous "stable fact", but the isomorphism holds also as above.) Let

$$0 \to \mathbf{R} \to \tilde{U}(M) \to U(M) \to 1$$

be the (topological) universal covering of U(M). It is known that U(M) is perfect (indeed simple up to centre [FH]). One may conjecture that  $\tilde{U}(M)$  is also perfect, namely that the short exact sequence above is still a covering in the algebraic sense of [Ker], and thus that there exists a surjective homomorphism of  $H_2(U(M))$  onto **R**. In any event it seems very unlikely that the group U(M) is acyclic.

#### **Appendix 1. About normal subgroups**

If X is an infinite countable set,  $\Sigma(X)$  has exactly two non trivial normal subgroups: the group  $\Sigma_f(X)$  of permutations of X with finite support and its derived group  $A_f(X)$  of even permutations [SU]. If X is any infinite set, normal subgroups of  $\Sigma(X)$  which are neither trivial nor  $A_f(X)$  are in bijection (via supports) with infinite cardinals smaller than the cardinal of X [B].

If  $(\Omega, \mathcal{B}, \mu)$  is a Lebesgue measure space with  $\mu$  infinite and non atomic,  $\mathcal{A}(\Omega)$  has exactly one non trivial normal subgroup consisting of those bi-measurable transformations  $\alpha$  with support  $\{\omega \in \Omega \mid \alpha(\omega) \neq \omega\}$  of finite measure [F1], [Ei].

If W is an infinite dimensional vector space over a field  $\mathbf{F}$ , normal subgroups of GL(W) have been studied in [R]; we present hereafter part of these results with different proofs inspired by [And], [Ep] and [Hi].

LEMMA A1. The group GL(W) is perfect.

**Proof.** If I is a set and if  $(W_i)_{i \in I}$  is a family of copies of W, we write any element in  $GL(\bigoplus W_i)$  as an  $(I \times I)$ -matrix with coefficients in End (W). If I is countable, we may identify  $\bigoplus W_i$  and W.

In  $GL(W \oplus W \oplus W)$  one has

$$\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for each  $x \in \text{End}(W)$ . It follows that any element of the form  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  in  $GL(W \oplus W)$  is a product of two commutators. In  $GL(\bigoplus_{i \in N} W_i)$ , one may apply the infinite repetition argument used in section 2. We write  $\gamma_1 \sim \gamma_2$  if two elements  $\gamma_1$ ,  $\gamma_2$  in a group  $\Gamma$  are conjugate. For any  $x \in GL(W)$  one has

in  $GL(\bigoplus_{i \in \mathbb{N}} W_i)$ . It follows that any element of the form  $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$  in  $GL(W \oplus W)$  is a commutator.

Let  $g \in GL(W)$ . Choose sequences  $(u_i)$  and  $(v_i)$  of vectors in W as follows:

$$u_1 \in W - \{0\}$$
  $u'_1 = g(u_1)$   $v_1 \in W - \text{span}(u_1, u'_1)$ 

and in general

$$u_{i+1} \in W - \operatorname{span} \begin{pmatrix} u_1 & v_1 & g^{-1}(v_1) \\ \cdot & \cdot & \cdot \\ u_i & v_i & g^{-1}(v_i) \end{pmatrix} \qquad u'_{i+1} = g(u_{i+1})$$
$$v_{i+1} \in W - \operatorname{span} \begin{pmatrix} u_1 & u'_1 & v_1 \\ \cdot & \cdot & \cdot \\ u_i & u'_i & v_i \\ u_{i+1} & u'_{i+1} \end{pmatrix}.$$

(The index *i* runs over  $N^*$  if the dimension of W is countable and over some

suitable set otherwise.) Define

$$U = \text{span}(u_1, u_2, ...) \qquad V_1 = \text{span}(v_1, v_3, ...)$$
$$V_2 = \text{span}(v_2, v_4, ...) \qquad V = V_1 \oplus V_2.$$

It is easy to check that  $U \cap V = \{0\}$  and  $g(U) \cap V = \{0\}$ . Thus there exists  $t \in GL(W)$  with  $tu'_i = u_i$  and  $tv_{2i} = v_{2i}$  for each *i*. As t = id on  $V_2$  one has  $t \sim \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(W \oplus W)$ ; as tg = id on *U* one has  $tg \sim \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \in GL(W \oplus W)$ . It follows from the beginning of the proof that g is a product of commutators in GL(W).

