Comment. Math. Helvetici 56 (1981) 272–296 0010–2571/81/002272–25$01.50+0.20/0

The Lefschetz number of an involution on the space of classes of positive definite quadratic forms

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Introduction

It is well known [2: §2] that the set of positive definite quadratic forms on $\mathbf{R}^n, n \in \mathbf{N}$, with determinant 1 can be identified with the symmetric space $X = \mathrm{SO}_n(\mathbf{R})\backslash \mathrm{Sl}_n(\mathbf{R})$. Here we use standard notation. Suppose that Γ is a subgroup of finite index in $\mathrm{Sl}_n(\mathbf{Z})$. Then two positive definite quadratic forms $q_1, q_2 : \mathbf{R}^n \to \mathbf{R}$ are called Γ-equivalent if there exists $\gamma \in \Gamma$ such that $q_1 = q_2 \circ \gamma$. The space of Γ-equivalence classes of positive definite quadratic forms with determinant 1 thus corresponds bijectively to X/Γ.

The map $A \mapsto {}^{\tau}A := {}^{t}A^{-1}$ sending a matrix A to the transpose of the inverse matrix induces an involution τ on X. If Γ is τ-stable we have an induced involution on X/Γ. Denote by $H^*(X/\Gamma, \mathbf{Q})$ the singular cohomology with coefficients $\mathbf{Q}$. It is well known that $H^*(X/\Gamma, \mathbf{Q}) = H^*(\Gamma, \mathbf{Q})$ where on the right side we have abstract group cohomology. The map τ induces maps $\tau^i : H^i(\Gamma, \mathbf{Q}) \to H^i(\Gamma, \mathbf{Q})$ with trace $\mathrm{tr}(\tau^i)$. We define the Lefschetz number of τ and Γ by

$$L(\tau, \Gamma) = \sum_{i=0}^{\infty} (-1)^i \,\mathrm{tr}(\tau^i).$$

For $\Gamma = \Gamma(m)$, $m \geq 3$, the full congruence subgroup mod m of $\mathrm{Sl}_n(\mathbf{Z})$ we will give an explicit formula for $L(\tau, \Gamma(m))$. If, for example, m and n are odd we get

$$L(\tau, \Gamma(m)) = m^{n(n-1)/2} \prod_{j=1}^{(n-1)/2} \Big(\prod_{p \mid m} (1-p^{-2j})\Big) \zeta(1-2j).$$

Here $\zeta(\ )$ denotes the Riemann zeta function. Hence the Lefschetz number is computable in terms of Bernoulli numbers. The second product has to be taken over all prime divisors of m. Similar formulas (see 5.6) hold in the other cases.

(1) Supported by the Sonderforschungsbereich "Theoretische Mathematik" at the University of Bonn.

Not too much is known about $H^*(\Gamma, \mathbf{Q})$ itself. Some information has been obtained by Harder [7], Borel and Serre [3], Borel [4], Lee and Szczarba [14, 15], Schwermer [20], Lee [16] and Soulé [25].

To prove the formula for the Lefschetz number we proceed as follows. In §1 we observe that the Lefschetz fixpoint formula

$$L(\tau, \Gamma) = \chi((X/\Gamma)^\tau)$$

holds. The right side denotes the Euler–Poincaré characteristic of the fixpoint set $(X/\Gamma)^\tau$ of the operation of τ *on* X/Γ. It is shown in 1.3 that the connected components of $(X/\Gamma)^\tau$ correspond bijectively to the non abelian cohomology $H^1(\mathfrak{g}, \Gamma)$, $\mathfrak{g} = \{1, \tau\}$. The cohomology set is nothing else but a set of certain equivalence classes of unimodular bilinear forms represented by symmetric matrices $\gamma \in \Gamma$. The fixpoint set $F(\gamma)$ determined by such a symmetric matrix γ is the image of the special orthogonal group SO $(\gamma)(\mathbf{R})$ belonging to γ under some map SO $(\gamma)(\mathbf{R}) \to \mathrm{Sl}_n(\mathbf{R}) \to X/\Gamma$. We determine $H^1(\mathfrak{g}, \Gamma)$ in §2 and §3. In §4 we compute volumes of some orthogonal groups in a local situation and determine Euler–Poincaré measures in the sense of Serre [23]. In §5 we use the formula of Minkowski–Siegel, the determination of $H^1(\mathfrak{g}, \Gamma)$ and the computation of the local volumes to obtain the final formula.

Generalisations of this procedure to get invariants of $H^*(\Gamma, \mathbf{Q})$ for other arithmetic groups Γ are possible. For galois-operations on Γ see [19]. Our result supports the idea that there should be a lifting from some automorphic forms of orthogonal groups to automorphic forms of the special linear group.

§0. Notation

0.1. Suppose that $f: M \to M$ is a map and M a set. Then we denote the fixpoint set of f by $M^f = \{m \in M / f(m) = m\}$. Let $|M|$ be the cardinality of M. The symbols $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$ are used in their usual meaning.

0.2. If $a \in \mathbf{Q}$ then $[a]$ denotes the biggest integer which is smaller than or equal to a. For integers m and n we write $m \mid n$ if m divides n and $m \nmid n$ if m does not divide n. We write $(m, n) = 1$ if m and n are relatively prime, and $m \equiv j \bmod n$ if m is of the form $m = j + k \cdot n$; $j, k \in \mathbf{Z}$.

0.3. If p is a prime number we denote by $\mathbf{Q}_p$ the field of p-adic numbers and by $\mathbf{Z}_p$ the ring of p-adic integers in $\mathbf{Q}_p$. The normalized absolut value $\| \ \|_p$ on $\mathbf{Q}_p$ has the property that $\|p\|_p = p^{-1}$. A typical place of $\mathbf{Q}$ is denoted by v. If v is not finite then $v = \infty$ and $\mathbf{Q}_v = \mathbf{R}$.

0.4. Let $M_n(R)$ be the R-algebra of $n\times n$ matrices with coefficients in the ring R. Denote by $\mathrm{Gl}_n(R)$ (resp. $\mathrm{Sl}_n(R)$) the group of elements of $M_n(R)$ with invertible determinant in R (resp. determinant 1). For $A\in M_n(R)$ we denote by ${}^tA$ the transpose of A. If $b\in \mathrm{Gl}_n(R)$ and $b={}^tb$ then we define the group of isometries with determinant 1 of the bilinear form b by $\mathrm{SO}(b)(R)=\{A\in \mathrm{Sl}_n(R)/{}^tAb^{-1}A=b^{-1}\}$. In order to simplify the notation later on we use in this definition b^{-1} instead of b. If $b=E$ is the identity matrix we write $\mathrm{SO}_n(R)$ instead of $\mathrm{SO}(E)(R)$.

0.5. We suppose that a finite group $\mathfrak{g}$ operates from the left on a group G by group automorphisms. For $\sigma\in\mathfrak{g}$ the operation is written in the form $g\mapsto {}^\sigma g$, $g\in G$. In [21: Chap. I §5] the first non commutative cohomology set $H^1(\mathfrak{g}, G)$ is explained. If $\gamma=\{\gamma_\sigma\}$, $\sigma\in\mathfrak{g}$, is a cocycle for $H^1(\mathfrak{g}, G)$ the group G together with a new twisted operation of $\mathfrak{g}$ given by $g\mapsto \gamma_\sigma{}^\sigma g\gamma_\sigma^{-1}$, $g\in G$ is denoted by ${}^\gamma G$. How to determine fibres of certain maps between cohomology sets is described in [21: Chap. I §5].

§1. Fixpoints of τ and non commutative cohomology

We set ${}^\tau A={}^tA^{-1}$ for $A\in \mathrm{Sl}_n(\mathbf{R})$. The map τ is just the standard Cartan involution with respect to $\mathrm{SO}_n(\mathbf{R})\subset \mathrm{Sl}_n(\mathbf{R})$. Let X be the symmetric space $\mathrm{SO}_n(\mathbf{R})\backslash \mathrm{Sl}_n(\mathbf{R})$. Then τ operates as an isometry on X. Here X is endowed with the right invariant Riemannian metric determined by the Cartan involution and the Killing form belonging to $\mathrm{Sl}_n(\mathbf{R})$ [8: p. 159].

Suppose that $\Gamma\subset \mathrm{Sl}_n(\mathbf{Z})$ is a torsion free τ-stable arithmetic subgroup. For example we may take $\Gamma=\Gamma(m)$, $m\geq 3$, a full congruence subgroup mod m of $\mathrm{Sl}_n(\mathbf{Z})$ [17: p. 203, p. 211]. Then X/Γ is a Riemannian manifold on which τ operates isometrically.

In this chapter we give a parametrisation of the connected components of the fixpoint set $(X/\Gamma)^\tau$ by certain classes of integral bilinear forms and show that the fixpoint components are locally symmetric spaces belonging to some orthogonal groups.

Some of the results and proofs of this paragraph extend to finite groups acting on a semisimple algebraic group and stabilizing an arithmetic subgroup. For galois actions see [19].

1.1. We define $\mathfrak{g}=\{1,\tau\}$ and consider a cocycle $b=(1,b)$ for the non commutative cohomology set $H^1(\mathfrak{g},\Gamma)$. The cohomology class determined by b will be again denoted by b. By definition $b=(1,b)$, $b\in\Gamma$, is a cocycle if $b\cdot{}^tb^{-1}=b\cdot{}^\tau b=1$, i.e., $b={}^tb$. Two cocycles $(1,b)$ and $(1,d)$ are equivalent if there exists a $c\in\Gamma$

with $d = c^{-1} \cdot b \cdot {}^{\tau}c$, i.e., $c \cdot d \cdot {}^{t}c = b$. If for example $\Gamma = \mathrm{Sl}_n(\mathbf{Z})$ then we get a bijection of $H^1(\mathfrak{g}, \Gamma)$ with the set of $\mathrm{Sl}_n(\mathbf{Z})$-equivalence classes of integral unimodular bilinear forms as follows. To every cocycle $(1, b)$ we associate the bilinear form $b(x, y) = {}^{t}x \cdot b \cdot y$. Here $x, y \in \mathbf{Z}^n$ are considered as row vectors.

