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On Grauert's conjecture and the characterization of Moishezon spaces

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0. Introduction

In this paper, we pursue the investigation of the global structure of 1-convex spaces which we begun in [1] [10a.b.c]. In this direction, we provide here an affirmative answer to the following:

Grauert's conjecture: Let S be an exceptional set of some 1-convex space X. Then there exists a coherent ideal sheaf $\tilde{J} \subset 0_X$ such that $\tilde{J} \mid S$ is weakly positive.

In the same token, Moishezon spaces can be characterized as follows:

A compact, irreducible $\mathbb{C}$-analytic space is Moishezon iff it carries a torsion free positive analytic coherent sheaf.

This result answers affirmatively a problem posed in [1], [9].

The organization of this paper goes as follows:

In Section 1, the equivalence of various notions of positivity for analytic coherent sheaves over compact $\mathbb{C}$-analytic spaces is established. The crucial vanishing theorem for arbitrary 1-convex spaces is proved in Section 2. In Section 3, we shall tackle Grauert's conjecture as well as the characterization of Moishezon spaces.

In the following, we shall use freely the basic definitions and notations employed in [1], [10a.b.c].

1. The positivity of analytic coherent sheaves

DEFINITION 1. Let S be a compact, irreducible $\mathbb{C}$-analytic space and let $\theta \in \mathrm{Coh}(S)$. Then θ is said to be

(i) *weakly positive* if the zero section of $L(\theta)$ admits a 1-convex neighbor-

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hood, where $L(\theta)$ is the linear fibre space associated to θ in the sense of Grauert [2].

(ii) *cohomologically positive* if $H^i(S, S^k(\theta)\otimes\mathscr{F})=0$ for $\forall i\geq 1$, $k\gg 0$ and $\mathscr{F}\in \mathrm{Coh}(S)$, where $S^k(\theta)$ denotes the k-th fold symmetric tensor product of θ.

(iii) *ample* if $S^k(\theta)\otimes\mathscr{F}$ is generated by its global sections for $k\gg 0$ and any $\mathscr{F}\in \mathrm{Coh}(S)$.

We are now in a position to prove the main result of this section.

THEOREM 1. *Let S be a compact, irreducible $\mathbb{C}$-analytic space and let $\theta\in \mathrm{Coh}(S)$. Then the following conditions are equivalent:*

(i) *θ is weakly positive*

(ii) *θ is cohomologically positive*

(iii) *θ is ample.*

Proof. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) are standard (see e.g. [2], [3], [4], [7]). It remains for us to show that (iii)$\Rightarrow$(i):

So let E be the linear fibre space associated to θ, let $\mathbb{P}(E)$ be the associated projective fibre space and let $L(E):=0_{\mathbb{P}(E)}(-1)$ be the tautological line bundle on $\mathbb{P}(E)$. Notice that there exists a canonical biholomorphism

$$\phi : E\backslash S \simeq L(E)\backslash \mathbb{P}(E) \tag{*}$$

Let $H:=L^*(E)$ and let H be the locally free sheaf associated to H. Then one has the following diagram

$$\begin{array}{ccc} H & & \theta \\ \downarrow & & \downarrow \\ \mathbb{P}(E) & \xrightarrow{p} & S \end{array}$$

and the following isomorphism

$$S^k(\theta)\otimes\mathscr{F}\simeq p_*(H^k\otimes p^*(\mathscr{F})) \tag{\#}$$

for $k\gg 0$ and $\forall \mathscr{F}\in \mathrm{Coh}(S)$ (see [9]).

Claim. H is ample.

In fact, for any $\hat{\mathscr{F}}\in \mathrm{Coh}(\mathbb{P}(E))$ there exists an integer $k\gg 0$, such that the following morphism

$$p^*p_*(\hat{\mathscr{F}}\otimes H^k)\to \hat{\mathscr{F}}\otimes H^k \tag{†}$$

is surjective (see e.g. [7]). Now let us replace $\hat{\mathscr{F}}$ by $p^*(\mathscr{F})$ in (†) where $\mathscr{F} := p_*(\hat{\mathscr{F}} \otimes H^n) \in \mathrm{Coh}(S)$. Thus we obtain the following surjective morphism of analytic coherent sheaves:

$$p^*p_*(p^*\mathscr{F} \otimes H^n) \to p^*\mathscr{F} \otimes H^n \quad \text{for} \quad n \gg 0 \tag{††}$$

Now (††) together with (#) and (†) give us the following surjective composite morphism:

$$p^*(S^N(\theta) \otimes \mathscr{F}) \to \hat{\mathscr{F}} \otimes H^N \quad \text{for} \quad N \gg 0$$

Since θ is ample, this implies the ampleness of the line bundle H. Hence our claim is proved.

