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Arcs omitted by support points of univalent functions

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This paper develops a general method for investigating certain geometric properties of the analytic arc omitted by a support point in the class S of univalent functions. The main result is that for every support point arising from point-evaluation of the derivative, the omitted arc has monotonic argument.

§1. Introduction

The class S consists of all functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots$$

analytic and univalent in the unit disk $|z|<1$. A *support point* of S is a function $f\in S$ which maximizes $\mathrm{Re}\{L\}$ for some complex-valued continuous linear functional L not constant on S. For instance, the Koebe function $k(z)=z(1-z)^{-2}$ and its rotations are support points of S because they maximize $\mathrm{Re}\{e^{i\theta}a_2\}$. There are other examples, but a complete description is not yet available.

Some general properties are well known (see *e.g.* [10], [4], or [5]). Each support point f maps the disk onto the complement of an analytic arc Γ which extends to infinity with increasing modulus and satisfies the differential equation

$$\Phi(w)\frac{dw^2}{w^2}>0, \qquad \Phi(w)=L\left(\frac{f^2}{f-w}\right). \tag{1}$$

This function Φ is analytic on Γ and has the property $\mathrm{Re}\{\Phi(w)\}>0$ there, except perhaps at the endpoints. At each point of Γ, the angle measured from the radial line to the tangent line is called the *radial angle* of Γ. In view of (1), the property $\mathrm{Re}\{\Phi(w)\}>0$ is equivalent to $\mathrm{Re}\{(dw/w)^2\}>0$, which says geometrically that the radial angle of Γ is less than $\pi/4$ in magnitude at each interior point. This is known as the $\pi/4$-*property*.

Pfluger [8] showed that $L(f^2) \neq 0$, or equivalently that the quadratic differential (1) has a simple pole at infinity. Brickman and Wilken [1] observed that Γ is tangent to the half-line

$$w = \frac{L(f^3)}{3L(f^2)} - L(f^2)t, \qquad t \geq 0 \tag{2}$$

at infinity. This is called the *asymptotic half-line*. It should be remarked that if the asymptotic half-line is a trajectory of the quadratic differential (1) near infinity, then Γ is itself a half-line which coincides with the asymptotic half-line near infinity. This is true because the quadratic differential has a unique trajectory extending to its simple pole at infinity.

For the support points associated with point-evaluation functionals $L(f) = f(\zeta)$, Brown [2] found that the omitted arc Γ has monotonic argument and monotonic radial angle. No rotation of the Koebe function can maximize $\mathrm{Re}\,\{f(\zeta)\}$ unless ζ lies on a certain segment of the real axis (*cf.* Schober [10], p. 84). For a certain choice of ζ on the negative real axis, Brown found numerically that the radial angle at the tip of Γ is approximately equal to $\pi/4$. This was a "numerical proof" that the upper bound $\pi/4$ in the general description of support points is best possible. For the functionals

$$L(f) = a_3 + \lambda a_2, \qquad \lambda \in \boldsymbol{C},$$

which typically exclude all rotations of the Koebe function as support points, Brown [3] again showed that Γ has monotonic argument.

More recently, Pearce [7] made the observation that for $0 < \zeta \leq \sin \pi/8$, the nonlinear problem of maximizing (or minimizing) $\arg f'(\zeta)$ is equivalent to a certain linear problem of the form

$$\max_{f \in S} \mathrm{Re}\,\{e^{-i\sigma} f'(\zeta)\}, \tag{3}$$

since the set of values $\{f'(\zeta) : f \in S\}$ then lies in the right half-plane and is supported by the radial line of maximal argument. Thus the functions

$$f(z) = \frac{z - \zeta z^2}{(1 - e^{i\theta} z)^2}, \qquad \theta = \cos^{-1} \zeta, \tag{4}$$

which are known to maximize $\arg f'(\zeta)$ over S for each $\zeta \leq 1/\sqrt{2}$ (see [6], p. 115), are support points of S if $\zeta \leq \sin \pi/8$. The arc Γ omitted by a function f of the form (4) is a half-line which for $\zeta = \sin \pi/8$ has a radial angle of exactly $\pi/4$ at its tip. In all cases, this arc Γ has monotonic argument and monotonic radial angle.

We shall generalize this result by proving that every support point arising from a problem of the type (3), for arbitrary real σ and $0<\zeta<1$, has an omitted arc Γ with monotonic argument. (There is no loss of generality in assuming that ζ is real and positive.) Without appeal to the rotation theorem, we shall observe independently that Γ is a nonradial half-line if and only if the extremal problem (3) is equivalent to that of maximizing or minimizing $\arg f'(\zeta)$. In a sense these results complement the work of Grad [9], who described the region of values of $f'(\zeta)$ as f ranges over S. We are concerned with the geometric properties of those functions f for which $f'(\zeta)$ lies on the boundary of the convex hull of this region.

Before turning to the problem (3), we shall develop some general techniques for using the differential equation (1) to deduce certain geometric properties of the arc Γ omitted by a support point. This method is then applied to establish the monotonic argument result for support points arising from both the point-evaluation problem and the derivative problem (3). In all cases it is the asymptotic half-line which distinguishes the trajectory Γ and makes this approach effective.