The proof above shows also the following fragmentation lemma: any element in GL(W) may be written as a product of finitely many elements similar to  $\begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix}$  in  $GL(W \oplus W)$ . Indeed, it remains to be checked that  $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$  has this property, and this is clear if one looks at

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

#### in $GL(W \oplus W \oplus W)$ .

Let  $N_{\max}$  be the normal subgroup of GL(W) containing those elements of the form  $\lambda + X$  with  $\lambda$  a homothety and X an endomorphism of W with rank strictly smaller than the dimension of W. Let  $g \in GL(W)$  with  $g \notin N_{\max}$ . Let us check that there exists a subspace V of W with V isomorphic to W/V and with  $V \cap g(V) = \{0\}$ .

One may choose a sequence  $(v_i)$  of vectors in W as follows:

 $v_1 \in W - \{0\}$  with  $g(v_1) \in W - \operatorname{span}(v_1)$ 

and in general

$$v_{i+1} \in W - \operatorname{span} \begin{pmatrix} v_1 & g(v_1) \\ \cdot & \cdot \\ v_i & g(v_i) \end{pmatrix} \quad \text{with} \quad g(v_{i+1}) \in W - \operatorname{span} \begin{pmatrix} v_1 & g(v_1) \\ \cdot & \cdot \\ v_i & g(v_i) \\ v_{i+1} \end{pmatrix}$$

Indeed, suppose one cannot find  $v_{i+1}$ . Let

$$F = \operatorname{span} \begin{pmatrix} v_1 \cdots v_i \\ g(v_1) \cdots g(v_i) \end{pmatrix}.$$

Then  $v \in W - F$  implies  $g(v) \in \text{span}(F, v)$ ; for any  $u \in F$ , one has also  $g(v+u) \in \text{span}(F, v)$ ; hence  $g(u) \in \text{span}(F, v)$ . It follows that F is invariant by g and that g induces a homothety on W/F. But this is ruled out by hypothesis.

Then  $V = \text{span}(v_1, v_2, \ldots)$  has the desired properties.

PROPOSITION A2. Any non trivial normal subgroup of GL(W) is contained in  $N_{max}$ .

**Proof.** Let N be a normal subgroup of GL(W) and assume that  $N \notin N_{max}$ . There exist  $f \in N$  and a subspace V of W with V isomorphic to W/V and with  $f(V) \cap V = \{0\}$ . We may thus view N as a normal subgroup of  $GL(W \oplus W)$  containing an element f of the form  $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ .

By the fragmentation lemma, it is enough to check that N contains any element of the form  $\begin{pmatrix} 1 & 0 \\ 0 & * \end{pmatrix}$ . Consider  $r, s \in GL(W)$  and define  $g = \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix}$ ,  $h = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix}$ . As N is normal, N contains  $\hat{h} = hfh^{-1}f^{-1}$  and  $g\hat{h}g^{-1}\hat{h}^{-1}$ . By a straightforward matrix computation, the latter is of the form

$$g\hat{h}g^{-1}\hat{h}^{-1} = \begin{pmatrix} 1 & * \\ 0 & rsr^{-1}s^{-1} \end{pmatrix}.$$

As GL(W) is perfect, it follows that, for any  $k \in GL(W)$ , there exists  $z \in End(W)$ with  $\begin{pmatrix} 1 & z \\ 0 & k \end{pmatrix} \in N$ .

