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Cyclic group actions on odd-dimensional spheres

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Abstract. We show that for any simple $(2q-1)$-knot k, $q>1$, and any positive integer n, the knot $\#_1^n k$ is the fixed-point set of a $\mathbf{Z}_n$-action on S^{2q+1}. Further, we show that for many values of n there are examples of $(2q-1)$-knots, $q \geqslant 2$, which are the fixed-point sets of inequivalent $\mathbf{Z}_n$-actions.

0. Introduction

An n-knot is a locally-flat PL pair (S^{n+2}, S^n), where S^n denotes the n-sphere. A $(2q-1)$-knot is *simple* if the complement of S^{2q-1} has the homotopy type of a circle up to but not including dimension q. For $q>1$ such knots have been classified in [L] in terms of the S-equivalence classes of their Seifert matrices, and in [K, T1, T2] in terms of their Blanchfield pairings. Using these classification results, for any simple $(2q-1)$-knot k, with $q>1$, and for any positive integer n, we construct a simple $(2q-1)$-knot k_n such that the n-fold cyclic cover of S^{2q+1} branched over k_n is again S^{2q+1}, and such that k_n lifts to $\#_1^n k$, the sum of n copies of k. An immediate corollary is that for any such k and n, there is a $\mathbf{Z}_n$-action on S^{2q+1} with fixed point set $\#_1^n k$.

The construction in this paper is purely algebraic, and may be contrasted with the geometric construction in [G], where for any m-knot k $(m \geqslant 2)$ Gordon constructs an m-knot which is the fixed-point set of a $\mathbf{Z}_n$-action and whose fundamental group is isomorphic to that of $\#_1^n k$.

As an application of our construction we are able for many values of n to find examples of $(2q-1)$-knots which are the fixed-point sets of inequivalent $\mathbf{Z}_n$-actions. The technique is to pick simple $(2q-1)$-knots k and l such that $\#_1^n k = \#_1^n l$, and such that $k_n \neq l_n$.

1. The main construction

Let k be a simple $(2q-1)$-knot, $q>1$, and $n>1$ an integer. Let A be a non-singular Seifert matrix of k, and set $\varepsilon = (-1)^q$. Following Trotter [T1], we set $S=(A+\varepsilon A')^{-1}$, $T=-\varepsilon A'A^{-1}$.

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PROPOSITION 1.1. *The pair (S, T) has the following properties.*

(i) *S is integral, unimodular, ε-symmetric.*
(ii) *$(I-T)^{-1}$ exists and is integral.*
(iii) *$T'ST=S$.*
(iv) *$A=(I-T)^{-1}S^{-1}$.*

Moreover, any pair of rational matrices (S, T) satisfying (i)–(iii) yields a Seifert matrix A by the formula (iv).

Proof. It is well known (see [L], [T1]) that $A+\varepsilon A'$ is unimodular, and so S is integral and unimodular. Clearly S is ε-symmetric.

Now $I-T=I+\varepsilon A'A^{-1}=(A+\varepsilon A')A^{-1}=S^{-1}A^{-1}$, from which (ii) and (iv) follow at once. Property (iii) is easily checked.

Now suppose that we are given a pair of rational matrices (S, T) satisfying (i)–(iii); then we can define the matrix $A=(I-T)^{-1}S^{-1}$, which by (i) and (ii) is a non-singular matrix over the integers. We have

$$\begin{aligned} A+\varepsilon A' &= (I-T)^{-1}S^{-1}+\varepsilon(S')^{-1}(I-T')^{-1} \\ &= (I-T)^{-1}S^{-1}+S^{-1}(I-ST^{-1}S^{-1})^{-1} \qquad \text{by (i), (iii)} \\ &= (I-T)^{-1}S^{-1}+S^{-1}S(I-T^{-1})^{-1}S^{-1} \\ &= [(I-T)^{-1}-(I-T)^{-1}T]S^{-1}=(I-T)^{-1}(I-T)S^{-1}=S^{-1} \end{aligned}$$

which is unimodular. It follows that A is a Seifert matrix. $\square$

Now we define matrices U, V by

$$U=\begin{pmatrix} 0 & \cdots & 0 & T \\ I & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & I & 0 \end{pmatrix}, \qquad V=\begin{pmatrix} S & & 0 \\ & \ddots & \\ 0 & & S \end{pmatrix},$$

there being $n\times n$ blocks in each case.