§2. General criteria for monotonicity

Let $f\in S$ be a support point which maximizes $\operatorname{Re}\{L\}$. Then f omits an analytic arc Γ which extends with increasing modulus from a point w_0 to infinity and satisfies the differential equation (1). Let Γ have a nonsingular parametrization $w=w(t)$, $0\le t\le\infty$, with $w(0)=w_0$ and $w(\infty)=\infty$. Then the relation

$$\frac{d}{dt}\arg w(t)=\operatorname{Im}\left\{\frac{w'(t)}{w(t)}\right\}$$

shows that $\arg w$ is increasing where $\operatorname{Im}\{w'/w\}>0$ and decreasing where $\operatorname{Im}\{w'/w\}<0$. On the other hand, $|\arg\{w'/w\}|<\pi/4$ by the $\pi/4$-property of Γ. Thus $\arg w$ is increasing or decreasing according as $\operatorname{Im}\{(w'/w)^2\}$ is positive or negative. In view of (1), this says that $\arg w$ increases where $\operatorname{Im}\{\Phi(w)\}<0$ and decreases where $\operatorname{Im}\{\Phi(w)\}>0$. If $\arg w$ ever reverses direction, it must do so on the set where $\Phi(w)>0$.

It is equivalent to the $\pi/4$-property that Γ lies entirely in the region where $\operatorname{Re}\{\Phi(w)\}>0$, except perhaps for the endpoint w_0. This will be called the *accessible region.* The set where $\operatorname{Re}\{\Phi(w)\}<0$ will be called the *forbidden region.* The range of f contains the forbidden region. The set where $\Phi(w)$ is real will be called the *critical set.*

It is clear that the intersection of the critical set with the accessible region (*i.e.*, the set of points where $\Phi(w)>0$) coincides with the set where trajectories are tangent to radial lines; while the part of the critical set in the forbidden region (the set of points where $\Phi(w)<0$) is the locus of points where trajectories are tangent to circles centered at the origin. The critical set is a collection of curves which divide the accessible region into subregions where the trajectories have strictly monotonic argument. In order to show that the trajectory Γ omitted by a support point has strictly monotonic argument, it is sufficient (in fact, equivalent) to show that Γ lies entirely in one of these subregions. It is here that the explicit knowledge of the asymptotic half-line becomes a powerful tool.

The monotonicity of the radial angle

$$\alpha(t)=\arg\left\{\frac{w'(t)}{w(t)}\right\}$$

can be discussed in a similar way. The differential equation (1) shows that

$$\arg \Phi(w(t))+2\alpha(t)=0,$$

and so

$$\alpha'(t)=-\tfrac{1}{2}\operatorname{Im}\left\{\frac{\Phi'(w)w'(t)}{\Phi(w)}\right\}.$$

Thus $\alpha'(t)=0$ if and only if

$$\left\{\frac{\Phi'(w)w'(t)}{\Phi(w)}\right\}^2\geq 0.$$

Comparing this with (1), we see that if the radial angle α ever reverses direction, it must do so on the set where

$$\frac{[\Phi'(w)]^2w^2}{[\Phi(w)]^3}\geq 0. \tag{5}$$

It should be remarked that for the trajectory Γ omitted by a support point, the monotonicity of the radial angle implies the monotonicity of the argument. This is true because the radial angle is bounded by $\pi/4$ and tends to zero at infinity as Γ approaches its asymptotic half-line (*cf.* Brown [2]).

§3. Applications to the point-evaluation problem

As a first illustration of the method, we now consider the point-evaluation functional $L(f)=f(\zeta)$ for fixed ζ with $|\zeta|<1$. Let $f\in S$ be a corresponding support point, and let Γ be its omitted arc. Then Γ satisfies the differential equation (1) with $\Phi(w)=B^2(B-w)^{-1}$ and $B=f(\zeta)$. Brown [2] integrated the differential equation to show that Γ is essentially the image under the Koebe function of a logarithmic spiral. This and the implicit determination of B as a function of ζ enabled him to show that Γ has monotonic argument and monotonic radial angle, and that Γ lies in a certain half-strip excluding the origin. We shall now use the tools developed in §2 to recapture some of these qualitative properties directly from the differential equation.

The asymptotic half-line (2) reduces to

$$w=\frac{B}{3}-B^2t, \qquad 0\le t<\infty.$$

The accessible region is a half-plane bounded by the line through B orthogonal to the asymptotic half-line. The critical set is the line

$$w=B-B^2t, \qquad -\infty<t<\infty,$$

through B parallel to the asymptotic half-line (see Figure 1). We may assume that $\operatorname{Im}\{B\}>0$, since the class S is preserved under conjugation and f is the Koebe function if B is real. The asymptotic half-line lies between the origin and the critical line. If Γ were to enter the quarter-plane I where $\operatorname{Im}\{\Phi(w)\}<0$, it would be forced to cross the critical line radially, violating its monotonic modulus property, in order to approach the asymptotic half-line. Thus Γ is confined to the quarter-plane II where $\operatorname{Im}\{\Phi(w)\}>0$, which contains the asymptotic half-line. In this region, $\arg w$ must decrease as $|w|$ increases. This proves that Γ has monotonic argument.

