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The fundamental group and the spectrum of the Laplacian

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It has been known for many years that there is a close relationship between the fundamental group of a closed manifold M and the Riemannian geometry of its universal cover $\tilde{M}$. Roughly speaking, a closed manifold picks out a family of metrics on its universal cover, namely, those metrics which are invariant under deck transformations. These metrics are all quasi-isometrically equivalent, because the original manifold is compact. Then "asymptotic properties" of the fundamental group are reflected in asymptotic geometric properties of the universal cover in this quasi-isometry class of metrics.

A well-known example of this is the theorem of Milnor–Švarč [14] which states that the fundamental group $\pi_1(M)$ has exponential growth in the sense of groups if and only if the volume $V(r)$ of a ball of radius r in $\tilde{M}$ grows exponentially in r.

The object of this paper is to add another example of this type. If N is any (possibly non-compact) Riemannian manifold, and Δ denotes the Laplace–Beltrami operator, with sign chosen so that it is a positive operator, we set

$$\lambda_0(N) = \inf \frac{\int_N \Delta f \cdot f}{\int_N f^2},$$

where f runs over all smooth functions on N with compact support. λ_0 has the following interpretation: it is well-known that the action of Δ on smooth functions with compact support extends to an unbounded positive, self-adjoint operator on $L^2(N)$. Then $\lambda_0(N)$ is the greatest lower bound of the spectrum of Δ (see [6]).

From the expression given for $\lambda_0(N)$, it seems that one derivative of the metric appears, since that is true of Δ. However, writing $\int_N \Delta f \cdot f = \int_N \|\operatorname{grad} f\|^2$ shows that λ_0 does not involve any derivatives in the metric, and one sees that the property $\lambda_0(N) = 0$ is unchanged when the metric on N is changed to a quasi-isometrically equivalent one.

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With this understood, our main result is that the condition $\lambda_0(\tilde{M})=0$ depends only on the fundamental group $\pi_1(M)$.

THEOREM 1. $\lambda_0(\tilde{M})=0$ *if and only if* $\pi_1(M)$ *is an amenable group.*

See [10] and §1 below for results concerning amenable groups.

An immediate consequence of this is the following:

THEOREM 2. *Let N be a simply connected Riemannian manifold, and Γ a group of isometries of N such that N/Γ is a compact manifold. Then Γ is amenable if and only if any other group of isometries satisfying the same condition is amenable.*

Recalling that a group containing a free group on two generators as a subgroup is not an amenable group, one might regard Theorem 2 as a generalization of a theorem of Tits [15], valid when N is a linear group.

The question of the existence of complete manifolds N such that $\lambda_0(N)>0$ has been studied by McKean ([13], see also [19]) who proved that if N is a complete, simply-connected manifold of negative sectional curvature bounded away from 0, then $\lambda_0(N)>0$. He also produced a lower bound for $\lambda_0(N)$ in terms of the bound on the sectional curvature, something that our topological methods cannot provide. It follows from McKean's theorem and Theorem 1 that the fundamental group of a compact manifold of negative sectional curvature is not amenable – this result is known from several points of view (see [3], [8], [9]).

The novelty of Theorem 1 is, however, that there are no curvature assumptions at all – in particular we find manifolds N which have the property that $\lambda_0(\tilde{N})>0$, which are not $K(\pi, 1)$'s or obtained from $K(\pi, 1)$'s in any geometric way. To see this, we observe that any finitely-presented group arises as the fundamental group of a compact manifold of dimension 4. In particular, this is true of the free group on two generators. The resulting manifold cannot be a $K(\mathbf{Z}*\mathbf{Z}, 1)$, since it has non-trivial H^4, and the figure-eight, which is a $K(\mathbf{Z}*\mathbf{Z}, 1)$, does not. Choosing any metric on this manifold N gives a metric on the universal cover $\tilde{N}$ with $\lambda_0(\tilde{N})>0$.

Theorem 1 also compares with the theorem of Milnor [14] by observing that a group with subexponential growth is necessarily amenable. This is well-known, and we provide a proof in §1. A consequence of Theorem 1 is then that if $\tilde{M}$ has subexponential growth, then $\lambda_0(\tilde{M})=0$. One may ask if this theorem is valid without the assumption that $\tilde{M}$ arises as the universal cover of a compact manifold. In fact, this result is valid for complete manifolds (see [2].) It is reasonable to expect that other theorems from group theory can be translated into Riemannian geometry with the aid of Theorem 1. We do not pursue these questions here.