1.2. We can construct points in $(X/\Gamma)^{\tau}$ as follows. If $b = (1, b)$ is a cocycle for $H^1(\mathfrak{g}, \Gamma)$ we define by ${}^{\bar\tau}A = b \cdot {}^{\tau}A \cdot b^{-1}$ a new operation of $\bar\tau$ on $\mathrm{Sl}_n(\mathbf{R})$ and on Γ. For $x \in X$ we define ${}^{\bar\tau}x = {}^{\tau}x \cdot b^{-1}$. For $x \in X$ and $g \in \mathrm{Sl}_n(\mathbf{R})$ we then have ${}^{\bar\tau}(x \cdot g) = {}^{\bar\tau}x \cdot {}^{\bar\tau}g$. Therefore we have an operation of $\bar\tau$ on X/Γ and this operation coincides with the old operation of τ. If $X(b)$ denotes the set of fixpoints of $\bar\tau$ on X and $\Gamma(b)$ the set of fixpoints of $\bar\tau$ in Γ then the image $F(b)$ of the natural map

$$X(b)/\Gamma(b) \to X/\Gamma$$

clearly lies in $(X/\Gamma)^{\tau}$ and depends only on the cohomology class represented by b. Moreover as Γ is torsion free it is easy to check that we have a natural isomorphism $X(b)/\Gamma(b) \xrightarrow{\sim} F(b)$. We will see that $F(b) \neq \emptyset$ and that all fixpoints arise from this construction.

PROPOSITION 1.3

$$(X/\Gamma)^{\tau} = \bigcup_{b \in H^1(\mathfrak{g}, \Gamma)} F(b).$$

Here we have a disjoint union of non empty and connected sets $F(b)$. The fixpoint component $F(b)$ is locally convex, its fundamental group is isomorphic to $\Gamma(b)$ and its universal cover is contractible.

Proof. We observed that $F(b) \in (X/\Gamma)^{\tau}$. If $\bar{x} \in (X/\Gamma)^{\tau}$ is represented by $x \in X$, then ${}^{\tau}x = x \cdot b$ with some $b \in \Gamma$. Because Γ is torsion free b is uniquely determined. We have ${}^{\tau}({}^{\tau}x) = x = {}^{\tau}(x \cdot b) = x \cdot b \cdot {}^{\tau}b$. Therefore $b \cdot {}^{\tau}b = 1$ and $(1, b)$ is a cocycle for $H^1(\mathfrak{g}, \Gamma)$. If we take another element $y = x \cdot c$, $c \in \Gamma$, representing $\bar{x} \in (X/\Gamma)^{\tau}$, then $c^{-1} \cdot b \cdot {}^{\tau}c$ is the cocycle associated to y. Therefore every $\bar{x} \in (X/\Gamma)^{\tau}$ determines uniquely a class in $H^1(\mathfrak{g}, \Gamma)$ and we have $(X/\Gamma)^{\tau} = \bigcup F(b)$, $b \in H^1(\mathfrak{g}, \Gamma)$ as a disjoint union. By $x \mapsto {}^{\tau}x \cdot b$, b as above, we have an isometric operation on X. From [8: I, 13.5] we then see that $F(b)$ is not empty. Any two points of $X(b)$ are joined by a unique geodesic of X [8: I, 13.3]. Therefore $X(b)$ is geodesially convex and all statements are proven. q.e.d.

Remark. (i) The proof above shows that more generally the following hold. Suppose that Γ is some group acting freely from the right on a set X. Let $\mathfrak{g}$ be a finite group acting from the left on Γ and on X in a compatible way, i.e.

${}^{\sigma}(x\gamma)={}^{\sigma}x{}^{\sigma}\gamma$ for $x\in X$, $\gamma\in\Gamma$, $\sigma\in\mathfrak{g}$. Suppose that for every cocycle c for $H^1(\mathfrak{g},\Gamma)$ the c-twisted $\mathfrak{g}$-action has a fixpoint in X. If $x_0\in X$ is a fixpoint of the $\mathfrak{g}$-action on X, then we have by $\gamma\mapsto x_0\cdot\gamma$ a map $\Gamma\to X$ and an exact sequence $1\to\Gamma\to X\to X/\Gamma\to 1$. We get an exact sequence of pointed sets

$$1\to\Gamma^{\mathfrak{g}}\to X^{\mathfrak{g}}\to(X/\Gamma)^{\mathfrak{g}}\xrightarrow{\partial} H^1(\mathfrak{g},\Gamma)\to 1,$$

which describes the fixpoint set $(X/\Gamma)^{\mathfrak{g}}$ completely. This can be applied to actions of $\mathfrak{g}$ on symmetric spaces, their compactifications and to actions on Bruhat-Tits buildings.

ii) If Γ does not act freely, the above method to construct fixpoints works as well. However there may be fixpoints which do not arise from that construction.

For every cocycle $(1,b)$, $b\in\Gamma$, we now determine $X(b)$.

LEMMA 1.4. *The natural inclusion* $\mathrm{SO}_n(\mathbf{R})\hookrightarrow\mathrm{Sl}_n(\mathbf{R})$ *induces a bijection* $H^1(\mathfrak{g},\mathrm{SO}_n(\mathbf{R}))\xrightarrow{i} H^1(\mathfrak{g},\mathrm{Sl}_n(\mathbf{R}))$.

Proof. If $(1,g)$ is a cocycle for $H^1(\mathfrak{g},\mathrm{Sl}_n(\mathbf{R}))$ we have a twisted operation given by $x\mapsto{}^{\tau}x\cdot g^{-1}$ on X. This operation is isometric and thus [8: I, 13.5] has a fixpoint $y\in X$. If $x_0\in X$ is the point corresponding to $\mathrm{SO}_n(\mathbf{R})$, then $y=x_0\cdot h^{-1}$ for some $h\in\mathrm{Sl}_n(\mathbf{R})$. Hence $y={}^{\tau}y\cdot g^{-1}=x_0\cdot h^{-1}=x_0\cdot{}^{\tau}h^{-1}\cdot g^{-1}$ and $h^{-1}\cdot g\cdot{}^{\tau}h\in\mathrm{SO}_n(\mathbf{R})$. This means that i is surjective.

To prove the injectivity of i we consider two cocycles $(1,k_1)$ and $(1,k_2)$, $k_1,k_2\in\mathrm{SO}_n(\mathbf{R})$, and suppose that there exists a $g\in\mathrm{Sl}_n(\mathbf{R})$ such that $k_2=g^{-1}\cdot k_1\cdot{}^{\tau}g$. From [8: III, 7.4] we have a unique Cartan decomposition $\mathrm{Sl}_n(\mathbf{R})=\mathrm{SO}_n(\mathbf{R})\cdot P$. Here P is a closed and τ-stable subset, $P=P^{-1}$, and for $k\in\mathrm{SO}_n(\mathbf{R})$ we have $k^{-1}\cdot P\cdot k=P$. Write $g=k\cdot p$ with $k\in\mathrm{SO}_n(\mathbf{R})$ and $p\in P$. Then $k_2=p^{-1}\cdot k^{-1}\cdot k_1\cdot{}^{\tau}k\cdot{}^{\tau}p=k^{-1}\cdot k_1\cdot{}^{\tau}k\cdot p'$ for some $p'\in P$. From the uniqueness of the Cartan decomposition we get $k_2=k^{-1}\cdot k_1\cdot{}^{\tau}k$ and thus i is injective. q.e.d.

Remark. The argument just given shows that an analog of Lemma 1.4 holds more generally for any finite group $\mathfrak{g}$ acting isometrically on a symmetric space. For some other special cases see [1: 6.8] and [19: 1.6]

We now compute $H^1(\mathfrak{g},\mathrm{Sl}_n(\mathbf{R}))$. In fact we just have to reinterpret the classical theorem of Sylvester giving the classification of real quadratic forms.

We recall the notion of a signature. If $b\in\mathrm{Gl}_n(\mathbf{R})$ is symmetric, then there exists an $A\in\mathrm{Gl}_n(\mathbf{R})$ such that ${}^{t}A\cdot b\cdot A=\eta_{r,s}$. Here ${}^{t}x\cdot\eta_{r,s}\cdot x=\sum_{i=1}^{r}x_i^2-\sum_{j=r+1}^{n}x_j^2$ for $x\in\mathbf{R}^n$ considered as row vector. The tuple (r,s) with $r+s=n$ is called the signature of b. By Sylvester's theorem we have a bijection $H^1(\mathfrak{g},\mathrm{Gl}_n(\mathbf{R}))\xrightarrow{\sim}\{(r,s)\in\mathbf{N}\times\mathbf{N}/r+s=n\}$ induced by $b\mapsto$ signature (b).

LEMMA 1.5. *The signature induces a bijection*

$$H^1(\mathfrak{g}, \mathrm{Sl}_n(\mathbf{R})) \xrightarrow{\sim} \{(r, s) \in \mathbf{N} \times \mathbf{N} / r+s=n, s \equiv 0 \bmod 2\}.$$

Proof. From the exact sequence $1 \to \mathrm{Sl}_n(\mathbf{R}) \to \mathrm{Gl}_n(\mathbf{R}) \xrightarrow{\det} \mathbf{R}^* \to 1$ we get an exact sequence

$$1 \to \mathrm{SO}_n(\mathbf{R}) \to \mathrm{O}_n(\mathbf{R}) \to \{\pm 1\} \xrightarrow{\partial} H^1(\mathfrak{g}, \mathrm{Sl}_n(\mathbf{R})) \xrightarrow{i} H^1(\mathfrak{g}, \mathrm{Gl}_n(\mathbf{R})) \to H^1(\mathfrak{g}, \mathbf{R}^*).$$

In order to show that i is injective we have to twist the whole situation with a cocycle $(1, b)$ for $H^1(\mathfrak{g}, \mathrm{Gl}_n(\mathbf{R}))$ and to verify that the new boundary map ∂_b is trivial. But this is obvious because every orthogonal group $\mathrm{O}(b)(\mathbf{R})$ contains a reflection of determinant -1. Thus i is injective and the result follows immediately. q.e.d.