If $\operatorname{Re}\{B\}\le 0$, then the origin lies in the forbidden half-plane or on its boundary (because $\Phi(0)=B$), and Γ is restricted to a half-plane bounded by the line

$$w=iB^2t, \qquad -\infty<t<\infty,$$

through the origin. The monotonicity of arg w then restricts Γ to the quarter-plane bounded by the rays $w=-B^2t$ and $w=-iB^2t$, $t\ge 0$. This quarter-plane

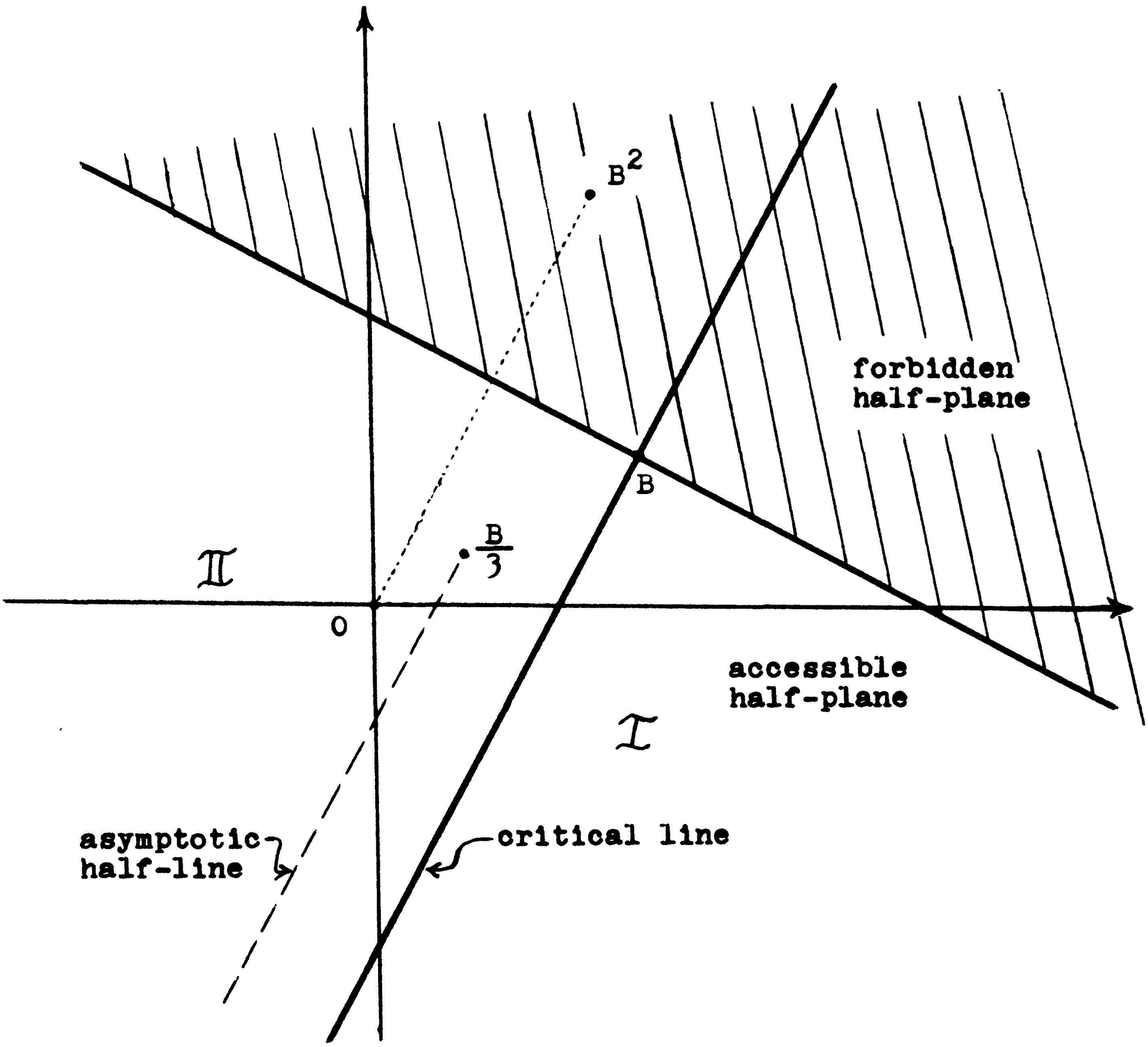

Figure 1

contains the asymptotic half-line near infinity. The case where $\operatorname{Re}\{B\}>0$ is more difficult.