The outline of this paper is as follows: In §1 we review the notion of an amenable group. In §2 we prove one direction of Theorem 1: under the assumption that $\pi_1(M)$ is amenable, we exhibit functions f_i with

$$\frac{\int f_i \cdot \Delta f_i}{\int f_i^2} \to 0 \quad \text{as} \quad i \to \infty.$$

Section 4 then gives a proof of the other direction of Theorem 1, after a brief discussion of integral currents in §3. In §5 we present some remarks on the heat kernel, connecting Theorem 1 with Kesten's Theorem [12].

This paper came about from several sources. Most immediately it arose from reading the inspiring paper of McKean [13], while searching for a geometric version of Kesten's Theorem [12]. Kesten's theorem was first brought to my attention by the Ph.D. thesis of F. Eisenberg [9], and I benefitted also by many conversations with Joel Cohen. The philosophy suggesting the possibility of Theorem 1 comes directly from [14].

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§1. Følner's theorem

Let G be a countably generated discrete group. G is said to be amenable if there is a finitely additive left-invariant mean μ on G, i.e., a bounded linear functional

$$\mu : L^\infty(G) \to \mathbf{R}$$

having the properties:

(i) $\inf_{g\in G} (f(g)) \leq \mu(f) \leq \sup_{g\in G} (f(g))$
(ii) for all $g \in G$, $\mu(g \cdot f) = \mu(f)$, where $g \cdot f(x) = f(g^{-1}x)$.

It is clear from the definition that any finite group is amenable: let

$$\mu(f) = \frac{1}{|G|} \sum_{g\in G} f(g).$$

However, in general, μ's constructed for infinite groups tend to be rather transcendental, their existence relying on some variant of the Axiom of Choice.

For applications, it is desirable to have alternative characterizations of this notion. We give one such characterization below, and one in §4.

The first of these is a combinatorial characterization due to Følner [17]. To state it, for any finite subset E of G, we let $\#(E)$ denote the cardinality of E, and $g \cdot E = \{gh : h \in E\}$ for $g \in G$.

THEOREM (Følner). *G is amenable if and only if, for every k in the interval* $0<k<1$, *and arbitrary, finitely many elements* $a_1, \dots, a_n$ *of G, there exists a finite subset E of G such that*

$$\#(E \cap a_i \cdot E) \geq k\, \#(E), \quad \textit{for} \quad i = 1, \dots, n.$$

As a simple application of this theorem, we prove:

PROPOSITION 1. *If G has subexponential growth, then G is amenable.*

Proof. Let $0<k<1$ be given, and $a_1, \dots, a_n$ a finite subset of G. Let E_r be the set of all words of length $\leq r$ in the elements $a_1, \dots, a_n$ and their inverses.

CLAIM. There exists an r such that for all i, $\#(E_r \cap a_i \cdot E_r) \geq k \cdot \#(E_r)$.

Proof. Indeed, we see that $E_{r-1} \subseteq E_r \cap a_i \cdot E_r$ for all i. If the claim is false, then $\#(E_{r-1}) < k\#(E_r)$, for all r; this implies that $\#(E_r) \geq (\text{const})(1/k)r$, and then G has exponential growth.

The proposition follows by setting $E = E_r$ for this r.

Remarks. This proposition is well-known. A proof along different lines may be found in [11], and a proof similar to the above for $G = \mathbf{Z}$ is given in [10].

From this theorem one can easily establish a large class of amenable groups. It follows that all abelian groups are amenable; one sees directly from the definition that subgroups and quotients of amenable groups are amenable, and the extension of an amenable group by an amenable group is amenable.

The best example of a non-amenable group is the free group on two generators – the fact that this is non-amenable follows from the Banach–Tarski–Hausdorff paradox. In particular any group containing a free subgroup on two generators is non-amenable, and this is how non-amenable groups arise geometrically.

We now give a geometric interpretation of Følner's theorem. To state it, let M be a compact manifold, and $\tilde{M}$ its universal cover. We denote by F a fundamental domain in M for the action of $\pi_1(M)$ which arises in the following way: we pick a smooth triangulation of M, and for each n-simplex in this triangulation, $n = \dim(M)$, we pick one simplex in $\tilde{M}$ covering this simplex. F is the union of all of these simplices – in particular, F is a union of finitely many smooth n-simplices, but F is not necessarily connected.