THEOREM 1.2. *The pair (V, U) determines a simple $(2q-1)$-knot k_n. The n-fold branched cyclic cover of k_n is the knot $\#_1^n k=k+\cdots+k$ (n times).*

Proof. We have to check that the pair (V, U) satisfies conditions (i)–(iii) of

Proposition 1.1. Clearly V satisfies (i), and it is easy to check that

$$(I-U)^{-1}=\begin{bmatrix} (I-T)^{-1} & T(I-T)^{-1} & \cdots & T(I-T)^{-1} \\ \vdots & \ddots & \ddots & T(I-T)^{-1} \\ (I-T)^{-1} & \cdots & \cdots & (I-T)^{-1} \end{bmatrix}.$$

But $T(I-T)^{-1}=-\varepsilon A'A^{-1}(I+\varepsilon A'A^{-1})^{-1}=-\varepsilon A'(A+\varepsilon A')^{-1}$, which is an integer matrix. Hence (ii) is satisfied. To check (iii) is a simple matrix multiplication.

Hence (V, U) determine a unique simple $(2q-1)$-knot k_n. A routine computation shows that

$$U^n=\begin{pmatrix} T & & 0 \\ & \ddots & \\ 0 & & T \end{pmatrix},$$

and hence the pair (V, U^n) satisfies (i)–(iii), and in fact represents the knot $\#_1^n k$.

Let K_n denote the complement of k_n, and $\tilde{K}_n$ the infinite cyclic cover of K_n. If u is a generator of the group of covering translations, then K_n is obtained from $\tilde{K}_n$ by quotienting out by the action of u. Similarly the n-fold cyclic cover of K_n is obtained from $\tilde{K}_n$ by quotienting out by the action of u^n.

Algebraically this can be described as follows, using Trotter's description of $H_q(\tilde{K}_n)$ in [T1]. Let $\mathbf{B}$ be a basis of $\mathbf{Q}^{rn}$ corresponding to (V, U) where T is an $r\times r$ matrix; then $H_q(\tilde{K}_n)$ is the $\mathbf{Z}[u, u^{-1}]$-module generated by $\mathbf{B}$, the action of u being given by U. The fact that $(1-u): H_q(\tilde{K}_n)\to H_q(\tilde{K}_n)$ is an isomorphism means that when we quotient out by the action of u we get a homology circle. But the form of U^n means that $(1-u^n): H_q(\tilde{K}_n)\to H_q(\tilde{K}_n)$ is also an isomorphism, and hence the n-fold cyclic cover of K_n is a homology circle. Therefore the n-fold branched cyclic cover of k_n is a homotopy sphere, and hence a sphere. □

COROLLARY 1.3. *If k is a simple* $(2q-1)$-knot, $q>1$, *then $\#_1^n k$ is the fixed point set of a $\mathbf{Z}_n$-action on S^{2q+1}.*

PROPOSITION 1.4. *Let B be the Seifert matrix of k_n corresponding to (V, U). Then*

$$B=\begin{bmatrix} A & -\varepsilon A' & \cdots & -\varepsilon A' \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & -\varepsilon A' \\ A & \cdots & \cdots & A \end{bmatrix}.$$

Proof:

$$B=(I-U)^{-1}V^{-1}=\begin{bmatrix}(I-T)^{-1} & T(I-T)^{-1} & \cdots & T(I-T)^{-1}\\ \vdots & \ddots & \ddots & \vdots\\ & & & T(I-T)^{-1}\\ (I-T)^{-1} & \cdots & \cdots & (I-T)^{-1}\end{bmatrix}\begin{bmatrix}S^{-1} & & 0\\ & \ddots & \\ 0 & & S^{-1}\end{bmatrix}$$

$$=\begin{bmatrix}(I-T)^{-1}S^{-1} & T(I-T)^{-1}S^{-1} & \cdots & T(I-T)^{-1}S^{-1}\\ \vdots & \ddots & \ddots & \vdots\\ & & & T(I-T)^{-1}S^{-1}\\ (I-T)^{-1}S^{-1} & \cdots & \cdots & (I-T)^{-1}S^{-1}\end{bmatrix}.$$

$$=\begin{bmatrix}A & TA & \cdots & TA\\ \vdots & \ddots & \ddots & \vdots\\ & & & TA\\ A & \cdots & \cdots & A\end{bmatrix}$$

$$=\begin{bmatrix}A & -\varepsilon A' & \cdots & -\varepsilon A'\\ \vdots & \ddots & \ddots & \vdots\\ & & & -\varepsilon A'\\ A & \cdots & \cdots & A\end{bmatrix}\square$$