By $\mathrm{SO}(r, s)(\mathbf{R})$ we denote the special orthogonal group over $\mathbf{R}$ belonging to the bilinear form $\eta_{r,s}$. Let $\mathrm{SO}(r, s)^0(\mathbf{R})$ be the connected component of the identity in $\mathrm{SO}(r, s)(\mathbf{R})$. We have $\mathrm{SO}(r, s)^0(\mathbf{R}) \cap \mathrm{SO}_n(\mathbf{R}) = \mathrm{SO}_r(\mathbf{R}) \times \mathrm{SO}_s(\mathbf{R})$. The inclusion $\mathrm{SO}(r, s)^0(\mathbf{R}) \hookrightarrow \mathrm{Sl}_n(\mathbf{R})$ induces a diffeomorphism of $\mathrm{SO}_r(\mathbf{R}) \times \mathrm{SO}_s(\mathbf{R}) \backslash \mathrm{SO}(r, s)^0(\mathbf{R})$ onto a closed submanifold of X.

PROPOSITION 1.6. *If $(1, b)$ is a cocycle for $H^1(\mathfrak{g}, \Gamma)$ and if (r, s) is the signature of b, then there exists a $g \in \mathrm{Sl}_n(\mathbf{R})$ such that*

$$X(b) \cong \mathrm{SO}_r(\mathbf{R}) \times \mathrm{SO}_s(\mathbf{R}) \backslash \mathrm{SO}(r, s)^0(\mathbf{R}) \cdot g.$$

Proof. We consider $(1, b)$ as cocycle for $H^1(\mathfrak{g}, \mathrm{Sl}_n(\mathbf{R}))$ and using 1.4 we suppose that $b \in \mathrm{SO}_n$. Then we have an exact sequence of pointed sets

$$1 \to {}^b\mathrm{SO}_n(\mathbf{R}) \to {}^b\mathrm{Sl}_n(\mathbf{R}) \to {}^bX \to 1,$$

where the upper index b indicates that we consider the $\mathfrak{g}$-operation obtained by twisting with b. We get an exact sequence $1 \to {}^b\mathrm{SO}_n(\mathbf{R})^\tau \to {}^b\mathrm{Sl}_n(\mathbf{R})^\tau \to {}^bX^\tau \xrightarrow{\partial} H^1(\mathfrak{g}, {}^b\mathrm{SO}_n(\mathbf{R})) \xrightarrow{i} H^1(\mathfrak{g}, {}^b\mathrm{Sl}_n(\mathbf{R}))$. Now ${}^bX^\tau = X(b)$, ${}^b\mathrm{Sl}_n(R)^\tau = \mathrm{SO}(b)(\mathbf{R})$ and ${}^b\mathrm{SO}_n(\mathbf{R})^\tau = \mathrm{SO}_n(\mathbf{R}) \cap \mathrm{SO}(b)(\mathbf{R})$. From 1.4 we deduce that i is a bijection. Hence ∂ is trivial. We know that $X(b)$ is connected. The connected component $\mathrm{SO}(b)^0(\mathbf{R})$ of the identity in $\mathrm{SO}(b)(\mathbf{R})$ operates transitively on $X(b)$. By 1.5 there is a $g \in \mathrm{Sl}_n(\mathbf{R})$ such that $b = {}^t g \cdot \eta_{r,s} \cdot g$. The desired result now follows. q.e.d.

Remark. Submanifolds like $X(b) \subset X$ occur also naturally in the reduction theory of indefinite quadratic forms [2: §5].

COROLLARY 1.7. *If b is a cocycle for $H^1(\mathfrak{g}, \Gamma)$ with signature (r, s) then the dimension of $F(b)$ is $r \cdot s$.*

Corollary 1.7 implies that $F(b)$ consists of one point if and only if b considered as a quadratic form is definite. From the compactness criterion [2: §8] we deduce:

COROLLARY 1.8. *The fixpoint set $F(b)$ is compact and different from a point if and only if b considered as a quadratic form is indefinite and anisotropic over* $\mathbf{Q}$.

If $n \geqslant 5$ then by the theorem of Minkowski-Hasse an indefinite quadratic form in n variables is isotropic over $\mathbf{Q}$ [22: p. 77].

It is well known that the abstract group cohomology $H^*(\Gamma, \mathbf{Q})$ coincides with the cohomology $H^*(X/\Gamma, \mathbf{Q})$ of the topological space X/Γ. The space X/Γ has the homotopy type of a compact manifold with boundary [7: 1.3.3.]. Therefore the $\mathbf{Q}$-dimension of $H^*(\Gamma, \mathbf{Q})$ is finite. The map $\tau^i: H^i(\Gamma, \mathbf{Q}) \to H^*(\Gamma, \mathbf{Q})$ induced by τ has a trace denoted by $\operatorname{tr}(\tau^i)$. We define the Lefschetz number of τ and Γ by $L(\tau, \Gamma) = \sum_{i=0}^{\infty} (-1)^i \operatorname{tr}(\tau^i)$. We can (and we will) consider this Lefschetz number as the Lefschetz number $L(\tau, X/\Gamma)$ of the continuous map $\tau: X/\Gamma \to X/\Gamma$. It is well known that $L(\tau, \Gamma) \in \mathbf{Z}$. The Euler–Poincaré characteristic of a topological space Y is enoted by $\chi(Y)$.

PROPOSITION 1.9 (*Lefschetz fixpoint formula*). *With the notation introduced above we have*

$$L(\tau, X/\Gamma) = \chi((X/\Gamma)^\tau).$$

Proof. Let $N = (n-1)(n+2)/2$ be the dimension of X. The involution τ fixes the point $x_0 \in X$ corresponding to $\mathrm{SO}_n(\mathbf{R})$ and induces the multiplication with -1 in the tangent space in x_0 at X. Clearly X/Γ is orientable. Denote by $H_c^*(\ ,\mathbf{Q})$ cohomology with compact supports and by $[X/\Gamma] \in H_c^N(X/\Gamma, \mathbf{Q})$ an orientation class. We write the isomorphism $H_c^N(X/\Gamma, \mathbf{Q}) \to \mathbf{Q}$ determined by $[X/\Gamma]$ in the form $\alpha \mapsto \langle \alpha, [X/\Gamma] \rangle$. We have $\tau^N([X/\Gamma]) = (-1)^N [X/\Gamma]$. By Poincaré duality [26: p. 139, p. 149] we have a non degenerate pairing

$$H_c^i(X/\Gamma, \mathbf{Q}) \times H^{n-i}(X/\Gamma, \mathbf{Q}) \to \mathbf{Q}$$

given by $(\alpha, \beta) \mapsto \langle \alpha \cup \beta, [X/\Gamma] \rangle$. Here $\alpha \cup \beta$ is the cup product of α and β. If $\tau_c^i: H_c^i(X/\Gamma, \mathbf{Q}) \to H_c^i(X/\Gamma, \mathbf{Q})$ is the map induced by τ then for its trace $\operatorname{tr}(\tau_c^i)$ we have $\operatorname{tr}(\tau_c^i) = (-1)^N \operatorname{tr}(\tau^{N-i})$. By the classical Lefschetz fixpoint formula

$L_c(\tau, X/\Gamma) = \chi_c((X/\Gamma)^\tau)$, see [28]. The index c on both sides of the equation indicates that we are using cohomology with compact supports. We know from 1.3 and 1.7 that $(X/\Gamma)^\tau$ is a disjoint union of even dimensional manifolds. Thus again by Poincaré duality $\chi_c((X/\Gamma)^\tau) = \chi((X/\Gamma)^\tau)$ and the desired formula follows. q.e.d.

§2. Equivalence classes of unimodular integral bilinear forms

In this chapter we describe the Γ-equivalence classes of certain unimodular integral bilinear forms in terms of their local data and the genera fixed by the local data. Again, as in §1, we use the language of non commutative cohomology.

2.1. For all places v of $\mathbf{Q}$ we consider $\mathrm{SL}_n(\mathbf{Q}_v)$ as a locally compact group in the topology induced by the topology of $\mathbf{Q}_v$. Suppose that for all finite places v of $\mathbf{Q}$ an open compact subgroup $\Gamma_v \subset \mathrm{Sl}_n(\mathbf{Q}_v)$ is given and that $\Gamma_v = \mathrm{Sl}_n(\mathbf{Z}_v)$ for almost all v. If $v = \infty$ set $\Gamma_v = \mathrm{Sl}_n(\mathbf{R})$. Then $\Gamma = \bigcap_v (\Gamma_v \cap \mathrm{Sl}_n(\mathbf{Q}))$ is a congruence subgroup of $\mathrm{Sl}_n(\mathbf{Q})$ and by the strong approximation property of $\mathrm{Sl}_n/\mathbf{Q}$ the closure of Γ in $\mathrm{Sl}_n(\mathbf{Q}_v)$ is Γ_v for all finite places v.

If Γ is τ-stable then all Γ_v are τ-stable and the inclusion $\Gamma \to \Gamma_v$ induces a map $h_v : H^1(\mathfrak{g}, \Gamma) \to H^1(\mathfrak{g}, \Gamma_v)$. We will see in 3.1 that $H^1(\mathfrak{g}, \Gamma_v) = \{1\}$ for almost all v. Therefore $\prod_v h_v = h$ induces a map

$$h : H^1(\mathfrak{g}, \Gamma) \to \coprod_v H^1(\mathfrak{g}, \Gamma_v).$$

For $b_v \in H^1(\mathfrak{g}, \Gamma_v)$ the Hasse–Witt invariant $\varepsilon_v(b_v) \in \{\pm 1\}$ is defined [22: p. 64]. If $\{b_v\} \in \coprod_v H^1(\mathfrak{g}, \Gamma_v)$ we define $\varepsilon(\{b_v\}) = \prod_v \varepsilon_v(b_v)$. This makes sense because almost all factors of the infinite product are 1.