PROPOSITION 2. *Let M be a compact manifold, and $\tilde{M}$ its universal cover. Then $\pi_1(M)$ is amenable if and only if, for every fundamental domain F as above, and for every ε, there is a finite subset E of $\pi_1(M)$ such that if*

$$H = \bigcup_{g \in E} g\,F$$

then H satisfies the isoperimetric inequality

$$\frac{\operatorname{area}(\partial H)}{\operatorname{vol}(H)} < \varepsilon.$$

Proof. First we assume that $\pi_1(M)$ is amenable, and pick F and ε as above. Let $a_1, \dots, a_n$ be the finite set of elements of $\pi_1(M)$ such that $a_i(F) \cap F \neq \varnothing$. Note that $a_1, \dots, a_n$ generate $\pi_1(M)$. Let $A = \operatorname{area}(\partial F)$, $V = \operatorname{vol}(F)$. Given k, we choose E a finite set as in Følner's theorem, and consider H as above. Then

$$\operatorname{vol}(H) = V \cdot \#(E),$$

and

$$\begin{aligned}\operatorname{area}(\partial H) &\leq A \cdot \#\{g \in E : a_i \cdot g \notin E \quad \text{for some} \quad i\}\\ &\leq A \cdot n \cdot (1-k)\#(E),\end{aligned}$$

where the last follows from the choice of E. Then

$$\frac{\operatorname{area}(\partial H)}{\operatorname{vol}(H)} \leq A \cdot n(1-k)$$

and choosing $k \geq 1 - \varepsilon/(A \cdot n)$ gives the desired inequality

Now assume that $\pi_1(M)$ satisfies the conditions of the proposition, and let $a_1, \dots, a_n$ and k be given as in Følner's theorem.

We choose a fundamental domain F so that $a_i \cdot F \cap F \neq \varnothing$, $i = 1, \dots, n$, which is particularly easy in view of the fact that we do not demand that F is connected. We may extend the set $\{a_1, \dots, a_n\}$ so that it includes all elements of G such that $g \cdot F \cap F \neq \varnothing$.

Now, given ε, we choose H and E as in the conclusion of the proposition. We want to estimate the set $\{g \in E : g \cdot F \cap \partial H \neq \varnothing\}$.

On the one hand, there is a constant l, depending only on F, such that if $g \cdot F \cap \partial H \neq \varnothing$, then $\operatorname{area}(g \cdot F \cap \partial H) \geq l$. Thus

$$\#\{g : g \cdot F \cap \partial H \neq \varnothing\} \leq \frac{\operatorname{area}(\partial H)}{l}.$$

On the other hand, this set contains the set $E - a_i \cdot E$ for each i.

Finally, if V denotes the volume of F, then

$$\#(E) = \frac{\operatorname{vol}(H)}{V}.$$

Then

$$\frac{\#(E \cap a_i \cdot E)}{\#(E)} = 1 - \frac{\#\{g : g \cdot F \cap \partial G \neq \varnothing\}}{\#(E)} \geq 1 - \frac{\operatorname{area}(\partial H)}{\operatorname{vol}(H)} \cdot \frac{V}{l}$$
$$\geq 1 - \varepsilon \cdot \frac{V}{l}.$$

Choosing ε small gives us a number $\geq k$, and so $\pi_1(M)$ satisfies the criterion of Følner's Theorem.

§2. Some test functions

In this section, we prove that if $\pi_1(M)$ is amenable, then $\lambda_0(\tilde{M}) = 0$. The idea of the proof is to find test functions f whose support lies in a union H of fundamental domains chosen according to Følner's Theorem, such that Δf is concentrated near ∂H.

We now fix for the discussion a fundamental domain F, two numbers $0 < \varepsilon_1 < \varepsilon_2 < 1$, which we will assume fixed but sufficiently small (depending on the shape of F), and $\psi : [0, 1] \to [0, 1]$ a smooth function such that $\psi(x) = 0$ for $x < \varepsilon_1$, $\psi(x) = 1$ for $x > \varepsilon_2$.

Now let $\varepsilon > 0$, and E a finite subset of $\pi_1(M)$ be given as in the conclusion of Proposition 2. Then let χ_H be the characteristic function of $H = \bigcup_{g \in E} g \cdot F$, and let

$$f_E(x) = \psi(\operatorname{dist}(x, H)) \cdot \chi_H.$$

f_E is smooth with support contained in H.

We now estimate

$$\frac{\int_{\tilde{M}} f_E \cdot \Delta f_E}{\int_{\tilde{M}} f_E^2}.$$

On the other hand, there is a C_1, depending only on ψ and F, such that

$$\int_{\tilde{M}} f_E \cdot \Delta f_E \leq C_1 \cdot \text{area}\,(\partial H)$$

since Δf_E has support only in an ε_2-neighborhood of ∂H, and here its values are determined in each fundamental domain $g \cdot F$ by ψ and the finitely many ways the fundamental domains adjoining $g \cdot F$ either occur or do not occur in H.