Next we prove a result which relates an Alexander matrix of k to one for k_n. Recall that an Alexander matrix $M(t)$ of k is a matrix over $\mathbf{Z}[t, t^{-1}]$ which presents $H_q(\tilde{K})$ as a $\mathbf{Z}[t, t^{-1}]$-module; that is, there is an exact sequence of $\mathbf{Z}[t, t^{-1}]$-modules

$$F \xrightarrow{M(t)} G \twoheadrightarrow H_q(\tilde{K})$$

where F and G are free $\mathbf{Z}[t, t^{-1}]$-modules.

PROPOSITION 1.5. *Let $M(t)$ be a square Alexander matrix for the knot k; then $M(t^n)$ is an Alexander matrix for k_n.*

Proof. We can describe the $\mathbf{Z}[u, u^{-1}]$-module structure of $H_q(\tilde{K}_n)$ in the following way. Let $L_1, \ldots, L_n$ be n copies of the $\mathbf{Z}[t, t^{-1}]$-module $H_q(\tilde{K})$. Then for $1 \leq i < n$, $u: L_i \to L_{i+1}$ is a $\mathbf{Z}[t, t^{-1}]$-isomorphism, and $u: L_n \to L_1$ is defined so that $u^n: L_1 \to L_2 \to \cdots \to L_n \to L_1$ coincides with $t: L_1 \to L_1$. Thus a presentation matrix for $H_q(\tilde{K}_n)$ as a $\mathbf{Z}[u, u^{-1}]$-module is

$$\begin{bmatrix}
M(u^n) & 0 & \cdots & \cdots & 0 \\
0 & \ddots & & & \vdots \\
\vdots & & \ddots & & 0 \\
0 & \cdots & \cdots & 0 & M(u^n) \\
uI & -I & 0 & \cdots & 0 \\
0 & uI & -I & 0 \cdots & 0 \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & & \ddots & \ddots & -I \\
0 & \cdots & \cdots & 0 & \\
-u^nI & 0 & \cdots & 0 & uI
\end{bmatrix}.$$

Two elementary row operations give

$$\begin{bmatrix}
M(u^n) & 0 & \cdots & \cdots & 0 \\
0 & \ddots & & \ddots & \vdots \\
\vdots & & \ddots & & \vdots \\
0 & & & M(u^n) & 0 \\
u^{n-1}M(u^n) & 0 & \cdots & 0 & \\
uI & -I & 0 & \cdots & 0 \\
0 & uI & -I & 0 \cdots & 0 \\
\vdots & & \ddots & & \\
-u^{n-1}I & 0 & \cdots & uI & 0 \\
-u^nI & 0 & \cdots & 0 & uI
\end{bmatrix}.$$

We eliminate the final row and column to give

$$\begin{bmatrix}
M(u^n) & 0 & \cdots & \cdots & 0\\
0 & M(u^n) & \ddots & & 0\\
\vdots & & \ddots & \ddots & \vdots\\
0 & \cdots & \cdots & 0 & M(u^n)\\
u^{n-1}M(u^n) & 0 & \cdots & \cdots & 0\\
uI & -I & 0 & \cdots & 0\\
0 & uI & -I & \ddots & \vdots\\
\vdots & & \ddots & \ddots & 0\\
0 & & & \ddots & -I\\
-u^{n-1}I & 0 & \cdots & 0 & uI
\end{bmatrix}.$$

Now subtract u^{n-1} times the first row from the n^{th} to obtain a row of zeros, which may be eliminated. Continuing in this way we eventually arrive at the matrix $M(u^n)$. □

THEOREM 1.6. *The knot k_n depends only on k and n, and not upon the choice of Seifert matrix A.*

Proof. Let A be an $r \times r$ matrix, and let $\Lambda = \mathbf{Z}[t, t^{-1}, (1-t)^{-1}]$, a subring of the field $\mathbf{Q}(t)$, the field of rational functions in one variable over the rationals. According to Trotter's viewpoint [T1], k gives rise to an ε-symmetric bilinear form [,] on $\mathbf{Q}^r$ represented by the matrix S, and a Λ-module M contained in $\mathbf{Q}^r$ where the action of t is represented by T. A choice of Seifert matrix corresponds to an admissible lattice contained in M (see [T1] for definitions). Although our construction is given in terms of the matrices S and T, it is clear that it could be phrased in terms of M and [,], and hence that it does not depend upon the choice of A.