Let $\mathbf{A}$ be the ring of adeles of $\mathbf{Q}$. If b is a nondegenerate bilinear form over $\mathbf{Q}$ then $\mathrm{SO}(b)(\mathbf{A})$ denotes the group of $\mathbf{A}$-valued points of $\mathrm{SO}(b)/\mathbf{Q}$. This group contains diagonally the group $\mathrm{SO}(b)(\mathbf{Q})$ and also the subgroup $\prod_v ({}^b\Gamma_v^\tau)$.

PROPOSITION 2.2. *Suppose that Γ is a τ-stable congruence subgroup of $\mathrm{Sl}_n(\mathbf{Q})$ and use the notation introduced in 2.1. Then there is an exact sequence of pointed sets*

$$1 \to \mathrm{SO}_n(\mathbf{Q}) \backslash \mathrm{SO}_n(\mathbf{A}) / \prod_v \Gamma_v^\tau \xrightarrow{\partial} H^1(\mathfrak{g}, \Gamma) \xrightarrow{h} \coprod_v H^1(\mathfrak{g}, \Gamma_v) \xrightarrow{\varepsilon} \{\pm 1\} \to 1.$$

The map ∂ is injective.

Proof. Consider the diagram

$$\begin{tikzcd}
1 \arrow[r] & \Gamma \arrow[r] \arrow[d, hook] & \mathrm{Sl}\,(\mathbf{Q}) \arrow[r] \arrow[d, hook] & \mathrm{Sl}_n\,(\mathbf{Q})/\Gamma \arrow[r] \arrow[d, hook] & 1 \\
1 \arrow[r] & \prod_v \Gamma_v \arrow[r] & \mathrm{Sl}_n\,(\mathbf{A}) \arrow[r] & \mathrm{Sl}_n\,(\mathbf{A}) \Big/ \prod_v \Gamma_v \arrow[r] & 1
\end{tikzcd}$$

From this we get the following commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & F \arrow[d, hook] \arrow[dddll, bend left] & \\
1 \arrow[r] & \mathrm{SO}_n\,(\mathbf{Q})/\Gamma^\tau \arrow[r] \arrow[d] & (\mathrm{Sl}_n\,(\mathbf{Q})/\Gamma)^\tau \arrow[r] \arrow[d] & H^1(\mathfrak{g}, \Gamma) \arrow[r] \arrow[d, "h'"] & H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Q})) \arrow[d, "h''"] \\
1 \arrow[r] & \mathrm{SO}_n\,(\mathbf{A}) \Big/ \prod_v \Gamma_v^\tau \arrow[r] \arrow[d] & (\mathrm{Sl}_n\,(\mathbf{A}) \Big/ \prod_v \Gamma_v)^\tau \arrow[r] \arrow[d] & H^1(\mathfrak{g}, \prod \Gamma_v) \arrow[r] & H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{A})) \\
 & \mathrm{SO}_n\,(\mathbf{Q})\backslash \mathrm{SO}_n\,(\mathbf{A}) \Big/ \prod_v \Gamma_v^\tau \arrow[r] \arrow[d] & (\mathrm{Sl}_n\,(\mathbf{Q})\backslash \mathrm{Sl}_n\,(\mathbf{A}) \Big/ \prod_v \Gamma_v)^\tau. & & \\
 & 1 & & &
\end{tikzcd}$$

Using $H^1(\mathfrak{g}, \Gamma_v) = \{1\}$ for almost all v we have $H^1(\mathfrak{g}, \prod_v \Gamma_v) = \coprod_v H^1(\mathfrak{g}, \Gamma_v)$ and $H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{A})) = \coprod_v H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Q}_v))$. Therefore h' coincides up to isomorphy with h. From the theorem of Hasse–Minkowski [22: p. 77] we get that h'' is injective. By strong approximation for $\mathrm{Sl}_n/\mathbf{Q}$ the set $\mathrm{Sl}_n\,(\mathbf{Q})\backslash \mathrm{Sl}_n\,(\mathbf{A})/\prod_v \Gamma_v$ consists of one point. If we denote by F the kernel of h' the existence and bijectivity of the snaked arrow follows by diagram chase.

We have to determine the cokernel of h'. If $\gamma \in H^1(\mathfrak{g}, \Gamma)$ then $\varepsilon h(\gamma) = 1$ by [22: p. 73]. Consider $\{\gamma_v\} \in \coprod_v H^1(\mathfrak{g}, \Gamma_v)$ with the property that $\prod_v \varepsilon_v(\gamma_v) = 1$. We have maps $\coprod_v H^1(\mathfrak{g}, \Gamma_v) \xrightarrow{\sim} H^1(\mathfrak{g}, \prod_v \Gamma_v) \to H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{A}))$. Thus $\{\gamma_v\}$ determines a class γ for $H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{A})) \xrightarrow{\sim} \coprod_v H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Q}_v))$ with trivial Hasse–Witt invariant. From [22: p. 78] then γ is in the image of $H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Q})) \to H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{A}))$. If $(1, \gamma_v)$ is a cocycle representing γ_v we therefore have a cocycle $(1, \xi)$, $\xi \in \mathrm{Sl}_n\,(\mathbf{Q})$, and an $a = \{a_v\} \in \mathrm{Sl}_n\,(\mathbf{A})$ such that $a_v^{-1} \cdot \xi \cdot {}^\tau a_v = \gamma_v$ for all v. By strong approximation $a = b \cdot c^{-1}$ with $b \in \mathrm{Sl}_n\,(\mathbf{Q})$, $c \in \prod_v \Gamma_v$. Hence $\eta = b^{-1} \cdot \xi \cdot {}^\tau b \in \Gamma$ represents a class η with $h(\eta) = \{\gamma_v\}$. The surjectivity of ε is easy to see. q.e.d.

Remark. i) This sequence determines the image of h and the fiber of h through the distinguished element of $H^1(\mathfrak{g}, \Gamma)$. If $(1, b)$ is a cocycle for $H^1(\mathfrak{g}, \Gamma)$ then the fiber of h through the element $\bar{b}$ represented by $(1, b)$ is obtained by twisting [21: I, 5.4]. We get $h^{-1}(\{h(\bar{b})\}) \cong \mathrm{SO}\,(b)(\mathbf{Q})\backslash\mathrm{SO}\,(b)(\mathbf{A})/\prod \Gamma_v\,(b)$. Here we abbreviate $\Gamma_v(b) = {}^b\Gamma_v^\tau$.

ii) It is well known that the class number of special orthogonal groups is finite. In the next chapter we see that $\coprod_v H^1(\mathfrak{g}, \Gamma_v)$ is finite. Hence $H^1(\mathfrak{g}, \Gamma)$ is finite. In [1: 3.8] there is a different and simpler proof of the finiteness of $H^1(\mathfrak{g}, \Gamma)$.

iii) Results similar to Proposition 2.2 hold more generally. For galois actions see [19: 3.1].

§3. Local equivalence classes of unimodular bilinear forms

Let p be a prime number, j a non negative integer and $n \geq 2$ an integer. We denote by $\Gamma_p(j)$ the kernel of the natural surjection $\mathrm{Sl}_n\,(\mathbf{Z}_p) \to \mathrm{Sl}_n\,(\mathbf{Z}/p^j\mathbf{Z})$ and observe that $\Gamma_p(j)$ is τ-stable. In this section we compute $H^1(\mathfrak{g}, \Gamma_p(j))$.

At first we consider $H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Z}_p))$. This set has been determined by Minkowski in a different language. To state the result we need some notation.

Denote by E, L, S the symmetric bilinear forms on $\mathbf{Z}_p^n$ given for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbf{Z}_p^n$ by $E(x, y) = \sum_{i=1}^n x_i y_i$, $L(x, y) = -x_1y_1 - x_2y_2 + \sum_{i=3}^n x_i y_i$, $S(x, y) = \sum_{i=1}^{n/2} x_{2i-1}y_{2i} + x_{2i}y_{2i-1}$ if n is even. The symmetric matrices representing E, L, S with respect to the standard basis are again denoted by E, L, S and E, L, S also denote the cohomology classes represented by E, L, S considered as cocycles.

PROPOSITION 3.1. *If $p \neq 2$ then $H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Z}_p)) = \{1\}$. If $p = 2$ and $n \not\equiv 0$ mod 4 then $H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Z}_2))$ consists of two classes which can be represented by E and L. If $p = 2$ and $n \equiv 0$ mod 4 then $H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Z}_2))$ consists of three classes which can be represented by E, L and S.*

Proof. The set $H^1(\mathfrak{g}, \mathrm{Gl}_n\,(\mathbf{Z}_p))$ has been described in [12, p. 91]. From the exact sequence $1 \to \mathrm{Sl}_n\,(\mathbf{Z}_p) \to \mathrm{Gl}_n\,(\mathbf{Z}_p) \to \mathbf{Z}_p^* \to 1$ we get the exact sequence

$$1 \to H^1(\mathfrak{g}, \mathrm{Sl}_n\,(\mathbf{Z}_p)) \xrightarrow{i} H^1(\mathfrak{g}, \mathrm{Gl}_n\,(\mathbf{Z}_p)) \to \mathbf{Z}_p^*/(\mathbf{Z}_p^*)^2 \to 1.$$

We twist the situation and show that i is actually injective. The result now follows. q.e.d.