On the other hand

$$\int_{\tilde{M}} f_E^2 \geq C_2 \cdot \text{vol}\,(H)$$

since in each fundamental domain $g \cdot F$ occurring in H, $f_E^2 \equiv 1$ except within an ε_2-neighborhood of $g \cdot \partial F$. Combining these two gives

$$\frac{\int_{\tilde{M}} f_E \cdot \Delta f_E}{\int_{\tilde{M}} f_E^2} \leq \frac{C_1}{C_2} \cdot \frac{\text{area}\,(\partial H)}{\text{vol}\,(H)},$$

and this last can be made arbitrarily small by Proposition 2. Thus,

$$\lambda_0(\tilde{M}) = \inf \frac{\int_{\tilde{M}} f \cdot \Delta f}{\int_{\tilde{M}} f^2} = 0.$$

§3. A variational problem

In this section, let M denote a bounded, open subset of an n-dimensional Riemannian manifold M', such that M has smooth boundary B. We consider the problem of minimizing the isoperimetric constant

$$h(M)=\inf_S \frac{\operatorname{area}(\partial S)}{\operatorname{vol}(S)}$$

where S runs over open submanifolds of M with smooth boundary. Our main result is:

THEOREM 3. *In the above situation, there is a non-zero integral current T realizing this minimum, such that the generalized mean curvature of ∂T is bounded in terms of $h(M)$ and the mean curvature of B.*

It is remarked in [4] that $h(M)>0$.

Let S_i be a sequence of such S such that $(\operatorname{area}(\partial S_i)/\operatorname{vol}(S_i))$ decreases to $h(M)$. According to [1], there exists positive constants a and b such that if $\operatorname{vol}(S_i)\leq a$, then

$$\operatorname{vol}(S_i)\leq b \operatorname{area}(\partial S_i)^{n/(n-1)},$$

so that

$$\left(\frac{\operatorname{area}(\partial S_i)}{\operatorname{vol}(S_i)}\right)^{n/(n-1)}(\operatorname{vol}(S_i))^{1/(n-1)}\geq\frac{1}{b}.$$

It follows that there is a constant c such that $\operatorname{vol}(S_i)>c$.

Since we also have that $\operatorname{vol}(S_i)<\operatorname{vol}(M)<\infty$, it follows that there exists d such that $N(S_i)=\operatorname{vol}(S_i)+\operatorname{vol}(\partial S_i)<d$, and we may apply [16, 4.2.17] to conclude the existence of an integral current T such that

$$\liminf_{i\to\infty}\operatorname{vol}(T-S_i)=0.$$

In particular, $\operatorname{vol}(T)\geq c$, and

$$\frac{\operatorname{area}(\partial T)}{\operatorname{vol}(T)}\leq\liminf_{i\to\infty}\frac{\operatorname{area}(\partial S_i)}{\operatorname{vol}(T)}=\liminf_{i\to\infty}\frac{\operatorname{area}(\partial S_i)}{\operatorname{vol}(\partial S_i)}=h(M).$$

Now let ξ be a compactly supported vector-field in M', and let h_t be the flow generated by ξ. We consider

$$\frac{d}{dt}\frac{\operatorname{area}(h_t(\partial T))}{\operatorname{vol}(h_t(T))};$$

it will be convenient to first handle the case where $\xi \restriction B$ is tangent to B. In this case, we have that $h_t(T)$ is again an integral current supported in M, so that by the minimizing property of T,

$$\frac{d}{dt}\frac{\operatorname{area}(h_t(\partial T))}{\operatorname{vol}(h_t(T))}=0 \tag{$*$}$$

Denote by ν the $\|\partial T\|$-measurable unit vector field satisfying

$$\frac{d}{dt}\operatorname{vol}(h_t(S))\big|_{t=0}=\int \xi\cdot\nu\, d\,\|\partial T\|;$$

ν is the unit "outward normal" to ∂T. If $\mathscr{m}$ denotes the generalized mean curvature of ∂T, then it is standard that we also have

$$\frac{d}{dt}\operatorname{area}(h_t(\partial T))=\int \xi\cdot \mathscr{m}\nu\, d\,\|\partial T\|$$

so that $(*)$ becomes

$$0=\frac{d}{dt}\frac{\operatorname{area}(h_t(\partial T))}{\operatorname{vol}(h_t(T))}=\frac{1}{\operatorname{vol}(h_t(T))}\int \xi\cdot\mathscr{m}\nu\, d\,\|\partial T\|-h(M)\int\xi\cdot\nu\, d\,\|\partial T|$$

$$=\frac{1}{\operatorname{vol}(h_t(T))}\int \xi\cdot(\mathscr{m}-h(M))\nu\, d\,\|\partial T\| \tag{$**$}$$

and we see that $\mathscr{m}\equiv h(M)$ on $\operatorname{supp}(\partial T)-B$.