Now consider the radial angle. The equation (5), which determines the locus of points where the radial angle may be stationary, takes the form

$$B^2(B-w)w^{-2} \geq 0.$$

This is a curve γ from B to ∞, passing through the origin. It may be analyzed by setting

$$B^2(B-w)w^{-2} = t > 0$$

and observing that w is then determined by the quadratic equation

$$tw^2+B^2w-B^3=0,$$

with solutions

$$w=-\frac{B^2}{2t}\left\{1\pm\left(1+\frac{4t}{B}\right)^{1/2}\right\}.$$

For small t these two solutions are

$$w=B-t+O(t^2)$$

and

$$w=-\frac{B^2}{t}-B+t+O(t^2).$$

As $t\to 0$, the first solution approaches B asymptotic to the half-line $w=B-t$, while the second tends to infinity asymptotic to the half-line $w=-B-B^2/t$. For large t the two solutions are

$$w=\pm B^{3/2}t^{-1/2}+O(t^{-1}), \qquad t\to\infty,$$

which approach the origin in opposite directions. Because Γ and γ have different (but parallel) asymptotes near infinity, it follows that they have no intersections outside a circle of sufficiently large radius. Thus the radial angle of Γ is eventually monotonic.

§4. Applications to the derivative problem

We now return to the linear problem (3) of maximizing $\mathrm{Re}\{e^{-i\sigma}f'(\zeta)\}$ over all functions $f\in S$, where σ is an arbitrary real number and ζ is an arbitrary point in the unit disk. We shall prove the following theorem.

THEOREM. *Let $f\in S$ be a function which maximizes* $\mathrm{Re}\{e^{-i\sigma}g'(\zeta)\}$ *among all $g\in S$, and let Γ be its omitted arc. Then Γ has monotonic argument.*

Proof. Since S is preserved under rotations, we may assume without loss of generality that $0<\zeta<1$. Specializing the general result for support points, we see that an extremal function f must map the disk onto the complement of an analytic arc Γ which satisfies

$$\Phi(w)\frac{dw^2}{w^2}>0, \tag{6}$$

where

$$\Phi(w)=\frac{e^{-i\sigma}CB(B-2w)}{(B-w)^2}, \qquad B=f(\zeta), \qquad C=f'(\zeta).$$

The asymptotic half-line (2) takes the form

$$w=\frac{B}{2}-e^{-i\sigma}CBt, \qquad t\geq 0.$$

If $e^{-i\sigma}C$ is real, the asymptotic half-line is radial and is a trajectory of the quadratic differential. This implies that Γ is a radial half-line and that f is a rotation of the Koebe function. Thus we may assume that $e^{-i\sigma}C$ is not real. In fact, we may assume without essential loss of generality that $\operatorname{Im}\{e^{-i\sigma}C\}>0$, since S is preserved under conjugation and so the region of values of $g'(\zeta)$ is symmetric with respect to the real axis. More specifically, f maximizes $\operatorname{Re}\{e^{-i\sigma}g'(\zeta)\}$ if and only if its conjugate $\bar{f}$ (defined by $\bar{f}(z)=\overline{f(\bar{z})}$) maximizes $\operatorname{Re}\{e^{i\sigma}g'(\zeta)\}$. Having made these reductions, we may set

$$e^{-i\sigma}C=e^{2i\gamma}\,|C|, \qquad 0<\gamma<\frac{\pi}{2}.$$

We also let

$$\omega=\xi+i\eta=\frac{e^{i\gamma}w}{B-w}, \tag{7}$$

whereupon a simple calculation gives

$$\Phi(w)=|C|\,\{e^{2i\gamma}-\omega^2\}.$$

The boundary of the accessible region is the curve $\operatorname{Re}\{\Phi(w)\}=0$, which in the ω-plane is the (possibly degenerate) hyperbola

$$\xi^2-\eta^2=\cos 2\gamma. \tag{8}$$

The critical curve $\operatorname{Im}\{\Phi(w)\}=0$ transforms to the hyperbola

$$2\xi\eta = \sin 2\gamma \tag{9}$$

in the ω-plane.

The variables w and ω are related by the linear fractional transformation (7) with the following table of values:

w	0	∞	$B/2$	B
ω	0	$-e^{i\gamma}$	$e^{i\gamma}$	∞

The line in the w-plane through 0 and B is mapped onto the line in the ω-plane through $e^{i\gamma}$ and $-e^{i\gamma}$. More generally, an arbitrary radial half-line in the w-plane is mapped onto a circular arc joining 0 and $-e^{i\gamma}$ in the ω-plane. Circles in the w-plane centered at the origin are mapped onto circles of Apollonius in the ω-plane with 0 and $-e^{i\gamma}$ as inverse points. The asymptotic half-line in the w-plane is mapped onto a circular arc in the ω-plane joining $e^{i\gamma}$ and $-e^{i\gamma}$. The typical situation is illustrated in Figures 2 and 3.