Alternatively, one can use the formula of Proposition 1.4 to show that if A is S-equivalent to A_1, then B is S-equivalent to B_1. □

2. Knots having distinct $\mathbf{Z}_n$-actions, n odd

In this section we shall show that for many odd integers n, there exist simple $(4q+1)$-knots $(q \geq 1)$, k and l, such that $\#_1^n k = \#_1^n l$ but $k_n \neq l_n$.

Let $\lambda_m(t)$ denote the m^{th} cyclotomic polynomial, where m is not a prime power. Let ζ be a primitive m^{th} root of unity, $K=\mathbf{Q}(\zeta)$ and $F=\mathbf{Q}(\zeta+\zeta^{-1})$, the fixed field of K under complex conjugation. Let h_K denote the class number of K, h_F that of F, and $h_-=h_K/h_F$. According to the work of Bayer [Ba: Example 6.2], the number of distinct simple $(4q+1)$-knots $(q\geq 1)$ with Alexander polynomial $\lambda_m(t)$ is h_-2^d if $m=2p^\alpha$ and h_-2^{d-1} otherwise, where $2d=[K:\mathbf{Q}]$. The factor h_- represents the number of isomorphism classes of $\mathbf{Z}[t,t^{-1}]$-modules supporting a Blanchfield pairing [Ba: Corollary 1.3], and the factor $2^d(2^{d-1})$ represents the number of non-isometric pairings which a given module supports. Note that Bayer's work is couched in terms of pairings on $\mathbf{Z}[\zeta]$-modules which are hermitian with respect to complex conjugation ($t\to t^{-1}$ becomes $\zeta\to\zeta^{-1}=\bar{\zeta}$), and we shall adopt this viewpoint.

Let U be the group of units of (the ring of integers of) K, U_0 the group of units of F, and $N:K\to F$ the norm. If I is a principal ideal, then let $\langle u\rangle$ denote the hermitian form h on I given by $h(a,b)=ua\bar{b}$. As in [Ba: Prop. 2.1], the set of isometry classes of unimodular hermitian forms on a given ideal (not necessarily principal) is in one-one correspondence with $U_0/N(U)$.

Now suppose that h_- has a factor $n>1$, where n is odd and $(m,n)=1$. Let a be an ideal of $\mathbf{Q}(\zeta)$ admitting a non-singular hermitian form h, with a being of order n in $\ker N:C_K\to C_F$. Then $\perp_1^n(a,h)$ has determinant $\langle u\rangle$ for some $u\in U_0/N(U)$; see [Ba: Definition 1.9] for the definition of determinant. Since the order of $U_0/N(U)$ is 2^d or 2^{d-1}, and n is odd, there exists $v\in U_0/N(U)$ such that $v^n=u$. Then $\perp_1^n\langle v\rangle$ has determinant $\langle v^n\rangle=\langle u\rangle$.

Set $K=(a,h)\perp(a,-h)$, $L=\langle v\rangle\perp\langle -v\rangle$. Then $\perp_1^n K$, $\perp_1^n L$ are indefinite and have the same rank, signatures and determinant. Hence by [Ba: Corollary 4.10] they are isometric. But K is not isometric to L, for the determinant of K is (a^2,α), and a^2 is non-zero in ker $N:C_K\to C_F$ since n is odd.

In fact, if k, l are the simple $(4q+1)$-knots corresponding to K, L respectively, we can show that $k_n\neq l_n$. For let $M(t)$ be an Alexander matrix of k, so that by Proposition 1.5 $M(t^n)$ is an Alexander matrix of k_n. The work of Fox and Smythe [F–S] enables us to obtain a row ideal class from the matrix $M(\zeta)$, and the work of Hillman [H: Chap. III, Theorem 12] identifies this with the ideal a^2 in the determinant of K. But the Alexander polynomial of k_n is $\lambda_m(t^n)$, which has $\lambda_m(t)$ as one of its factors since $(m,n)=1$. Let τ be a primitive m^{th} root of unity such that $\tau^n=\zeta$. Setting $t=\tau$ in the Alexander matrix $M(t^n)$, we obtain $M(\tau^n)=M(\zeta)$, and hence obtain a Fox-Smythe invariant a^2 again. In the case of l_n, these ideal invariants are all trivial, hence $k_n\neq l_n$.