In view of result in [3], [4] $L(E)$ is a weakly negative line bundle in the sense of [2]. Consequently (*) tells us that θ is weakly positive. *Q.E.D.*

Remarks. (i) Notice that Theorem 1 is well known in the special case where θ is locally free [2], [3], [4]. Furthermore, Theorem 1 does not hold in general if S is not assumed to be compact.

(ii) From now on, any analytic coherent sheaf satisfying one of the equivalent conditions in Theorem 1, will be called simply positive coherent sheaf.

2. Vanishing theorem for 1-convex spaces

In [10a] we established a vanishing theorem for "embeddable" 1-convex spaces. We would like to present here a crucial vanishing theorem for arbitrary 1-convex spaces.

THEOREM 2. *Let (X, S) be a 1-convex space. Then there exists a coherent ideal sheaf $J \subset 0_X$ supporting on S such that*

$$H^i(X, J^k \otimes \mathscr{F}) = 0 \quad \text{for } \forall i \geq 1, k \gg 0 \textit{ and } \mathscr{F} \in \mathrm{Coh}(X)$$

Proof. Step 1. Since X is 1-convex, there exist a Stein space Y and a proper, surjective and holomorphic morphism $\pi: X \to Y$ inducing a biholomorphism $X \backslash S \simeq Y \backslash T$ where $T := \pi(S)$ (see e.g. [2]). Now the main result in [5] tells us that there exist:

(i) a coherent ideal sheaf $m_T \subset 0_Y$ supporting on T

(ii) a monoidal transformation $\chi:\tilde{X}\to Y$ with respect to m_T inducing a biholomorphism $\tilde{X}\backslash\tilde{S}\simeq Y\backslash T$ where $\tilde{S}:=\chi^{-1}(T)$

(iii) a proper surjective and holomorphic morphism $p:\tilde{X}\to X$ such that the following diagram

$$\begin{tikzcd} \tilde{X} \arrow[r, dashed, "p"] \arrow[dr, dashed, "\chi"'] & X \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

is commutative.

Now let $I:=m_T\cdot 0_{\tilde{X}}$ (resp. $J:=m_T\cdot 0_X$) be the inverse image ideal sheaf of m_T by χ (resp. by π). Since T consists of finitely may points, it follows from [6] that

(a) p is a monoidal transformation with respect to J

(b) $I\,|\,\tilde{S}$ is an ample line bundle

Step 2. In view of (b), by using a standard spectral sequence argument, exactly as in the proof of Theorem 1 in [10b] one can show that

$$R^ip_*(I^k\otimes p^*\mathcal{F})=0 \quad \text{for} \quad \forall i\geq 1,\, k\gg 0 \text{ and } \mathcal{F}\in \mathrm{Coh}\,(X) \qquad (\dagger)$$

Claim. The natural sheaf morphism

$$J^k\otimes\mathcal{F}\to p_*(I^k\otimes p^*\mathcal{F}) \qquad (\S)$$

is an isomorphism for $k\gg 0$, $\mathcal{F}\in\mathrm{Coh}\,(X)$ along S.

In fact in view of the following exact sequence

$$0\to \mathrm{Ker}\cdot f=:\kappa\to 0_X\xrightarrow{f} p_*0_{\tilde{X}}$$

and the compactness of S, one can find an integer $k\gg 0$, such that $J^k\cdot\kappa=0$; hence in view of Artin–Rees lemma, one has $J^k\cap\kappa=0$. Consequently, in view of (a), a result in [3] (Chap. III Théorème 2.3.1) tells us that the sheaf morphism

$$J^k\to p_*(I^k) \qquad (*)$$

is an isomorphism along S, for $k\gg 0$.

Now for any $x\in S$, let us consider the following local resolution of coherent sheaves for any $\mathcal{F}\in\mathrm{Coh}\,(X)$

$$0_X^m\to 0_X^q\to\mathcal{F}\to 0 \qquad (\dagger\dagger)$$

From (†) and (††), one obtains the following commutative diagram of analytic coherent sheaves with exact rows and with k arbitrary large:

$$\begin{tikzcd}
{} \arrow[r] & p_*(p^*0_X^m \otimes I^k) \arrow[r] & p_*(p^*0_X^q \otimes I^k) \arrow[r] & p_*(p^*\mathcal{F} \otimes I^k) \arrow[r] & 0 \\
{} \arrow[r] & 0_X^m \otimes J^k \arrow[r] \arrow[u, dashed, "\alpha"] & 0_X^q \otimes J^k \arrow[r] \arrow[u, dashed, "\beta"] & \mathcal{F} \otimes J^k \arrow[r] \arrow[u, dashed, "\gamma"] & 0
\end{tikzcd}$$

In view of (*), it follows readily that α and β are isomorphism; consequently so does γ and our claim is proved.