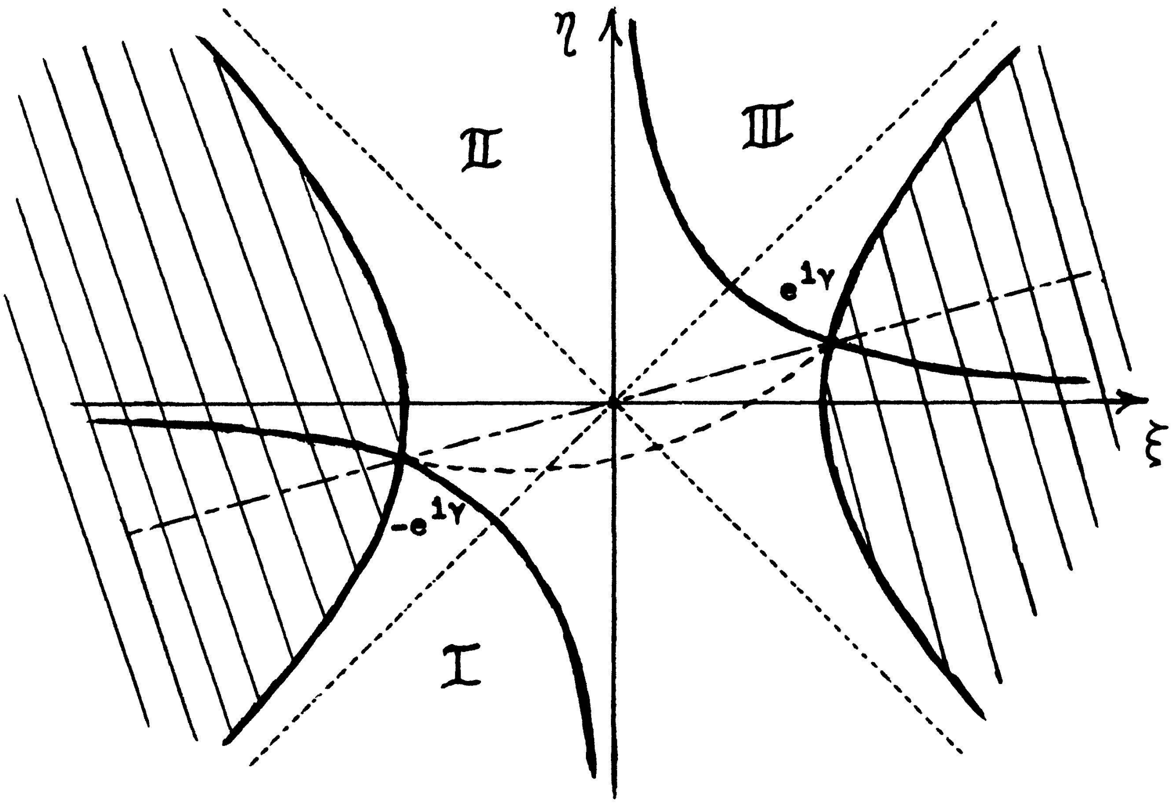

Figure 2. ω-plane, $0<\gamma<\pi/4$.