Taking the n-fold branched cyclic covers of k_n, l_n we obtain respectively the knots $\#_1^n k$, $\#_1^n l$. Since $\perp_1^n K$ is isometric to $\perp_1^n L$, we have $\#_1^n k=\#_1^n l$.

Many examples may be obtained from the tables in [Sch].

For the case of $(4q-1)$-knots, $q \geqslant 1$, and $m \neq 2p^r, p^r$, where p is a prime, then as in [Ba: §5], $\zeta - \zeta^{-1}$ is a unit and so we can multiply all the pairings above by $\zeta - \zeta^{-1}$ to obtain skew-hermitian pairings. The argument then goes through as before. We are grateful to Dr. Bayer for pointing out this extension to the case of $(4q-1)$-knots.

3. Number theory

This section deals with some results from algebraic number theory, which will be used in the next section to deal with the case $n=2$.

Let K be an algebraic number field, $R = \mathrm{int}(K)$ its ring of integers, $\mathbf{Z}_{(p)}$ the p-adic integers, $R_p = R \otimes \mathbf{Z}_{(p)}$, $K_p = K \otimes \mathbf{Z}_{(p)}$, $U(R) = \prod_P R_p^\times$ and $J(K) = U(R) \cdot \coprod_p K_p^\times$, where $\coprod$ denotes the direct sum. $K^\times$ is considered as a subgroup of $J(K)$ under the "diagonal" map. If $C(K)$ denotes the ideal class group of K, then we have $C(K) \cong J(K)/U(R) \cdot K^\times$ an isomorphism which is natural with respect to ring extensions.

Now suppose that L is an algebraic number field, Γ a group of automorphisms of L, $S = \mathrm{int}(L)$, $K = L^\Gamma$ the subfield of L fixed under Γ, and $R = \mathrm{int}(K) = S^\Gamma$.

LEMMA 3.1. $\ker[C(R) \to C(S)] \cong \ker[H^1(\Gamma, S^\times) \to H^1(\Gamma, U(S))]$, *where the first map is induced by ring extension, the second by the "diagonal" map* $S^\times \to U(S)$.

Proof. Consider the exact sequence

$$0 \to U(S) \cdot L^\times \to J(L) \to C(S) \to 0,$$

Since $J(L)^\Gamma = J(K)$, we obtain a commutative diagram

$$\begin{array}{ccccccc} 0 \to & U(R) \cdot K^\times & \to & J(K) & \to & C(R) & \to 0 \\ & \downarrow & & \| & & \downarrow & \\ 0 \to & (U(S) \cdot L^\times)^\Gamma & \to & J(L)^\Gamma & \to & C(S)^\Gamma & \end{array}$$

Applying the Snake Lemma [Bass: p. 26] we find that

$$\begin{aligned} \ker[C(R) \to C(S)] &= \ker[C(R) \to C(S)^\Gamma] \\ &\cong \mathrm{coker}[U(R) \cdot K^\times \to (U(S) \cdot L^\times)^\Gamma]. \end{aligned}$$

Now consider the exact sequence

$$0 \to S^\times \to U(S) \oplus L^\times \to U(S) \cdot L^\times \to 0$$

where the first map is $s \mapsto (s, s^{-1})$. From cohomology theory we obtain the exact sequence

$$0 \to R^\times \to U(R) \oplus K^\times \to (U(S) \cdot L^\times)^\Gamma \to H^1(\Gamma, S^\times) \to H^1(\Gamma, U(S)) \oplus H^1(\Gamma, L^\times).$$

Since by Hilbert 90, $H^1(\Gamma, L^\times) = 0$, we have

$$\begin{aligned} \operatorname{coker}[U(R) \oplus K^\times \to (U(S) \cdot L^\times)^\Gamma] &= \operatorname{coker}[U(R) \cdot K^\times \to (U(S) \cdot L^\times)^\Gamma] \\ &\cong \ker[H^1(\Gamma, S^\times) \to H^1(\Gamma, U(S))], \end{aligned}$$

and the result follows. $\square$

Now let

$L = \mathbf{Q}(\sqrt{-123}, \sqrt{-31})$	$S = \operatorname{int}(L)$	
$K = \mathbf{Q}(\sqrt{-123})$	$\Gamma = \operatorname{Gal}(L/K)$,	$R = \operatorname{int}(K)$
$K' = \mathbf{Q}(\sqrt{3813})$	$\Gamma' = \operatorname{Gal}(L/K')$	$R' = \operatorname{int}(K')$.