Now let $x\in\operatorname{supp}(\partial T)\cap B$, and let U be a neighborhood about x. Let ρ be the signed distance function from B

$$\rho(y)=\operatorname{dist}(y,B)\quad\text{if}\quad t\in U-M$$

$$\rho(y)=-\operatorname{dist}(y,B)\quad\text{if}\quad y\in U\cap M.$$

Then $\operatorname{grad}(\rho)$ is a unit vector field pointing outward from B. If f is a smooth function with compact support in U, let $\mu(f)=(d/dt)(\operatorname{area}(g_t(\partial T))$ where g_t is the flow generated by $f\cdot\operatorname{grad}(\rho)$. We wish to estimate $\mu(f)$ in terms of $|f|$ and $|H|$, where H is the mean curvature vector of B, since this will give us the desired bound for the mean curvature of ∂T at x.

To state our main result, we will need the following related quantities:

$$\mu_1(f)=\int_{M-B} f \operatorname{grad}(\rho)\cdot \nu \, d\,\|\partial T\|$$

$$\mu_2(f)=\int_B f\, d\,\|\partial T\|$$

$$\mu_3(f)=\int_B fH\cdot \nu\, d\,\|\partial T\|$$

$$\mu_4(f)=\lim_{h\downarrow 0}\int_{-h<\rho(x)\le 0} f[1-|\nu\cdot \operatorname{grad}(\rho)|^2]\, d\,\|\partial T\|.$$

Our main observation is:

LEMMA. $\mu(f)=h(M)\cdot\mu_1(f)+\mu_3(f)+\mu_4(f)$.

In particular, this shows that the limit defining μ_4 exist.

Proof. Let φ_h be the Lipshitz function given by

$$\begin{aligned}\varphi_h(x)&=1 && \text{if}\quad x\ge 0\\ &=1+\frac{1}{h}x && \text{if}\quad -h\le x\le 0\\ &=0 && \text{if}\quad x\le -h.\end{aligned}$$

Then

$$\mu(f)=\mu((1-\psi_h)f)+\mu(\psi_h f)\quad \text{where}\quad \psi_h=\varphi_h\circ\rho.$$

Then $\mu((1-\psi_h)f)=h(M)\int_M(1-\psi_f)\cdot f \operatorname{grad}\rho\cdot\nu\, d\,\|dT\|$ since $(1-\psi_h)f$ is supported away from B. As $h\to 0$, this becomes $h(M)\cdot\mu_1(f)$. Now $\mu((\psi_h f))=\int_{M\cup B}\nabla(\psi_h\cdot f\operatorname{grad}\rho)\cdot\pi_T\, d\,\|\partial T\|$ where π_T denotes orthogonal projection onto the tangent bundle of ∂T, see [18]. On the other hand,

$$\begin{aligned}\nabla(\psi_h\cdot f\operatorname{grad}\rho)=(\varphi_h'\circ\rho)f\,d\rho\cdot\operatorname{grad}\rho+(\varphi_h\circ\rho)\,df\\ \cdot\operatorname{grad}(\rho)+(\varphi_h\circ\rho)f\nabla\operatorname{grad}(\rho).\end{aligned}$$

The first term then gives

$$\int_{M\cup B}(\varphi_h'\circ\rho)f\,d\rho\cdot\operatorname{grad}\rho=\frac{1}{h}\int_{1-h<x<0} f\cdot[1-|\nu\cdot\operatorname{grad}\rho|^2]\,d\,\|\partial T\|$$

The second term tends to 0 as $h \to 0$, since because $\operatorname{grad}(\rho) = \nu \; \|\partial S\|$-almost everywhere on B, the contribution of the boundary is 0, while as $h \to 0$ the contribution of $M - B$ vanishes.

The third term is then easily seen to be $\int_B fH \cdot \operatorname{grad} \rho \, d\,\|\partial T\|$ as $h \to 0$, and the lemma is established.

Of these terms, only μ_4 is not estimated in terms of our desired quantities. To handle this, we observe that

$$\frac{d}{dt} \operatorname{vol}(g_t(T)) = \mu_1(f) + \mu_2(f).$$

If $f \geq 0$ everywhere, then $-f \cdot \operatorname{grad}(\rho)$ points everywhere interior to M, so that by the minimizing property of T,

$$\frac{d}{dt}\left(\frac{\operatorname{area}(g_t(\partial T))}{\operatorname{vol}(g_t(T))}\right) = \frac{1}{\operatorname{vol}(T)}([\mu(-f) - h(M)[\mu_1(-f) + \mu_2(-f)]) \geq 0$$

and we get

$$\mu_3(-f) + \mu_4(-f) \geq h(M)\mu_2(-f)$$

and hence

$$0 \leq \mu_4(f) \leq h(M)\mu_2(f) - \mu_3(f)$$

when $f \geq 0$. Now we have, for general f,

$$\begin{aligned}
|\mu(f)| &= |h(M)\mu_1(f) + \mu_3(f) + \mu_4(f)| \\
&\leq |h(M)\mu_1(f)| + |\mu_3(f)| + \mu_4(f^+) + \mu_4(f^-) \\
&\leq |h(M)\mu_1(f)| + |h(M)\mu_2(|f|)| + 2\,|\mu_3(|f|)| \\
&\leq h(M) \int_{M \cup B} |f|\, \rho \cdot \nu \,|\, d\,\|\partial T\| + 2 \int_B |f| \cdot |H| \, d\,\|\partial T\|
\end{aligned}$$

from which we see that the mean curvature of ∂T is bounded in absolute value by $1 + 2\,|H|$, and the theorem is established.