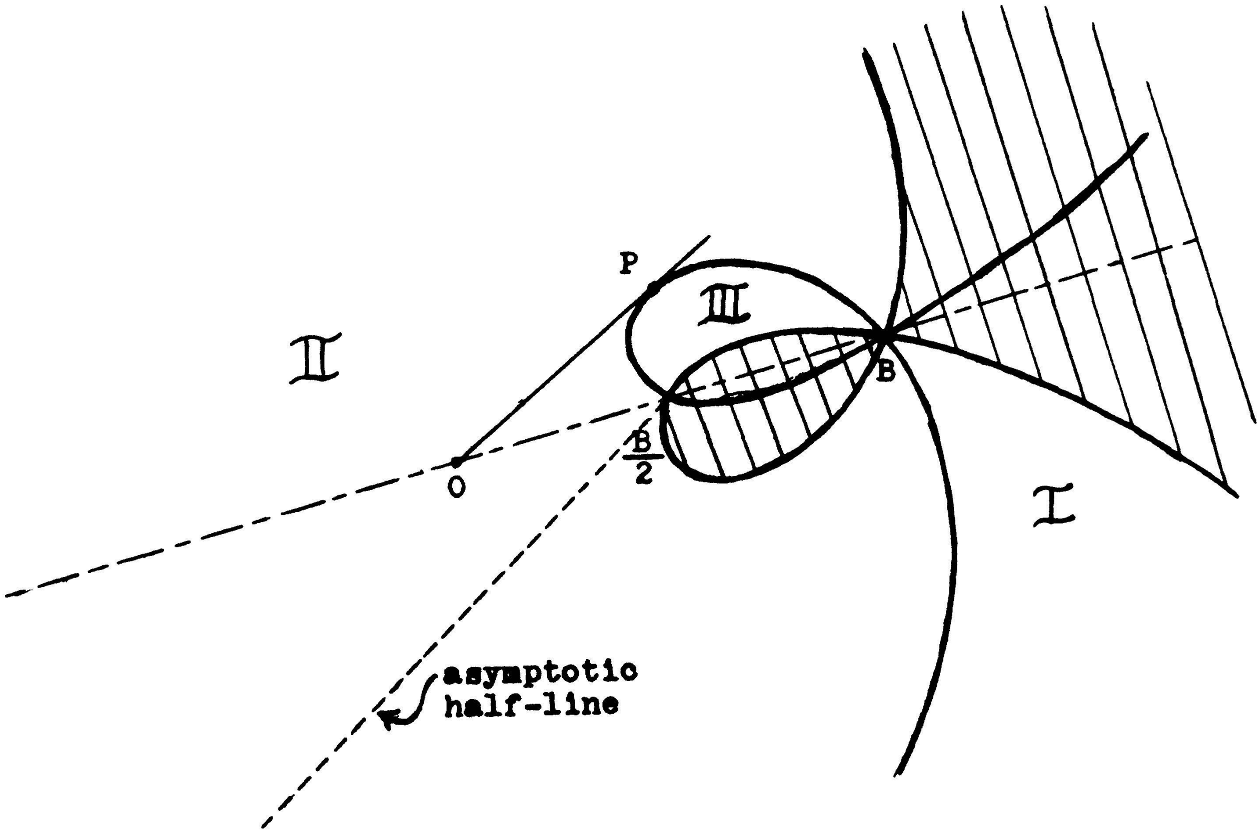

Figure 3. w-plane, $0<\gamma<\pi/4$.

Suppose first that $0<\gamma<\pi/4$, so that $\cos 2\gamma>0$ and $\sin 2\gamma>0$. The hyperbolas (8) and (9) are then as shown in Figure 2. Observe that the two hyperbolas intersect at the points $\pm e^{i\gamma}$. The image of the forbidden region in the ω-plane is the shaded region determined by the inequality

$$\cos 2\gamma+\eta^2<\xi^2.$$

The critical curve divides the accessible region into three subregions which are labeled I, II, and III in Figure 3. Observe that $\operatorname{Im}\{\Phi(w)\}<0$ in Regions I and III, while $\operatorname{Im}\{\Phi(w)\}>0$ in Region II. According to our previous remarks (§2), this implies that as Γ is traversed with increasing modulus, its argument must decrease in Region II and increase in Regions I and III. Should Γ ever pass from one region to another, it must have radial direction as it crosses the critical curve.

Our first observation is that the asymptotic half-line

$$w=\frac{B}{2}-e^{2i\gamma}Bt, \qquad t\geq 0, \tag{10}$$

lies entirely in Region II, except perhaps for an initial segment near $B/2$. Indeed, its image in the ω-plane is a circular arc from $e^{i\gamma}$ to $-e^{i\gamma}$ which makes an angle of 2γ with the line joining these two points, while the hyperbola (9) has slope $d\eta/d\xi = -\tan\gamma$ at $-e^{i\gamma}$. Referring to Figure 2, we conclude that the circular arc is tangent to (9) at $-e^{i\gamma}$, which implies that the critical curve is tangent to the asymptotic half-line at ∞.

We claim that Γ lies entirely in Region II and so has decreasing argument. Suppose first that Γ has points in Region I. Then, in order to reach the asymptotic half-line at infinity, it must either cross the critical curve radially into Region II or approach the critical curve asymptotically. If Γ crosses the critical curve from Region I to Region II, it must do so with decreasing modulus, because each ray intersects this portion of the critical set at most once. (This follows from the fact that the image of such a ray is a circular arc in the ω-plane from 0 to $-e^{i\gamma}$, which can have at most one other intersection with the lower branch of the hyperbola (9).) If Γ approaches the critical curve asymptotically at infinity within Region I, it must do so with decreasing argument. (This is again seen by considering its image in the ω-plane, which would approach the point $-e^{i\gamma}$ within the image of Region I and would be tangent to the critical hyperbola there.) In Region I, however, we know that Γ must have *increasing* argument. This shows that no part of Γ can lie in Region I.

Suppose next that Γ has points in Region III. Then, in order to approach the asymptotic half-line at infinity, it must eventually cross the critical curve (radially) into Region II. To see that this is impossible, we observe that the critical curve which separates Region II from Region III is divided into two subarcs by a point P where the ray from the origin is tangent to the curve. At every point on the open subarc from $B/2$ to P, rays from the origin pass from Region II to Region III; while on the subarc from P to B, rays pass from Region III to Region II. (Again consider the image of such a ray in the ω-plane, which is a circular arc from 0 to $-e^{i\gamma}$. This circle intersects the upper branch of the hyperbola (9) at most twice.) If Γ crosses the critical curve from Region III to Region II, it must cross radially along the subarc from P to B (possibly at P), because of its increasing modulus. In Region II, however, the argument of Γ must decrease. Therefore, after Γ passes into Region II, it is blocked away from the asymptotic half-line by a curve which extends along the critical set from P to B, then radially from B to ∞. The radial half-line from B to ∞ lies in the forbidden region. This shows that no part of Γ can lie in Region III. Thus Γ lies entirely in Region II, where its argument is strictly decreasing. This concludes the proof that Γ has monotonic argument if $0<\gamma<\pi/4$.