We collect some simple facts. Let $\mathrm{M}_n\,(A)$ be set of $n \times n$-matrices with coefficients in a ring A. For $B \in M_n\,(A)$ define ${}^\tau B = -{}^tB$. Denote by $G_p(j)$ the kernel of the natural surjection $\mathrm{Gl}_n\,(\mathbf{Z}_p) \to \mathrm{Gl}_n\,(\mathbf{Z}/p^j\mathbf{Z})$. Then the following hold.

i) For all primes p and all $j \geq 1$ there is an isomorphism of commutative groups with τ-operation

$$\mathrm{M}_n\,(\mathbf{Z}/p\mathbf{Z}) \xrightarrow{\sim} G_p(j)/G_p(j+1)$$

given by $\bar{B} \mapsto 1+p^jB \bmod G_p(j+1)$. Here $B\in \mathrm{M}_n\,(\mathbf{Z}_p)$ is a matrix with $B\equiv \bar{B} \bmod p\cdot \mathrm{M}_n\,(\mathbf{Z}_p)$.

ii) If $B\in p^j\cdot \mathrm{M}_n\,(\mathbf{Z}_p)$ and if $p\neq 2$ and $j\geq 1$ or $p=2$ and $j\geq 2$ then $B^n/n!\in p^{2j-1}\cdot \mathrm{M}_n\,(\mathbf{Z}_p)$. The series $\exp(B)=\sum_{n=0}^{\infty} B^n/n!$ converges and defines a τ-equivariant bijection

$$\exp: p^j\cdot \mathrm{M}_n\,(\mathbf{Z}_p)\to G_p(j).$$

PROPOSITION 3.2. *Let b be either the cocycle E or the cocycle L. Then for all $j\geq 1$ the natural map $\Gamma_2(j)\to \Gamma_2(j)/\Gamma_2(j+1)$ induces a bijection*

$$H^1(\mathfrak{g}, {}^b\Gamma_2(j)) \xrightarrow{\sim} H^1(\mathfrak{g}, {}^b(\Gamma_2(j)/\Gamma_2(j+1))) = (\mathbf{Z}/2\mathbf{Z})^{n-1}.$$

The cocycles for $H^1(\mathfrak{g}, {}^b\Gamma_2(j))$ representing the different classes can be chosen as diagonal matrices with only integers out of $\{u, u(1+2^j)\}$, $u\in(1+2^j\cdot \mathbf{Z}_2)^2$ on the diagonal and $(1+2^j)u$ occurring an even number of times. For $p\neq 2$ and all $j\geq 1$ we have

$$H^1(\mathfrak{g}, \Gamma_p(j)) = \{1\}.$$

Proof. We have the exact sequence $1\to G_p(j+1)\to G_p(j)\to G_p(j)/G_p(j+1)\to 1$ and thus for every cocycle d for $H^1(\mathfrak{g}, \mathrm{Gl}_n\,(\mathbf{Z}_p))$ an exact sequence

$$H^1(\mathfrak{g}, {}^dG_p(j+1)) \xrightarrow{u} H^1(\mathfrak{g}, {}^dG_p(j)) \xrightarrow{i} H^1(\mathfrak{g}, {}^d(G_p(j)/G_p(j+1))).$$

Now ${}^d(G_p(j)/G_p(j+1)) \xrightarrow{\sim} {}^d\mathrm{M}_n\,(\mathbf{Z}/p\mathbf{Z}) \xrightarrow{\sim} \mathrm{M}_n\,(\mathbf{Z}/p\mathbf{Z})$ as groups with τ-operation. The last isomorphism is given by $B\mapsto B\cdot d$. Cocycles for $H^1(\mathfrak{g}, \mathrm{M}_n\,(\mathbf{Z}/p\mathbf{Z}))$ are matrices $A\in \mathrm{M}_n\,(\mathbf{Z}/p\mathbf{Z})$ with $A-{}^tA=0$ and elements of the form $C+{}^tC$ are coboundaries. We get.

$$H^1(\mathfrak{g}, {}^d(G_p(j)/G_p(j+1)) = \begin{cases} (\mathbf{Z}/2\mathbf{Z})^n & \text{if } p=2 \\ 0 & \text{if } p\neq 2 \end{cases}.$$

If $d=E$ or $d=L$ all cocycles can be represented by diagonal matrices where only integers out of $\{1, 1+2^j\}$ occur on the diagonal.

If $p \neq 2$ and $j \geq 1$ or if $p=2$ and $j \geq 2$ the map u is trivial. To see this we take a cocycle for $H^1(\mathfrak{g}, {}^d(G_p(j+1)))$ represented by a matrix $A \in G_p(j+1)$ with $A = d \cdot {}^tA \cdot d^{-1}$. If $A = \exp(B)$, $B \in p^{j+1} \cdot \mathrm{M}_n(\mathbf{Z}_p)$, then $B = d \cdot {}^tB \cdot d^{-1}$. For $C = \exp(-B/2)$ then $C \in G_p(j)$ and $C^{-1} \cdot d \cdot {}^\tau C \cdot d^{-1} = A$. Similarily the natural map $H^1(\mathfrak{g}, {}^d\Gamma_p(j+1)) \to H^1(\mathfrak{g}, {}^d\Gamma_p(j))$ is trivial.

For $p \neq 2$ and $j \geq 1$ we get $H^1(\mathfrak{g}, G_p(j)) = \{1\}$. We write $U(j)$ instead of $1 + p^j\mathbf{Z}_p$. Then from the exact sequence $1 \to \Gamma_p(j) \to G_p(j) \to U(j) \to 1$ we get the exact sequence

$$G_p(j)^\tau \to U(j)^\tau \xrightarrow{\partial} H^1(\mathfrak{g}, \Gamma_p(j)) \to 1.$$

We observe that ∂ is trivial. The result for $p \neq 2$ and $j \geq 1$ follows.

We have an exact sequence of commutative groups

$$0 \to \Gamma_2(j)/\Gamma_2(j+1) \to G_2(j)/G_2(j+1) \to U(j)/U(j+1) \to 0.$$

and $U(j)/U(j+1) = \mathbf{Z}/2\mathbf{Z}$ with trivial τ-action. We get an exact sequence

$$(G_2(j)/G_2(j+1))^\tau \to \mathbf{Z}/2\mathbf{Z} \xrightarrow{\partial} H^1(\mathfrak{g}, \Gamma_2(j)/\Gamma_2(j+1)) \to$$
$$H^1(\mathfrak{g}, G_2(j)/G_2(j+1)) \to H^1(\mathfrak{g}, \mathbf{Z}/2\mathbf{Z}).$$

Now ∂ is trivial $H^1(\mathfrak{g}, \mathbf{Z}/2\mathbf{Z}) = \mathbf{Z}/2\mathbf{Z}$ and the last arrow is non trivial. Therefore $H^1(\mathfrak{g}, \Gamma_2(j)/\Gamma_2(j+1)) = H^1(\mathfrak{g}, {}^L(\Gamma_2(j)/\Gamma_2(j+1))) \cong (\mathbf{Z}/2\mathbf{Z})^{n-1}$. If $b = E$ or $b = L$ then from the description of cocycles for $H^1(\mathfrak{g}, \Gamma_2(j)/\Gamma_2(j+1))$ we see that the natural map

$$H^1(\mathfrak{g}, {}^b\Gamma_2(j)) \to H^1(\mathfrak{g}, {}^b(\Gamma_2(j)/\Gamma_2(j+1)))$$

is surjective and our assertion follows for $p=2$ and $j \geq 2$.

From the exact sequence $1 \to \Gamma_2(2) \to \Gamma_2(1) \to \Gamma_2(1)/\Gamma_2(2) \to 1$ we get a map ${}^d(\Gamma_2(1)/\Gamma_2(2))^\tau \xrightarrow{\partial} H^1(\mathfrak{g}, {}^d\Gamma_2(2))$. We show that ∂ is surjective. If $A \in \Gamma_2(2)$, $A = d \cdot {}^tA \cdot d^{-1}$, $A = \exp(B)$, $B \in 4 \cdot \mathrm{M}_n(\mathbf{Z}_2)$ then $B = d \cdot {}^tB \cdot d^{-1}$ and $C = 1 - B/2$ represents an element in ${}^d(G_2(1)/G_2(2))^\tau$ which lies in the image of ${}^d(\Gamma_2(1)/\Gamma_2(2))^\tau$. Now $C^{-1} \cdot d \cdot {}^tC^{-1} \cdot d^{-1} \equiv (1+B/2)d(1+{}^tB/2)d^{-1} \equiv 1 + B + B^2/4 \equiv 1 + B + B^2/2 \equiv \exp(B) \bmod 8\,\mathrm{M}_n(\mathbf{Z}_2)$. The natural map $H^1(\mathfrak{g}, {}^d\Gamma_2(2)) \to H^1(\mathfrak{g}, {}^d(\Gamma_2(2)/\Gamma_2(3)))$ is injective. Therefore the last equations mean that A is in the image of ∂. We get that the natural map $H^1(\mathfrak{g}, {}^d\Gamma_2(1)) \to H^1(\mathfrak{g}, {}^d(\Gamma_2(1)/\Gamma_2(2)))$ is injective. Thus the claimed result holds also for $p=2$ and $j=1$. q.e.d.

Later on we need information on the fibers of the natural map $H^1(\mathfrak{g}, \Gamma_2(1)) \to H^1(\mathfrak{g}, \mathrm{Sl}_n(\mathbf{Z}_2))$. We use the following simple combinatorial facts.

The number of subsets of even cardinality out of n elements is 2^{n-1}. This follows for example from the equation $2^n = (1+1)^n$ and $0 = (1-1)^n$. We set

$$e(E) = \sum_j \binom{n}{4j} \quad \text{and} \quad e(L) = \sum_j \binom{n}{4j+2}.$$

From consideration of $(1-i)^n$, $i = (-1)^{1/2}$, we see that $e(E) = \frac{1}{2}(2^{n-1} + 2^{n/2} \cdot \cos(\pi n/4))$ and $e(L) = \frac{1}{2}(2^{n-1} - 2^{n/2} \cdot \cos(\pi n/4))$.