The action of the non-trivial elements of Γ, Γ' will be denoted respectively by $\tilde{\ }, \bar{\ }$. Our immediate purpose is to show that $C(R) \to C(S)$ is injective.

LEMMA 3.2. *The fundamental unit of R' is* $v = 247 + 4\sqrt{3813}$.

Proof. Certainly $247^2 - 16.3813 = 1$, so v is a unit of R'. If v is not the fundamental unit, then there exist positive integers a, b, c, d such that $(a + b\sqrt{N})(c + d\sqrt{N}) = 4v$, where $N = 3813$. Thus

$$ac + bdN + (ad + bc)\sqrt{N} = 4(247 + 4\sqrt{N}).$$

But $ac + bdN \geq N > 4.247$, so this is impossible. $\square$

By the Dirichlet Unit Theorem, rank $(S^\times) = 1$ and $S^\times = \langle \pm 1 \rangle \times \langle u \rangle$ for some u (a fundamental unit).

LEMMA 3.3. *S has $u=\sqrt{-123}+2\sqrt{-31}$ as a fundamental unit.*

Proof. Note that $u\bar{u}=v$, so $u\in S^\times$. If $u=\pm w^n$ for some $w\in S$ (± 1 are the only units of finite order) then $w\bar{w}\in R'^\times$ and so $w\bar{w}=\pm v^m$ for some $m\in\mathbf{Z}$. But then $(v^m)^n=\pm v$ whence $m=n=\pm 1$. Hence the result. □

LEMMA 3.4. *$H^1(\Gamma, S^\times)\to H^1(\Gamma, S_{31}^\times)$ is injective, and hence so is $H^1(\Gamma, S^\times)\to H^1(\Gamma, U(S))$.*

Proof. For an abelian Γ-group A we use the representation

$$H^1(\Gamma, A)\cong\frac{\{a\in A: a\tilde{a}=1\}}{\{a/\tilde{a}: a\in A\}}.$$

This representation is natural with respect to extension of A. Now

$$u\tilde{u}=(\sqrt{-123}+2\sqrt{-31})(\sqrt{-123}-2\sqrt{-31})=-123+124=1,$$

and $u/\tilde{u}=u^2/u\tilde{u}=u^2$ so

$$H^1(\Gamma, S^\times)=S^\times/\langle u^2\rangle=\{(1),(-1),(u),(-u)\}.$$

We must show that none of -1, u, $-u$ is of the form $s/\tilde{s}$ for some $s\in S_{31}^\times$.

If for some $s\in S_{31}^\times$, $s/\tilde{s}=-1$, then $s=-\tilde{s}$ and so $s=r\sqrt{-31}$ for some $r\in R_{31}$. Hence s is not a unit.

If for some $s\in S_{31}^\times$, $s/\tilde{s}=u$, then, as $S_{31}=R_{31}[u]$, $s=a+bu$ with $a, b\in R_{31}$, and so $a+bu=(a+b\tilde{u})u=au+b$. Hence $a=b$ and $s=a(1+u)$. As s is a unit, $N_{L/\mathbf{Q}}(s)$ is also a unit. But $N_{L/\mathbf{Q}}(s)=(a\bar{a})^2(1+u)(1+\tilde{u})(1+\bar{u})(1+\bar{\tilde{u}})=(a\bar{a})^2\cdot 16.31$ and this is not a unit in $\mathbf{Z}_{31}$.