Since X is 1-convex, (†) and (§) imply that

$$H^i(X, J^k \otimes \mathcal{F}) = 0 \quad \text{for} \quad \forall i \geq 1, k \gg 0 \text{ and } \mathcal{F} \in \mathrm{Coh}(X) \quad Q.E.D.$$

COROLLARY 3. *Let X be a 1-convex manifold. Let us assume that its exceptional set is non singular and of pure codimension one. Then there exists a torsion free $J \in \mathrm{Coh}(X)$ such that $J \mid S$ is positive.*

Proof. Since S is non singular and of pure codimension 1, it follows from Theorem 2 that, for $\forall i \geq 1$, $k \gg 0$ and $\mathcal{F} \in \mathrm{Coh}(S)$,

$$H^i(X, J^k \otimes \mathcal{F}) \simeq H^i(S, J^k/J^{k+1} \otimes \mathcal{F}) \simeq H^i(S, S^k(J/J^2 \otimes \mathcal{F}) \simeq 0$$

(Notice that since both X and S are non singular with $\dim X = \dim S + 1$, the ideal J is locally principal hence $S^k(J/J^2) \simeq J^k/J^{k+1}$ for $k \gg 0$). Consequently in view of Theorem 1, it follows readily that $J \mid S$ is positive. *Q.E.D.*

Remarks. As far as Corollary 3 is concerned, we would like to point out 2 facts:

(i) It seems likely that the exceptional set S is projective algebraic; however at this writing, we are not able to prove it yet, so we would like to come back in the future.

(ii) In constrast to the case where $\dim \cdot X = 2$, [10c] $J \mid S$ in general, is not the normal bundle of S in X (see [2], [8]).

3. The existence of positive analytic coherent sheaves

Our main goal here is to strengthen Corollary 3. First of all the following result will be needed.

LEMMA 4 [2]. *Let S be a compact $\mathbb{C}$-analytic space and let us assume that, for every positive dimensional subspace T of S, there exist an n and a non zero section of $L^n \otimes 0_T$ which vanishes at some point of T. Then S is projective algebraic and L is ample.*

We are now in a position to establish the main result of this section.

Theorem 5. *Let (X, S) be a 1-convex space. Then there exists a coherent ideal sheaf $\mathcal{F} \subset 0_X$ supporting on S such that $\mathcal{F} \mid S$ is positive.*

Proof. Let $x, y \in S$ with $x \neq y$ and let $I_{x,y}$ be the coherent ideal sheaf of germs of holomorphic functions vanishing at x and y. Now Theorem 2 and the compactness of S tell us that, for any points $x \neq y \in S$, there exists $K \gg 0$, such that the restriction map

$$H^0(S, \theta) \to \theta_x \oplus \theta_y$$

is surjective where $\theta := J^K \otimes 0_S$. In view of the surjectivity of $\theta \otimes S^{n-1}(\theta) \to S^n(\theta)$, one obtains the epimorphism

$$H^0(S, S^n(\theta)) \to S^n(\theta)_x \oplus S^n(\theta)_y \tag{$*$}$$

for any pair of points $x \neq y$ in S.

Let us consider the following diagram and let us use the same notations as in the proof of Theorem 1 above:

$$\begin{array}{ccc} H & & \theta \\ \downarrow & & \downarrow \\ \mathbb{P}(E) & \xrightarrow{p} & S \end{array}$$

where E is the linear fibre space associated to

Claim. H is an ample line bundle.

In fact, in view of Lemma 4, it suffices to show that, for any closed analytic subvariety $T \subset \mathbb{P}(E)$ which is not a point, there exists a section $\sigma \in \Gamma(T, H^n \mid T)$ which vanishes at some point of T, but does not vanish identically on T.

However this is obvious if T is contained in the fibre of p; so let $t_1, t_2 \in T$ be such that $p(t_1) =: x \neq y := p(t_2)$. Since H is relatively ample [7], one can find a section $\tau \in \Gamma(p^{-1}(x), H^n) = p_*(H^n)_x$ such that $\tau(t_1) \neq 0$ for some $n \gg 0$. In view of

(*) and the isomorphism $S^n(\theta) \simeq p_*(H^n)$ for $n \gg 0$, the pull back section map

$$H^0(S, p_*(H^n)) \to H^0(\mathbb{P}(E), H^n)$$

will provide us a global section $s \in H^0(\mathbb{P}(E), H^n)$ with $s(t_1) = \tau(t_1) \neq 0$ and $s(t_2) = 0$. Now $\sigma := s \mid T$ will be our desired section and our claim is proved.