Let now  $a, b \in GL(W)$  with a+b=1. (One may define a as an infinite direct sum of automorphisms of a vector space of dimension two, each represented by  $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ , and similarly for b with  $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$ .) There exist  $x, y \in End(W)$  with  $\begin{pmatrix} 1 & x \\ 0 & a^{-1} \end{pmatrix}$  and  $\begin{pmatrix} 1 & y \\ 0 & b^{-1} \end{pmatrix}$ 

in N. Then

$$\binom{1 & -z}{0 & 1}\binom{1 & x}{0 & a^{-1}}\binom{1 & z}{0 & 1}\binom{1 & -xa}{0 & a} = \binom{1 & z(a-1)}{0 & 1} \in N$$

and

$$\binom{1}{0} \frac{z(a-1)}{1} \binom{1}{0} \frac{z(b-1)}{1} = \binom{1}{0} \frac{-z}{1} \in N.$$

It follows that

$$\begin{pmatrix} 1 & z \\ 0 & k \end{pmatrix} \begin{pmatrix} 1 & -z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix} \in N$$

the proof is complete.

It would be easy to prove by similar arguments all of theorem B (and thus also theorem A) in [R].

Let now V be an infinite dimensional Hilbert space over the reals, complexes or quaternions and GL(V) be as in the introduction. Let GE(V, C) be the normal subgroup of GL(V) containing those elements of the form  $\lambda + x$  with  $\lambda$  a homothety and X a compact operator (we assume V to be separable). It is quite easy to check that GL(V) is perfect (see problems 191 and 192 in [Hal]). There is a fragmentation lemma which follows straightforwardly from polar decomposition and spectral theorem. Any  $g \in GL(V)$  with  $g \notin GE(V, C)$  is similar to an element of the form  $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$  in  $GL(V \oplus V)$ : this is corollary 3.4 in [BP] or theorem 1 in [AnS]. Hence the proof above applies, and is very much simpler than that of [H1]. The subgroup of GL(V) containing all bijective isometries of V can be handled either as in [H1] or as suggested in [H3], and we have proved the following result.

PROPOSITION A3. Any non trivial normal subgroup of GL(V) is contained in GE(V, C). Any non trivial normal subgroup of U(V) is contained in  $UE(V, C) = U(V) \cap GE(V, C)$ .

For normal subgroups of GL(M) and U(M), when M is a properly infinite von Neumann algebra, see [H3] and papers reviewed there.

COROLLARY A4. Let G be one of the groups described in the introduction and let N be a non trivial normal subgroup of G. Then N is of uncountable index in G.

Let G be as above and let  $N_{\text{max}}$  be the maximal normal subgroup of G. There are cases for which we have information about the homology of  $N_{\text{max}}$ : see works

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by Nakaoka and Priddy [P] if  $G = \sum (X)$  and  $N_{\max} = \sum_f (X)$  with X infinite countable, the papers on group cohomology in [E] if G = GL(W), or [BHS] if G = GL(V). In each case our main theorem provides corresponding information about the homology of the quotient  $G/N_{\max}$ .

#### Appendix 2. About monoids of monomorphisms

Each of the acyclic groups of automorphisms considered above is the group of units in a corresponding monoid (or semigroup) of monomorphisms. For example,  $\sum (X)$  is the group of units in the monoid M(X) formed by all injective maps from X to X. One can form the classifying space BM of a monoid in exactly the same way as that of a group; see [Se]. In particular, the Eilenberg-MacLane homology groups  $H_i(M; \mathbb{Z})$  are just the integral homology groups of the space BM. Quillen pointed out in an unpublished version of [Q1] that the classifying spaces of monoids such as M(X) are contractible. Of course, this implies that the monoids are acyclic.

Here is a sketch of his argument. Say two homomorphisms  $f, g: M \to M$  are semi-conjugate if there is  $m \in M$  such that mf(n) = g(n)m for all  $n \in M$ . The argument is based on the fact that two homomorphisms which are semi-conjugate induce homotopic maps on BM; see [Q1] §1. Choose  $p \in M(X)$  so that the image p(X) of X under p is in Gr. Define  $f: M(X) \to M(X)$  by  $f(n)(x) = pnp^{-1}(x)$  if  $x \in p(X)$  and by f(n)(x) = x otherwise. Then f is semi-conjugate both to the identity homomorphism and to the trivial homomorphism which takes every  $n \in M(X)$  to the identity element. It follows that BM(X) is contractible.

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Section de mathématiques, C.P. 124, 1211 Genève 24

Department of Mathematics SUNY, Stony Brook, NY 11794

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