The case $\pi/4<\gamma<\pi/2$ is somewhat similar. The hyperbola (9) remains in the same position, but the hyperbola (8) now has a vertical axis. The image in the ω-plane of the forbidden region now lies between the two branches of the

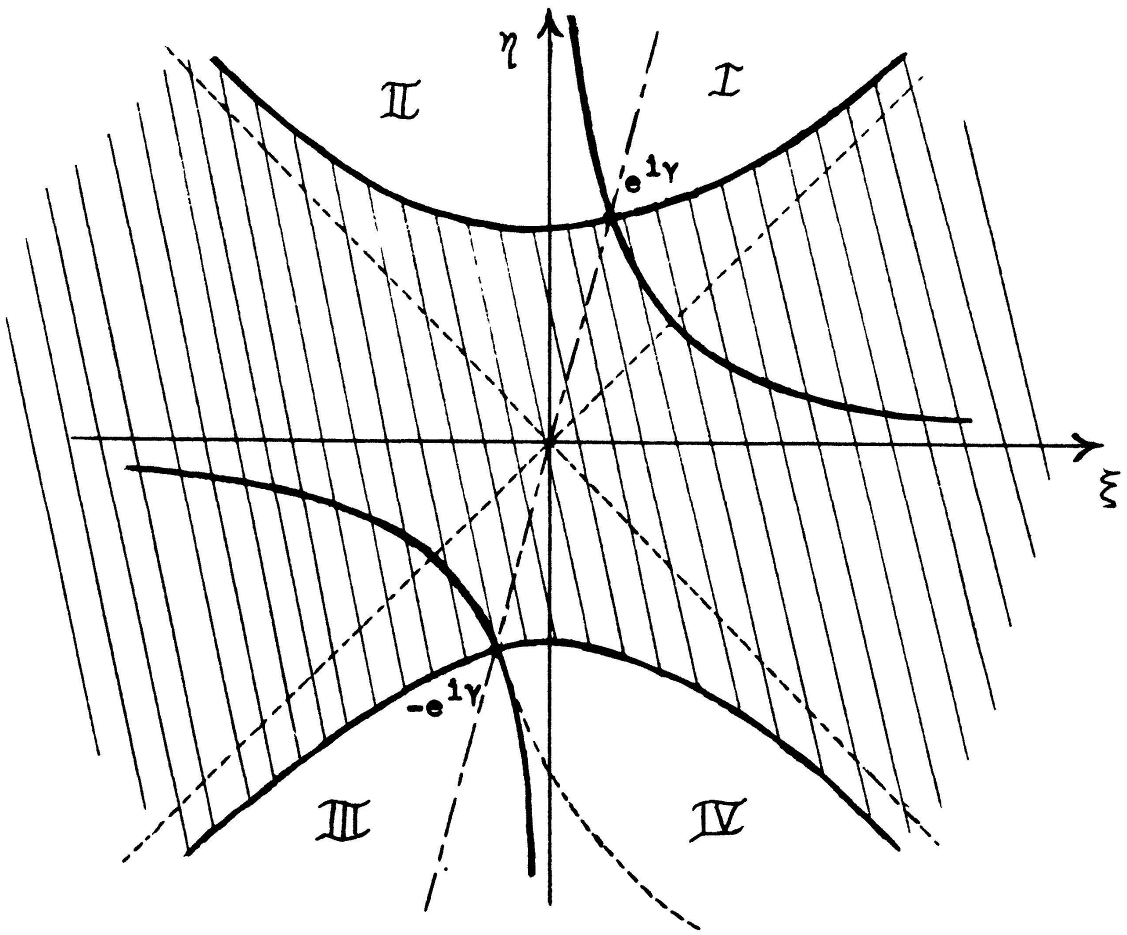

Figure 4. ω-plane, $\pi/4<\gamma<\pi/2$.