COROLLARY 3.3. *The classes represented by E and by L form the image of the natural map $H^1(\mathfrak{g}, \Gamma_2(1)) \to H^1(\mathfrak{g}, \mathrm{Sl}_n(\mathbf{Z}_2))$. In $H^1(\mathfrak{g}, \Gamma_2(1))$ there are $e(E)$ classes with image E and $e(L)$ classes with image L.*

Proof. Clearly E and L are in the image. The bilinear form S has the property that $S(x, x)/2 \in \mathbf{Z}_2$ for all $x \in \mathbf{Z}_2^n$. As no bilinear form in the image of $H^1(\mathfrak{g}, \Gamma_2(1))$ has this property S is not in the image. We have Hasse–Witt invariants $\varepsilon_2(E) = 1$ and $\varepsilon_2(L) = -1$. If $2 \cdot t$ is the number of integers $1+2$ occurring in the diagonal representation of a cocycle $A \in \Gamma_2(1)$ for $H^1(\mathfrak{g}, \Gamma_2(1))$ then $\varepsilon_2(A) = (-1)^t$. The result now follows from the combinatorial meaning of $e(E)$ and $e(L)$. q.e.d.

§4. Volume computations

Let $b \in \mathrm{Sl}_n(\mathbf{Z})$, $n \geq 2$, be a symmetric matrix. In this chapter we explicitly chose biinvariant measures ω_v on $\mathrm{SO}(b)(\mathbf{Q}_v)$ for all places v of $\mathbf{Q}$ such that $\prod_v \omega_v$ is the Tamagawa measure [7] on $\mathrm{SO}(b)(\mathbf{A})$ with $\mathbf{A}$ the ring of adeles over $\mathbf{Q}$. We compare ω_∞ with the Euler-Poincaré measure e_χ in the sense of Serre [23: §3] on $\mathrm{SO}(b)(\mathbf{R})$.

The volumes of congruence subgroups of $\mathrm{SO}(b)(\mathbf{Q}_p)$ are computed. We will explain in the beginning of the next chapter how such results are useful.

4.1. We recall some facts on Lie algebras of orthogonal groups. Let $\mathrm{so}(b)(\mathbf{Q}) = \{A \in \mathrm{M}_n(\mathbf{Q}) / A + b^t A b^{-1} = 0\}$. If $X, Y \in \mathrm{so}(b)(\mathbf{Q})$ then by $e(X, Y) = -\mathrm{tr}(X \cdot Y)/2$ we have a non degenerate bilinear form on $\mathrm{so}(b)(\mathbf{Q})$ and it is well known that $(X, Y) \mapsto -2(n-2)e(x, y)$ is the Killing form on $\mathrm{so}(b)(\mathbf{Q})$. If $b = E$ we write $\mathrm{so}_n(\ )$ instead of $\mathrm{so}(E)(\ )$. Clearly $\mathrm{so}_n(\mathbf{Q})$ consists of skew symmetric matrices. We have a natural basis $\{u_{i,j}\}_{i<j}$ for $\mathrm{so}_n(\mathbf{Q})$ consisting of skew symmetric matrices whose coefficients above the diagonal different from the one coefficient in the j-th place of the i-th column, which is one, are zero. One has $\mathrm{so}_{n-1}(\mathbf{Q}) \subset \mathrm{so}_n(\mathbf{Q})$ in a natural way. By induction on n we see that $\det(e(u_{i,j}, u_{k,l})_{i<j,k<l}) = 1$. This property is one reason for the choice of e. We have an isomorphism of vector

spaces $\mathrm{so}(b)(\mathbf{Q}) \xrightarrow{\sim} \mathrm{so}_n(\mathbf{Q})$ given by $A \mapsto A \cdot b$. Therefore we have a natural basis $e_{i,j} := u_{i,j} \cdot b^{-1}$, $i<j$, of $\mathrm{so}(b)(\mathbf{Q})$ and because $\det(b)=1$ we find $\det(e(e_{i,j}, e_{k,l})_{i<j,k<l}) = \pm 1$. We denote the dual basis of $e_{i,j}$, $i<j$, by $de_{i,j}$, $i<j$.

4.2. Set $d=n(n-1)/2$ and choose some order $e_1, \ldots, e_d$ of the basis elements of $\mathrm{so}(b)(\mathbf{Q})$. Then $\omega = de_1 \wedge \cdots \wedge de_d$ is a form of highest degree on $\mathrm{so}(b)(\mathbf{Q})$ which can be extended by left translation to a differential form on the algebraic group $\mathrm{SO}(b)/\mathbf{Q}$. If $X_1, \ldots, X_d \in \mathrm{so}(b)(\mathbf{Q})$ are linear independent, then

$$\omega(X_1, \ldots, X_d) = \mathrm{sign}(X_1, \ldots, X_d)(+|\det(e(X_i, X_j)_{i,j=1,\ldots,d}|^{1/2}).$$

Here $\mathrm{sign}(X_1, \ldots, X_d) = 1$ if $X_1, \ldots, X_d$ determine the same orientation as $e_1, \ldots, e_d$ and $\mathrm{sign}(X_1, \ldots, X_d) = -1$ otherwise. This formula has to be checked only for the basis $e_1, \ldots, e_d$ and follows then from 4.1. From the last equation we see that ω is invariant under the adjoint action of $\mathrm{SO}(b)/\mathbf{Q}$. Therefore ω is a biinvariant differential form.

4.3. The differential form ω determines for each place v of $\mathbf{Q}$ a biinvariant measure ω_v on the locally compact group $\mathrm{SO}(b)(\mathbf{Q}_v)$. This is explained in [7]. We can describe the measures ω_v as follows.

i) If $v=\infty$ then on $\mathrm{SO}(b)(\mathbf{R})$ the measure ω_∞ is the measure which in the sense of differential geometry belongs to the (possibly non positive) metric induced by $e(,)$.

ii) Suppose that $v=p$. The exponential map exp defines a bianalytic map of a neighbourhood V of 0 in $\mathrm{so}(b)(\mathbf{Q}_v)$ onto a neighbourhood U of the identity element in $\mathrm{SO}(b)(\mathbf{Q}_v)$. For $Y \in V$ we identify the tangentspace in $\exp(Y)$ at $\mathrm{SO}(b)(\mathbf{Q}_v)$ by left translation with the Lie algebra. Therefore the determinant of the differential of exp in the point Y $\det(\mathrm{Dexp}|_Y)$ makes sense. If V is small enough then $\|\det(\mathrm{Dexp}|_Y)\|_p = 1$ because $\mathrm{Dexp}|_0 = E$. We consider ω as constant differential form on V. Then by the transformation rule of integrals

$$\int_U \omega_v = \int_V \omega.$$

If $X_1, \ldots, X_d \in \mathrm{so}(b)(\mathbf{Q}_v)$ are linear independent and if $P = \mathbf{Z}_p \cdot X_1 + \cdots + \mathbf{Z}_p \cdot X_d \subset V$ then $\int_{\exp(P)} \omega_v = \|\det e(X_i, X_j)_{i,j}\|_p^{1/2}$.

4.4. Let $\mathbf{A}$ be the adele ring of $\mathbf{Q}$. Then $\mathrm{SO}(b)(\mathbf{A})$ is a locally compact group and ω determines a biinvariant measure denoted by $\omega = \prod \omega_v$ on $\mathrm{SO}(b)(\mathbf{A})$. This and the notation $\omega = \prod_v \omega_v$ is explained in [7]. On $\mathrm{SO}(b)(\mathbf{Q})\backslash\mathrm{SO}(b)(\mathbf{A})$ there is a measure induced by ω and by the formula of Minkowski–Siegel we have the volume $\mathrm{vol}_\omega\,(\mathrm{SO}\,(b)(\mathbf{Q})\backslash \mathrm{SO}\,(b)(\mathbf{A})) = 2$. If $n \geq 3$ see [27] if $n = 2$ see [18: p. 128]. This result is for $n = 2$ nothing else but a translation of the analytical class number formula for quadratic extensions of $\mathbf{Q}$.

We now compare the Gauß-Bonnet form with ω_∞. In principle such a comparison has been done by Siegel [24: III, 79, §3, §4].

PROPOSITION 4.5. *Suppose that b is a rational non degenerate bilinear form of signature $(r, 2j)$, $r + 2j = n$. Set $l = [n/2]$ and define a biinvariant measure e_X on* $\mathrm{SO}\,(b)(\mathbf{R})$ *by*

$$e_X = (-1)^{r \cdot j} \binom{l}{j} \mathrm{vol}_{\omega_\infty} (\mathrm{SO}_n\,(\mathbf{R}))^{-1} \cdot \omega_\infty$$

Then for every torsion free arithmetic subgroup Γ of $\mathrm{SO}\,(b)(\mathbf{Q})$ *the following holds*

$$\int_{\mathrm{SO}\,(b)(\mathbf{R})/\Gamma} e_X = \chi(\Gamma).$$

Proof. If $j = 0$ or $2j = n$ then $\mathrm{SO}\,(b)(\mathbf{R})$ is compact, $\Gamma = \{1\}$, and the assertion is trivially true. Suppose now that $j \neq 0$ and $2j \neq n$. Then $\mathrm{SO}\,(b)(\mathbf{R})$ is non compact and has two connected components [8: p. 346]. We consider the symmetric space $X = K\backslash \mathrm{SO}\,(b)(\mathbf{R})$ where K is a maximal compact subgroup of $\mathrm{SO}\,(b)(\mathbf{R})$. The group K meets both components of $\mathrm{SO}\,(b)(\mathbf{R})$ and the connected components of the identity K^0 of K is isomorphic to $\mathrm{SO}_r\,(\mathbf{R}) \times \mathrm{SO}_{2j}\,(\mathbf{R})$. This group contains a compact torus of maximal rank $[n/2] = [r/2] + j$. From Harder's Gauß–Bonnet theorem [7] we see that there exists a $\mathrm{SO}\,(b)(\mathbf{R})$-right invariant differential form ω_X of highest degree on X such that

$$\int_{X/\Gamma} \omega_X = \chi(\Gamma).$$

for all arithmetic torsion free subgroups $\Gamma \subset \mathrm{SO}\,(b)(\mathbf{Q})$. Moreover it is well known that ω_X can be computed in a universal way from the curvature tensor R of X [13: p. 318].