We remark that it seems likely that this bound can be improved. Perhaps a more careful argument would give the mean curvature of ∂T at $\operatorname{supp}(\partial T) \cap B$ to be bounded by the mean curvature of B.

§4. Proof of Theorem 1

In this section we complete the proof of Theorem 1 by showing that if $\lambda_0(\tilde{M})=0$, then $\pi_1(M)$ is amenable. The idea of the proof is to use the assumption that $\lambda_0(\tilde{M})=0$ to find a sequence of functions f_i with compact support in $\tilde{M}$ such that

$$\frac{\int f_i \cdot \Delta f_i}{\int f_i^2} = \varepsilon_i \to 0 \quad \text{as} \quad i \to \infty.$$

From this we conclude from an argument of Cheeger ([4], and see below) the existence of hypersurfaces S_i dividing $\tilde{M}$ into a bounded and an unbounded component, such that the isoperimetric ratio

$$\frac{\operatorname{area}(S_i)}{\operatorname{vol}(\operatorname{int}(S_i))}$$

(where $\operatorname{int}(S_i)$ denotes the bounded component of M) goes to 0 as $i \to \infty$.

Then we would like to replace $\operatorname{int}(S_i)$ with the fundamental domains which intersect $\operatorname{int}(S_i)$ in hopes of applying Proposition 2, and concluding that $\pi_1(M)$ is amenable.

A moment's thought, however, will convince one that this approach does not quite work. Given a hypersurface S_i, one may attach to it a long, thin spike which will increase both the surface area and the enclosed volume by an arbitrarily small amount, but which will wander through a large number of fundamental domains.

To overcome this difficulty, we apply the theorem of §3 to replace the hypersurfaces S_i by integral currents T_i, whose isoperimetric ratio is not greater than that of S_i, but whose mean curvature is bounded independent of i. It will be seen below that this is sufficient to imply that the isoperimetric ratio of the union of fundamental domains meeting $\operatorname{supp}(T_i)$ tends to 0 as $i \to \infty$, and so Proposition 2 applies to conclude that $\pi_1(M)$ is amenable.

We begin first with a review of Cheeger's inequality [4] from our present viewpoint.

Let N be a compact manifold with non-empty boundary, and let f be a function satisfying Dirichlet boundary conditions (for us it will suffice to assume that $f \equiv 0$ in a neighborhood of ∂N). For almost all $t>0$, the level surface S_t is a smooth hypersurface in N, by Sard's Theorem; S_t divides N into two components corresponding to $f(x)>t$ and $f(x)<t$. Let $\operatorname{int}(S_t)$ denote the first of these components.

Then Cheeger's inequality says that, for some t, we have the inequality

$$\frac{\int f \cdot \Delta f}{\int f^2} \geq \frac{1}{4}\left(\frac{\text{area}\,(S_t)}{\text{vol}\,(\text{int}\, S_t)}\right)^2.$$

In particular, taking the infimum of both sides, and allowing the hypersurface S to range over all possible hypersurfaces, gives the lower bound $\lambda_0(N) \geq \frac{1}{4}h^2$ for the spectrum of Δ on N with Dirichlet boundary conditions, in terms of the isoperimetric constant

$$h = \inf_S \frac{\text{area}\,(S)}{\text{vol}\,(\text{int}\,(S))}.$$

In the situation at hand, we are given that $\lambda_0(\tilde{M}) = 0$; it follows from the definitions that there are functions f_i with compact support with the property that

$$\frac{\int_{\tilde{M}} f_i \cdot \Delta f_i}{\int_{\tilde{M}} (f_i)^2} = \varepsilon_i \to 0 \quad \text{as} \quad i \to \infty.$$

We now pick a fundamental domain F for the action of $\pi_1(M)$, and let H_i be a finite union of translates of F so that $\text{supp}\,(f_i) \subset H_i$. It follows from Cheeger's inequality that there exist hypersurfaces S_i contained in H_i such that

$$\frac{\text{area}\,(S_i)}{\text{vol}\,(\text{int}\,(S_i))} \to 0 \quad \text{as} \quad i \to \infty.$$

In order to apply the results of §3, we need a uniform way of "smoothing the boundary" of H_i, so that the mean curvature of the boundary is uniformly bounded. This is easily done – cover a neighborhood of the boundary of F by open sets U_i, and for each U_i choose smooth functions g_i which are $\equiv 1$ for $x \in F$, and which are $\equiv 0$ whenever $\text{dist}\,(x, F) \geq \varepsilon_1$, where ε_1 is an arbitrarily small constant. If U_i contains a corner of F, so that U_i meets more than two translates of F, we choose one such g_i for all the possible ways each of these translates may either be included or not included in H_i.