A similar argument disposes of the possibility $s/\tilde{s}=-u$ and the result is proved. □

Remark 3.5. We can now see that the primes of S above 31 are principal. By our calculation above, $N_{L/\mathbf{Q}}((1+u)/2)=31$ and so, since $(1+u)/2$ is integral $(N_{L/K}((1+u)/2=(1+\sqrt{-123})/2\in R)$, $((1+u)/2)_S$ is a prime of S above 31. But $L/\mathbf{Q}$ is galois so all the primes of S above 31 are conjugate and hence principal. In fact, $(31)_S=((1+u)/2)_S^2((1-u)/2)_S^2$.

PROPOSITION 3.6. *$(3,\sqrt{-123})_{S[1/31]}$ is not principal.*

Proof. $(3,\sqrt{-123})_R$ is clearly not principal, for the equation $a^2+123b^2=12$ has no solutions over $\mathbf{Z}$. Since $C(R)\to C(S)$ is injective by Lemmas 3.1 and 3.4, $(3,\sqrt{-123})_S$ is not principal.

In passing from $C(S)$ to $C(S[1/31])$ we kill off the ideal classes represented by ideals dividing $(31)_S$; since by Remark 3.5 these ideals are all principal, the map $C(S) \to C(S[1/31])$ is injective. The result follows. □

4. The case $n = 2$

In this section we construct two simple $(4q+1)$-knots k and l such that $k+k=l+l$ but $k_2 \neq l_2$.

LEMMA 4.1. *Let* $\Delta(t)=31t^2-61t+31$. *Then* $\mathbf{Q}(\sqrt{-123}, \sqrt{-31})$ *is a splitting field for* $\Delta(t^2)$.

Proof. Let τ be a root of $\Delta(t^2)$; then we can take $\tau^2=(61+\sqrt{-123})/62$, so that $\Delta(t)$ splits in $\mathbf{Q}(\sqrt{-123})$. Now $31\tau^2=(61+\sqrt{-123})/2=-[(1-\sqrt{-123})/2]^2$, and so $\tau=(1-\sqrt{-123})/2\sqrt{-31}$. Hence $\tau \in \mathbf{Q}(\sqrt{-123}, \sqrt{-31})$. But the conjugates of τ are τ, $-\tau$, $\bar{\tau}=1/\tau$ and $-1/\tau$, so $\Delta(t^2)$ splits in $\mathbf{Q}(\sqrt{-123}, \sqrt{-31})$. □

Let J denote the ideal $(3, \sqrt{-123})$ over the ring $\mathbf{Z}[\tau^2, \tau^{-2}]=R[1/31]$ in the notation of Section 3. Note that $J=\bar{J}$ and $J\bar{J}=(3)$, where $\bar{\ }$ here denotes complex conjugation. Hence we can define a non-singular hermitian form $b: J \times J \to R[1/31]$ by $b(\alpha, \beta)=\alpha\bar{\beta}/3$. Let $(J \oplus J, B)$ denote the orthogonal direct sum $(J, b) \perp (J, b)$, and set

$$e=((6+\sqrt{-123})/31, (51+\sqrt{-123})/31)$$
$$f=((51-\sqrt{-123})/31, (-6+\sqrt{-123})/31).$$

It is easily checked that $B(e, e)=B(f, f)=1$ and that $B(e, f)=0$. Hence $(J, b) \perp (J, b) \cong \langle 1 \rangle \perp \langle 1 \rangle$.

Let k be the simple $(4q+1)$-knot $(q \geq 1)$ represented by (J, b) and l the corresponding knot represented by $\langle 1 \rangle$. Then $k+k=l+l$, but since J is a non-principal ideal by Proposition 3.6, $k \neq l$. Let $M(t)$ be a square Alexander matrix for k; then by Proposition 1.5, $M(t^2)$ is an Alexander matrix for k_2. The Fox–Smythe row ideal class of k_2 is obtained from the matrix $M(\tau^2)$ over the ring $\mathbf{Z}[\tau, \tau^{-1}]=S[1/31]$, and by [H: Chap. III, Theorem 12] this is the ideal $J_{S[1/31]}$. By Proposition 3.6, this ideal is non-principal. Since the corresponding invariant for l_2 is trivial, we have $k_2 \neq l_2$.

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Note added in proof: The second author has recently shown that for any integer n there is an integer m, prime to n and not a prime power, such that, if ζ is an mth root of 1, there is an ideal class in $C_{Q[\zeta]}$ of order n with norm 1 in $C_{Q[\zeta+\zeta^{-1}]}$. Thus the results of section 2 are valid for any odd n.