To compute ω_X we at first recall the Hirzebruch proportionality principle [9: p. 134]. Let $\mathfrak{l} = \mathfrak{so}\,(b)(\mathbf{R})$ be the Lie algebra of $\mathrm{SO}\,(b)(\mathbf{R})$, let $\mathfrak{k} \subset \mathfrak{l}$ be the subalgebra corresponding to K and let $\mathfrak{l} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. If $x_0 \in X$ is the point corresponding to K then $\mathfrak{p}$ is the tangent space of X at x_0. The subspace $\mathfrak{k}$ is orthogonal to $\mathfrak{p}$ with respect to the bilinear form $e(,)$. Thus $-e(,)$ induces by right translation a Riemannian metric on X. The connected real Lie subgroup of $SO(b)(\mathbf{C})$ corresponding to the real subalgebra $\mathfrak{k} \oplus i\mathfrak{p}$ is denoted by G_0. Consider $\mathfrak{l} \oplus_{\mathbf{R}} \mathbf{C}$ as a real Lie algebra and $\mathfrak{l} \subset \mathfrak{l} \oplus_{\mathbf{R}} \mathbf{C}$ as a subalgebra. Then there is a natural extension of $e(,)$ to a real bilinear form on $\mathfrak{l} \oplus_{\mathbf{R}} \mathbf{C}$ which we again denote by $e(,)$. This e induces a Riemannian metric on the compact symmetric space $X_0 = K^0 \backslash G_0$ and the map $Y \mapsto i \cdot Y$ from $\mathfrak{p}$ to $i\mathfrak{p}$ induces an open immersion $j: X \to X_0$ which is an isometry on the tangent spaces at the point corresponding to K^0.

It is known that the curvature tensors R resp. R_0 on X resp. X_0 are $SO(b)(\mathbf{R})$-resp. G_0-right invariant and that for $U, V, W \in \mathfrak{p}$ we have $R(U, V, W) = -[[U, V], W]$ resp. $R_0(iU, iV, iW) = -[[iU, iV], iW]$. Hence $-j^*|_{x_0} R_0 = R|_{x_0}$. The G_0-right invariant Gauß–Bonnet form ω_{0_x} on X_0 has the form $\omega_{0_x} = c \cdot v_0$ where $c \in \mathbf{R}$ and where v_0 is the right invariant volume form determined by e on X_0. On the other hand ω_X and ω_{0_x} are given as universal polynomials in the curvature tensors. Using $-j^*|_{x_0} R_0 = R|_{x_0}$ we see that $\omega_X|_{x_0} = (-1)^{r \cdot j} \cdot c \cdot v|_{x_0}$. Here v is the volume form on X determined by e. Hopf and Samelson proved that $\chi(X_0) = |W_{G_0}| \cdot |W_{K^0}|^{-1}$ where $|W_{G_0}|$ and $|W_{K^0}|$ are the orders of the Weyl groups of G_0 and K^0. Therefore $c = |W_{G_0}|\,|W_{K^0}|^{-1} \operatorname{vol}_{v_0}(K^0 \backslash G_0)^{-1}$. Now

$$\int_{X/\Gamma} v = \operatorname{vol}(K)^{-1} \int_{SO(b)(\mathbf{R})/\Gamma} \omega_\infty.$$

We get

$$|W_{G_0}| \cdot |W_{K^0}|^{-1} \cdot \operatorname{vol}_{v_0}(K^0 \backslash G_0)^{-1} \int_{SO(b)(\mathbf{R})/\Gamma} \omega_\infty = \chi(\Gamma).$$

We have to compute the factors on the left hand side. Obviously $\mathfrak{k} \oplus i\mathfrak{p}$ is a real compact form for $\mathfrak{l} \oplus_{\mathbf{R}} \mathbf{C}$. Therefore [8: III, 7.3] the group G_0 is conjugate in $SO(b)(\mathbf{C})$ to $SO_n(\mathbf{R})$ and $K^0 \cong SO_r(\mathbf{R}) \times SO_{2j}(\mathbf{R})$. Thus $|W_{G_0}|$ and $|W_{K^0}|$ are known. We have $\operatorname{vol}(K) = 2 \cdot \operatorname{vol}(K^0)$. Because G_0 and $SO_n(\mathbf{R})$ are conjugate in $SO(b)(\mathbf{C})$ and because the volume elements on both groups are induced from the same $SO(b)(\mathbf{C})$-invariant bilinear form on $\mathfrak{l} \oplus_{\mathbf{R}} \mathbf{C}$ we have $\operatorname{vol}(G_0) = \operatorname{vol}(SO_n(\mathbf{R}))$. The desired formula follows now easily. q.e.d.

If $K_v \subset SO(b)(\mathbf{Q}_v)$ is a compact subset we write $\operatorname{vol}(K_v)$ for the volume of K_v with respect to the measures introduced in 4.3.

LEMMA 4.6. *Denote by* $\operatorname{vol}(S^j)$, $j \geq 1$, *the volume of the standard* j-

dimensional sphere of radius 1 *with respect to the euclidean volume element. Then*

$$\mathrm{vol}\,(\mathrm{SO}_n\,(\mathbf{R})) = \mathrm{vol}\,(S^1)\cdot\,\cdots\,\cdot\mathrm{vol}\,(S^{n-1}).$$

We have

$$\mathrm{vol}\,(\mathrm{SO}_{2l}\,(\mathbf{R})) = \frac{(2\pi)^l}{(l-1)!}\prod_{j=1}^{l-1}\frac{(2\pi)^{2j}}{(2j-1)!}$$

$$\mathrm{vol}\,(\mathrm{SO}_{2l+1}(\mathbf{R})) = 2l\prod_{j=1}^{l}\frac{(2\pi)^{2j}}{(2j-1)!}.$$

Proof. We have a fibration $\mathrm{SO}_{n-1}\,(\mathbf{R}) \to \mathrm{SO}_n\,(\mathbf{R}) \xrightarrow{f} S^{n-1}$. If $N = (1, 0, \ldots, 0) \in \mathbf{R}^n$ then for $A \in \mathrm{SO}_n\,(\mathbf{R})$ we define $f(A) = A(N) = A\cdot{}^tN$. On $\mathrm{so}_n\,(\mathbf{R})/\mathrm{so}_{n-1}\,(\mathbf{R})$ we have an induced scalar product and f induces an isometry of $\mathrm{so}_n\,(\mathbf{R})/\mathrm{so}_{n-1}\,(\mathbf{R})$ with the tangent space of S^{n-1} at N. The tangent space here is equipped with the euclidean scalar product. The result now follows by induction and by well known formulas for $\mathrm{vol}\,(S^i)$. q.e.d.

PROPOSITION 4.7. *With the notation introduced in* §3 *the following formulas hold.*

i)

$$\mathrm{vol}\,(\mathrm{SO}_n\,(\mathbf{Z}_2) = e(E)^{-1}\cdot\prod_{j=1}^{[(n-1)/2]}(1-2^{-2j})$$

If $p \neq 2$ *then*

$$\mathrm{vol}\,(\mathrm{SO}_n\,(\mathbf{Z}_p)) = \begin{cases} \displaystyle\prod_{j=1}^{[(n-1)/2]}(1-p^{-2j}) & \textit{if} \quad n \equiv 1 \bmod 2 \\ \displaystyle(1-p^{-n/2})\prod_{j=1}^{[(n-1)/2]}(1-p^{-2j}) & \textit{if} \quad n \equiv 0 \bmod 4 \\ \displaystyle(1-\chi(p)p^{-n/2})\prod_{j=1}^{[(n-1)/2]}(1-p^{-2j}) & \textit{if} \quad n \equiv 2 \bmod 4 \end{cases}$$

Here we have $\chi(p) = (-1)^{(p-1)/2}$. *For all primes* p

$$\mathrm{vol}\,(\Gamma_p(j)^\tau) = p^{-j\cdot n(n-1)/2} \quad \textit{if} \quad j \geq 1.$$

ii)

$$\mathrm{vol}\,(\mathrm{SO}\,(L)(\mathbf{Z}_2)) = e(L)^{-1} \prod_{j=1}^{[(n-1)/2]} (1-2^{-2j})$$

$$\mathrm{vol}\,({}^L\Gamma_2(1)^\tau) = 2^{-n(n-1)/2}$$

iii)

$$\mathrm{vol}\,(\mathrm{SO}\,(S)(\mathbf{Z}_2)) = (1-2^{-n/2}) \prod_{j=1}^{[(n-1)/2]} (1-2^{-2j}) \quad \textit{if} \quad n \equiv 0 \ \textit{mod}\ 4$$

Proof. Let b be one of the matrices E, L, S. Suppose that $p \neq 2$ and $j \geq 1$ or $p=2$ and $j \geq 2$. Then we have a bijection exp: $p^j \cdot M_n(\mathbf{Z}_p) \tilde{\to} \Gamma_p(j)$ and ${}^b\Gamma_p(j)^\tau \cong \{A \in p^j \cdot \mathrm{M}_n\,(\mathbf{Z}_p)/A \cdot b + {}^t(A \cdot b) = 0\} \tilde{\to} \{B \in p^j \cdot \mathrm{M}_n(\mathbf{Z}_p)/B + {}^tB = 0\}$. The last isomorphism is given by $A \mapsto A \cdot b$ and is an isomorphism of free $\mathbf{Z}_p$-modules. By definition of our measure and 4.3 we get $\mathrm{vol}\,({}^b\Gamma_p(j)^\tau) = p^{-j \cdot n(n-1)/2}$.

We consider now the case that $p=2$ and $j=1$ and $b=E$ or $b=L$. Then we have an exact sequence

$$1 \to {}^b\Gamma_2(1)^\tau/{}^b\Gamma_2(2)^\tau \to {}^b(\Gamma_2(1)/\Gamma_2(2))^\tau \xrightarrow{\partial} H^1(\mathfrak{g}, {}^b\Gamma_2(2)) \to H^1(\mathfrak{g}, {}^b\Gamma_2(1))$$

From 3.2 we know that ∂ is surjective and that $|H^1(\mathfrak{g}, {}^b\Gamma_2(2))| = 2^{n-1}$. It is easy to see that $|{}^b(\Gamma_2(1)/\Gamma_2(2))^\tau| = 2^{n-1} \cdot 2^{n(n-1)/2}$ and we find that the index of ${}^b\Gamma_2(2)^\tau$ in ${}^b\Gamma_2(1)^\tau$ is $2^{n(n-1)/2}$. Together with the above computation we now have proven the last formulas of i) and ii).