We now cover H_i by translates of the U_i's, choose g_i's as above, and glue them together by a partition of unity to get a smooth function g. For some ε_2, then, $g^{-1}(1-\varepsilon_2)$ will be a smooth hypersurface of $\tilde{M}$, and we set $\tilde{H}_i = H_i \cup g^{-1}((1-\varepsilon_2, 1))$, $\partial\tilde{H}_i = g^{-1}(1-\varepsilon_2)$. The result of §3 now applies to give:

LEMMA. *There are integral currents T_i in $\tilde{M}$ such that*:

(i) $\dfrac{\text{area}(\partial T_i)}{\text{vol}(T_i)} \to 0$ as $i \to \infty$

(ii) *The mean curvature of ∂T_i is bounded in absolute value independent of i.*

We are now ready to apply Proposition 2 to conclude the amenability of $\pi_1(M)$. To that end, let $\varepsilon > 0$ and fundamental domain F be given.

For each T_i, let K_i be the union of translates of F meeting T_i

$$K_i = \bigcup_{g \cdot F \cap T_i \neq \varnothing} g \cdot F$$

PROPOSITION 3. *There is a constant κ, depending only on F, such that*

$$\frac{\text{area}(\partial K_i)}{\text{vol}(K_i)} \leq \kappa \cdot \frac{\text{area}(\partial T_i)}{\text{vol}(T_i)}.$$

Given Prop. 3, Proposition 2 implies the theorem, since we may choose i sufficiently large so that

$$\kappa \cdot \frac{\text{area}(T_i)}{\text{vol}(\text{int}(T_i))} < \varepsilon.$$

Proof of Proposition 3. It is clear that $\text{vol}(K_i) \geq \text{vol}(T_i)$, since $T_i \subseteq K_i$, so it suffices to show that

$$\text{area}(\partial K_i) \leq \kappa \cdot \text{area}(\partial T_i).$$

To see this, we cover M with finitely many geodesic balls B_δ of radius δ, where 3δ is less than the injectivity radius of M. We lift these balls to a covering B_δ^k of $\tilde{M}$.

For each ball B_δ^k such that $B_\delta^k \cap T_i \neq \varnothing$, let x_k be a point in this intersection. Then the balls $B_{2\delta}(x_k)$ cover T_i.

CLAIM. There is a constant K such that

$$\text{area}(B_{2\delta}(x_k) \cap T_i) \geq K.$$

Proof. It follows from the Monotonicity Theorem ([18], Remark 3.15, or [1]), that there is a constant M such that

$$\frac{\text{area}\,(B_r(x))e^{Mr}}{r^{n-1}}$$

is monotone increasing in r, where area $(B_r(x))$ is the volume of a geodesic ball of radius r about x in T_i, and M can be estimated from the mean curvature of T_i and the second fundamental form of some isometric embedding of $\tilde{M}$ into $\mathbf{R}^N$. Furthermore, as $r \to 0$ the limit of this expression is $\geq \alpha(n-1)$, the volume of a unit ball in $\mathbf{R}^{n-1}$, whenever $x \in T_i$.

It follows that area $(B_{2\delta}(x_k)) \cap T^i \geq e^{-M(2\delta)}(2\delta)^{n-1}\alpha(n-1)$, establishing the claim.

It follows from the compactness of M that there is a number L so that at most L disks $B_{2\delta}(x_k)$ meet at any given point of $\tilde{M}$. It similarly follows that there is a number m such that any $B_{2\delta}(x_k)$ lies in at most m fundamental domains.

Setting $A = \text{area}\,(\partial F)$, and letting N denote the number of balls $B_{2\delta}(x_k)$ which meet T_i, we have

$$\text{area}\,(T_i) \geq \frac{N \cdot K}{L} \geq \frac{K}{L} m \frac{\text{area}\,(\partial K_i)}{A}$$

and setting const $= \dfrac{AL}{Km}$ establishes the theorem.

§5. The heat equation

We now give an alternate description of amenable groups, due to Kesten [12].

For G a countable discrete group, a probability distribution P on G is a function $P: G \to [0, 1]$ such that $\sum_{g \in G} P(g) = 1$. We will assume that P is symmetric – i.e., $P(g) = P(g^{-1})$ – and we will also assume that $P(e) > 0$.