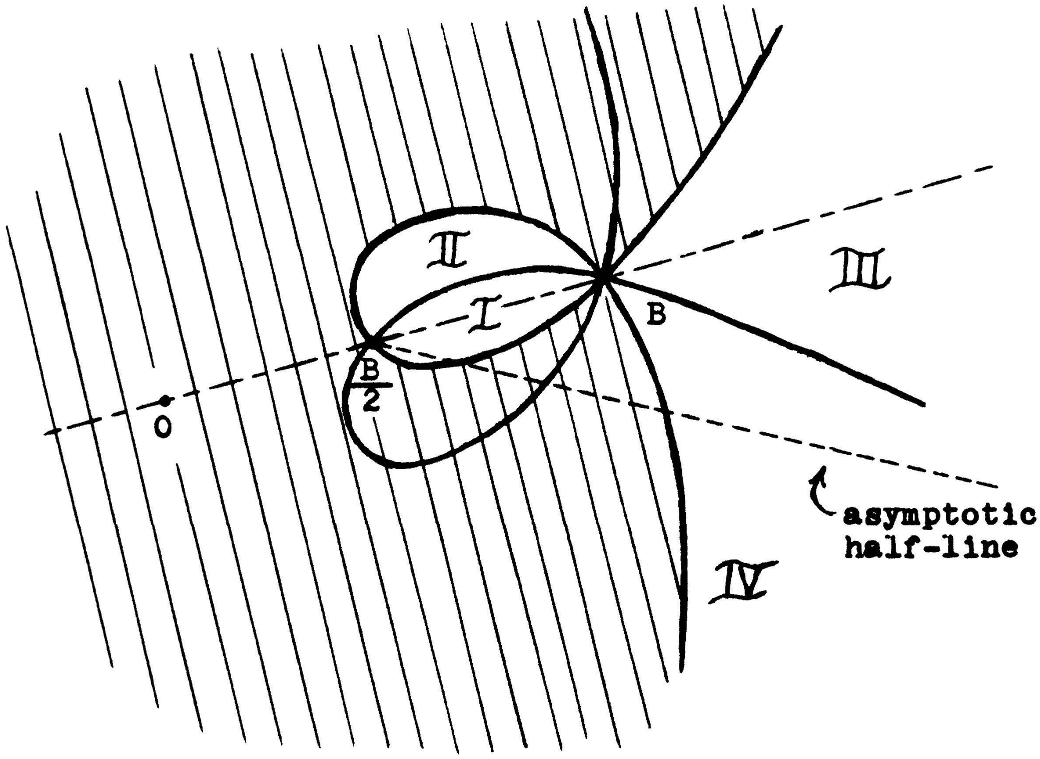

Figure 5. w-plane, $\pi/4<\gamma<\pi/2$.

hyperbola (8). The critical curve corresponds to the hyperbola (9) and divides the accessible region into four subregions. These are called Regions I, II, III, and IV, as indicated in Figures 4 and 5. Now $\operatorname{Im}\{\Phi(w)\}<0$ in Regions I and III, while $\operatorname{Im}\{\Phi(w)\}>0$ in Regions II and IV. The asymptotic half-line lies eventually in Region IV and is tangent to the critical curve at infinity. (This is seen by considering the image of the asymptotic half-line in the ω-plane, an arc of a circle through $e^{i\gamma}$ and $-e^{i\gamma}$ which makes an angle of $\pi-2\gamma$ with the line through these two points. Geometric considerations reveal that this circular arc is tangent to the lower branch of the hyperbola (9) at $-e^{i\gamma}$.) No part of Γ can lie in Region I or II, because these are bounded regions entirely surrounded by the forbidden region. (It is clear that $B\notin\Gamma$, since $f(\zeta)=B$.)

If any part of Γ lies in Region III, then eventually either Γ crosses the critical curve (radially) into Region IV, or Γ approaches the asymptotic half-line within Region III. However, a ray from the origin may meet this part of the critical curve at most once, where it crosses from Region IV into Region III. (This is seen by observing that its image in the ω-plane is a circular arc through 0 and $-e^{i\gamma}$ which meets the lower branch of the hyperbola (9) at $-e^{i\gamma}$ and at one other point at most, where it must cross.) Thus if Γ crosses from Region III to Region IV, it must do so with decreasing modulus, which is impossible. Finally, it is clear from Figure 5 that if Γ approaches the asymptotic half-line within Region III, it must eventually have decreasing argument. But this is impossible, since $\operatorname{Im}\{\Phi(w)\}<0$ in Region III, and so Γ has increasing argument there. Thus Γ is confined entirely to Region IV, where it has decreasing argument.

The final case $\gamma=\pi/4$ is quite special. Here $e^{-i\sigma}C=i\,|C|$, and it is easily verified that the asymptotic half-line

$$w=\frac{B}{2}-iBt, \qquad t>0,$$

satisfies the differential equation (6) for the omitted arc Γ. As we have already remarked (§1), this implies that Γ is itself a half-line which coincides with the asymptotic half-line near infinity. Since a half-line clearly has monotonic argument, this completes the proof of the theorem.

The last case merits closer inspection. What is the most general situation in which Γ is a half-line? This occurs if and only if the asymptotic half-line (10) is a trajectory (near infinity) of the quadratic differential (6). A direct calculation shows that this is true if and only if $e^{4i\gamma}$ is real. If $e^{4i\gamma}=1$, then $e^{2i\gamma}=\pm 1$ and Γ is a radial half-line, which implies that f is a rotation of the Koebe function. If $e^{4i\gamma}=-1$, then $e^{2i\gamma}=\pm i$ and $e^{-i\sigma}C=\pm i\,|C|$. On the other hand,

$$\max_{g\in S}\operatorname{Re}\{e^{-i\sigma}g'(\zeta)\}=\max_{g\in S}\operatorname{Im}\{ie^{-i\sigma}g'(\zeta)\}=0$$

if and only if $C=-ie^{i\sigma}\,|C|$, and it is clear geometrically that this occurs if and only if the extremal problem is equivalent to that of finding the maximum of $\arg g'(\zeta)$. Similarly, $C=ie^{i\sigma}\,|C|$ if and only if the extremal problem is equivalent to the minimum argument problem. In other words, the only cases (aside from rotations of the Koebe function) in which Γ is linear are those identified by Pearce [7], where the extremal problem (3) is equivalent to the maximum or minimum argument problem. It should be remarked that we have been led to this result without appeal to the rotation theorem and without prior knowledge of its specific extremal functions in the case $|\zeta|\leq 1/\sqrt{2}$. In fact, our approach demonstrates *a priori* that whenever ζ is chosen so that $\mathrm{Re}\{g'(\zeta)\}\geq 0$ for all $g\in S$, the maximum and minimum of $\arg g'(\zeta)$ are attained *only* by the elementary rational functions which map the disk onto the complement of a half-line.

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