Consequently, as in the proof of Theorem 1, the ampleness of H implies the weakly positivity (hence the positivity) for θ. *Q.E.D.*

Remark. This result provides an affirmative answer to a conjecture posed by Grauert [2] (see also [1]).

COROLLARY 6. *Let S be a compact, irreducible $\mathbb{C}$-analytic space. Then S is Moishezon iff S carries a positive torsion free analytic coherent sheaf θ.*

Proof. If S carries a positive torsion free sheaf θ, then θ is weakly positive in view of Theorem 1. Therefore S is Moishezon (see e.g. [1]).

Now if S is Moishezon, a main result in [1] tells us that S can be realized as an exceptional set of some 1-convex space X. Hence Theorem 5 will imply the existence of a positive coherent and torsion free sheaf θ on S. *Q.E.D.*

COROLLARY 7. (*Blowing down problem*).

Let S be a compact $\mathbb{C}$-analytic subvariety of some $\mathbb{C}$-analytic space X. Then S is exceptional iff there exists an ideal sheaf $J \subset 0_X$ supporting on S such that the analytic coherent sheaf J/J^2 is positive.

Proof. Let us assume that there exists an ideal sheaf $J \subset 0_X$ supporting on S such that J/J^2 is positive. Hence, following Theorem 1, J/J^2 is weakly positive. A result in [2] tells us that S is exceptional.

Now if S is exceptional, it follows from an analytical version of Chow's Lemma [5] and Theorem 5 above that there exists an analytic coherent ideal sheaf $J \subset 0_X$ such that J/J^2 is positive. *Q.E.D.*

Comments. (i) An algebraic version of Corollary 7 is well known in Algebraic Geometry (see e.g. J. Mazur; Conditions for the existence of contractions in the category of algebraic spaces. Trans. AMS 209 (1975) p. 259–265).

(ii) Let X, S and J be as in Corollary 7. Let $\pi : \tilde{X} \to X$ be the blowing up of X with respect to J and let $\tilde{S}$ be an effective Cartier divisor on $\tilde{X}$ determined by $I := J \cdot 0_{\tilde{X}}$. Now let E be the linear fibre space determined by $\theta := J/J^2$, let $\mathbb{P}(E)$ be the projectivization of E and let $L(E)$ be the tautological line bundle on $\mathbb{P}(E)$.

Hence in view of (*) in Theorem 1, it follows readily that

θ is positive iff $L(E)$ is weakly negative (†)

Now let $\tilde{L}$ be the line bundle on $\tilde{X}$, determined by $\tilde{S}$, then one check easily that:

$$\tilde{S} \subset \mathbb{P}(E) \cap \tilde{X} \quad \text{and} \quad \tilde{N} := \tilde{L} \mid \tilde{S} \simeq L(E) \mid \tilde{S} \tag{††}$$

Recently, T. Peternell gave an another proof for Corollary 6 (Über exzeptionelle Mengen; Manus. Math., 37 (1982) p. 19–26). However, his proof contains a serious gap. He claimed that (Satz 3) θ is weakly positive iff $\tilde{N}$ is weakly negative. But this is simply not true; in fact, it follows from (††) that the weak negativity of $\tilde{N}$ only implies the weak negativity of $L(E)$ when restricted to $\tilde{S}$ which is, in general, merely a subspace (or a primary component, in the sense of [9]) of $\mathbb{P}(E)$. Consequently, in view of (†), this does not imply the positivity for θ!

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(ii) The author also would like to express his sincere thanks to the referee for his thoughtful comments and suggestions.

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Note added in proof: By using the same notations as in comments ii) above, let $\tau : \tilde{N} \to C$ be the blowing down of $\tilde{N}$ along $\pi |_{\tilde{S}} : \tilde{S} \to S$ and let I be the ideal sheaf in 0_C determined by S. In order to patch up his previous gap (loc. cit.) Peternell (Erratum et Addendum zu der Arbeit: Über exzeptionelle Mengen; Manus. Math., 42 (1983) p. 259–263) proposed another proof which is based on the following erroneous claim, among others:

$$0_C \simeq I^{k+1} \oplus (\oplus I^{\nu}/I^{\nu+1}) \quad \text{for} \quad \forall k \in \mathbb{N} \tag{$*$}$$

He referred to Grauert's paper [2] for a proof of (*) which does not exist!