In a natural way, P determines a left-invariant random walk on G, by the following construction: given an element $x \in G$, the probability, when taking a step from x, of landing on xg, is precisely $P(g)$.

If $\{g_i\}$ denotes some indexing of the group G, let M denote the infinite matrix

$$M = (m_{i,j}) \quad \text{where} \quad m_{i,j} = P(g_i^{-1} g_j)$$

M is clearly a symmetric matrix, all of whose entries are positive, and whose rows and columns sum to 1. Let also $P^{(k)}$ denote the function

$$P^{(k)}(g)=\sum_{g_1\cdot g_2\cdots g_k=g} P(g_1)\cdot P(g_2)\cdots P(g_k).$$

Then $P^{(k)}$ denotes the probability distribution associated to the random walk corresponding to taking k steps in the random walk determined by P; the associated matrix $M^{(k)}$ is obtained from M by k-fold matrix multiplication.

We define the Kesten number $\kappa(P)$ by the formula

$$\kappa(P)=\lim_{k\to\infty}(m_{g,g}^{k})^{1/k}$$

where g is any element of G, for instance e.

$\kappa(P)$ has a number of interpretations. One sees readily that if $\lambda_1<\kappa(P)<\lambda_2$, then for sufficiently large k we have $(\lambda_1)^k<P^{(k)}(e)<(\lambda_2)^k$ – thus $\kappa(P)$ is an asymptotic estimate on the probability that after k steps of the random walk P, one has returned to the starting point.

Another interpretation arises by allowing P to act on $l^2(G)$ according to the convolution law

$$(P*f)(g)=\sum_{g'} f(gg')\cdot P(g').$$

Then $\kappa(P)$ is precisely the spectral radius of this operator P.

With this understood, we state the following:

THEOREM (Kesten) [12]. *Assuming that the support of P generates the group G, then $\kappa(P)=1$ if and only if G is amenable.*

In particular, whether or not $\kappa(P)=1$ does not depend on P. The value of $\kappa(P)$, however, doesn't have an intrinsic group-theoretic meaning, and depends on P.

Now we introduce a probability distribution on $\pi_1(M)$ in the following geometric way:

Let $H_t(x, y)$ denote the fundamental solution for the heat equation in $\tilde{M}$ – the existence and uniqueness of $H_t(x, y)$ is established in [7], see also [5]. We also fix for the discussion a fundamental domain F in $\tilde{M}$.

For each t, let $P_t(g)$ be the probability distribution defined by the formula

$$P_t(g) = \frac{1}{\text{vol}(F)} \int_{F \times g \cdot F} H_t(x, y)\, dx\, dy.$$

It is evident that $P_t(g)$ is a symmetric probability distribution on $\pi_1(M)$ whose support is all of $\pi_1(M)$. Let κ_t denote the Kesten number associated to P_t.

Intuitively, P_t describes the following random walk on $\tilde{M}$: at time 0, a unit amount of heat is uniformly distributed throughout F. The heat then distributes itself about $\tilde{M}$, according to the heat equation, until at time t. At this point, the heat is redistributed evenly within each fundamental domain. Then the heat flows according to the heat equation until time $2t$, at which time the heat is again redistributed evenly in each fundamental domain, and the process continues. Then κ_t is an asymptotic measure of the heat remaining in F.

We compare this with the random walk on $\tilde{M}$ which is given by the undisrupted flow of heat for all time. We define a number κ_∞, the analogue of the Kesten number in this case, by the formula

$$\kappa_\infty = \lim_{t \to \infty} \left(\int_{F \times F} H_t(x, y)\, dx\, dy \right)^{1/t} = e^{-\lambda_0(\tilde{M})}.$$

where the last equality follows from the spectral representation of $H_t(x, y)$. Thus Theorem 1 and Kesten's Theorem give

THEOREM 4. *For any t, $\kappa_t = 1$ if and only if $\lambda_0(\tilde{M}) = 0$.*

More precisely, one sees easily that

$$(P_t(e))^{1/t} = \left(\frac{1}{\text{vol}(F)} \int_{F \times F} H_t(x, y)\, dx\, dy \right)^{1/t} \to e^{-\lambda_0(\tilde{M})} \quad \text{as} \quad t \to \infty,$$

and that $\kappa_t \geq P_t(e)$, so that $\lim_{t \to \infty} (\kappa_t)^{1/t} = e^{-\lambda_0(M)}$. Hence it is reasonable to ask:

Question. Do we have $\lim_{t \to \infty} (\kappa_t)^{1/t} = e^{-\lambda_0(M)}$?

An affirmative answer would give a new proof of the result of §2. If one could also show that the right-hand side cannot be 1 without each term in the limit on the left-hand side being 1, then one would have a new proof of Theorem 1.

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