If $p \neq 2$ then from 3.1 we have an isomorphism $\mathrm{SO}_n\,(\mathbf{Z}_p)/\Gamma_p(1)^\tau \tilde{\to} \mathrm{SO}_n\,(\mathbf{Z}/p\mathbf{Z})$. The order of $\mathrm{SO}_n\,(\mathbf{Z}/p\mathbf{Z})$ is known [6: 25.4] and the second formula of i) follows.

We now prove the first formulas of i) and ii). Let b be the matrix E or L. We consider the diagram

$$\begin{array}{l} 1 \to {}^b(\Gamma_2(1)/\Gamma_2(2))^\tau \to {}^b(\Gamma_2/\Gamma_2(2))^\tau \to {}^b(\Gamma_2/\Gamma_2(1))^\tau \\ \qquad\qquad \xrightarrow{\partial} H^1(\mathfrak{g}, {}^b(\Gamma_2(1)/\Gamma_2(2))) \to H^1(\mathfrak{g}, {}^b(\Gamma_2/\Gamma_2(2))) \\ \qquad\qquad\qquad\qquad \uparrow\wr \qquad\qquad\qquad\qquad\qquad \uparrow\wr \\ \qquad\qquad\qquad H^1(\mathfrak{g}, {}^b\Gamma_2(1)) \qquad\qquad\qquad H^1(\mathfrak{g}, {}^b\Gamma_2). \end{array}$$

From 3.3 we know that $\mathrm{im}\,(\partial)$ has $e(b)$ elements. Now ${}^b(\Gamma_2(1)/\Gamma_2(2))^\tau$ has $2^{n-1} \cdot 2^{n(n-1)/2}$ elements. Therefore ${}^b(\Gamma_2/\Gamma_2(2))^\tau$ has $2^{n-1} \cdot 2^{n(n-1)/2}\,|(\Gamma_2/\Gamma_2(1))^\tau|\, e(b)^{-1}$

elements. From [5: p. 63] we deduce that

$$|(\Gamma_2/\Gamma_2(1))^\tau| = 2^{n(n-1)/2} \prod_{j=1}^{[(n-1)/2]} (1-2^{-2j}).$$

With the exact sequence

$$1 \to {}^b\Gamma_2^\tau/{}^b\Gamma_2(2)^\tau \to {}^b(\Gamma_2/\Gamma_2(2))^\tau \to H^1(\mathfrak{g}, {}^b\Gamma_2(2)) \to 1,$$

the formulas $|H^1(\mathfrak{g}, {}^b\Gamma_2(2))| = 2^{n-1}$ and $\mathrm{vol}\,({}^b\Gamma_2(2)^\tau) = 2^{-n(n-1)}$ we find the desired result.

We define the quadratic form $q_S(x) = S(x, x)/2$, $x \in \mathbf{Z}_2^n$. The orthogonal group $\mathrm{O}\,(q_S)/\mathbf{Z}_2$ then is smooth. By Hensels lemma we find

$$\mathrm{O}\,(q_S)/{}^S\Gamma_2(2)^\tau \xrightarrow{\sim} \mathrm{O}\,(q_S)(\mathbf{Z}/4\mathbf{Z}).$$

From [6: 25.1] we have $|\mathrm{O}\,(q_s)(\mathbf{Z}/4\mathbf{Z})| = 2 \cdot 2^{n(n-1)} \cdot (1-2^{-l}) \cdot \prod_{j=1}^{l-1} (1-2^{-2j})$, $l = n/2$. In the beginning of this proof we saw $\mathrm{vol}\,({}^S\Gamma_2(2)^\tau) = 2^{-n(n-1)}$. Therefore iii) follows. q.e.d.

Remark. We compare our local volumes with the local factors occurring in the classical Minkowski-Siegel formula. We use Siegels notation [24: I p. 367] and get.

$$A_\infty(E, E) = 2^{-n+1}\,\mathrm{vol}\,(\mathrm{SO}_n\,(\mathbf{R})).$$

If b is one of the matrices E, L, S then

$$\alpha_2(b, b) = 2^{n-1}\,\mathrm{vol}\,(\mathrm{SO}\,(b)(\mathbf{Z}_2))$$
$$\alpha_p(b, b) = \mathrm{vol}\,(\mathrm{SO}\,(b)(\mathbf{Z}_p) \quad \text{for} \quad p \neq 2.$$

§5. The Euler–Poincaré characteristics of the fixpoint set

We know from 1.3 that for each class $b \in H^1(\mathfrak{g}, \Gamma)$ we have a fixpoint set $F(b) \subset (X/\Gamma)^\tau$. If $(1, b)$ is a cocycle representing b, then

$$\chi(F(b)) = \int_{\mathrm{SO}\,(b)(\mathbf{R})/\Gamma(b)} e_X$$

by 4.5. In order to apply 4.5 we have to know vol $(\mathrm{SO}(b)(\mathbf{R})/\Gamma(b))$. It turns out to be easier (and sufficient) to compute the volume of certain unions of such spaces. To do this we use the Minkowski–Siegel formula, i.e., the fact that the Tamagawa number of an orthogonal group is 2. Using all the local results we have obtained and the classification of §2 we finish this paper with an explicit formula for the Lefschetz number $L(\tau, \Gamma(m))$.

5.1. If $(1, b)$ is a cocycle for $H^1(\mathfrak{g}, \Gamma)$ we get by twisting from 2.2 an exact sequence

$$1 \to \mathrm{SO}(b)(\mathbf{Q})\backslash \mathrm{SO}(b)(\mathbf{A}) \Big/ \prod_v \Gamma_v(b) \xrightarrow{i} H^1(\mathfrak{g}, {}^b\Gamma) \xrightarrow{h_b} \coprod_v H^1(\mathfrak{g}, {}^b\Gamma_v).$$

Here we use the notation $\Gamma_v(b) = {}^b\Gamma_v^\tau$.

We describe the fixpoint set corresponding to a class c in the kernel of h_b. If $x \in \mathrm{SO}(b)(\mathbf{A})$ represents a class c then x also represents an element in ${}^b(\mathrm{Sl}_n(\mathbf{A})/\prod_v \Gamma_v)^\tau$. By strong approximation write $x = g \cdot \eta$ with $g \in \mathrm{Sl}_n(\mathbf{Q})$, $\eta \in \prod_v \Gamma_v$. Then $\gamma = g^{-1} \cdot b \cdot {}^\tau g \cdot b^{-1} \in \Gamma$ gives a cocycle for $H^1(\mathfrak{g}, {}^b\Gamma)$ which by definition of i represents $i(c)$. We get

$$\begin{aligned} {}^b\Gamma(\gamma) &= \{u \in \Gamma / b \cdot {}^\tau(g \cdot u \cdot g^{-1}) b^{-1} = g \cdot u \cdot g^{-1}\} = \\ &= g^{-1} \cdot \mathrm{SO}(b)(\mathbf{Q}) \cdot g \cap \Gamma = g^{-1}(g\Gamma g^{-1} \cap \mathrm{SO}(b)(\mathbf{Q}))g. \end{aligned}$$

Clearly $\chi({}^b\Gamma(\gamma)) = \chi(g \cdot {}^b\Gamma(\gamma) \cdot g^{-1})$. Therefore it suffices to compute vol $(\mathrm{SO}(b)(\mathbf{R})/g \cdot \Gamma \cdot g^{-1} \cap \mathrm{SO}(b)(\mathbf{R}))$.

5.2. We get a sum of such volumes as follows. Write $\mathrm{SO}(b)(\mathbf{A}) = \bigcup_{i=1}^{n(b)} \prod_v \Gamma_v(b) \cdot x_i^{-1} \cdot \mathrm{SO}(b)(\mathbf{Q})$ as a disjoint finite union with $x_i \in \mathrm{SO}(b)(A)$, $i = 1, \dots, n(b)$, representing the different classes in $\mathrm{SO}(b)(Q)\backslash \mathrm{SO}(b)(A)/\prod_v \Gamma_v(b)$. Now $\prod_v \Gamma_v(b) x_i^{-1} \cdot \mathrm{SO}(b)(\mathbf{Q}) = x_i^{-1} \cdot x_i \prod_v \Gamma_v(b) \cdot x_i^{-1} \cdot \mathrm{SO}(b)(\mathbf{Q})$ and $x_i(\prod_v \Gamma_v(b)) x_i^{-1} \cap \mathrm{SO}(b)(\mathbf{Q}) = x_i(\prod_v \Gamma_v) x_i^{-1} \cap \mathrm{SO}(b)(\mathbf{Q})$. As above we write $x_i = g_i \cdot u_i$ with $g_i \in \mathrm{Sl}_n(\mathbf{Q})$ and $u_i \in \prod_v \Gamma_v$. We get $\mathrm{SO}(b)(\mathbf{Q}) \cap x_i \prod_v \Gamma_v(b) x_i^{-1} = \mathrm{SO}(b)(\mathbf{Q}) \cap g_i \Gamma g_i^{-1}$ and have a decomposition

$$\begin{aligned} \prod_v \Gamma_v(b) \cdot x_i^{-1} \cdot \mathrm{SO}(b)(\mathbf{Q}) = x_i^{-1}(\mathrm{SO}(b)(\mathbf{R}) \times \prod_p x_{i,p} \cdot \Gamma_p(b) \cdot x_{i,p}^{-1}/\mathrm{SO}(b)(\mathbf{Q}) \\ \cap g_i \Gamma g_i^{-1}) \times \mathrm{SO}(b)(\mathbf{Q}). \end{aligned}$$

Here $x_{i,p}$ is the component at the place p of x_i. If ω is the Tamagawa measure then $\mathrm{vol}_\omega\,(\prod_v \Gamma_v(b) \cdot x_i^{-1} \cdot \mathrm{SO}(b)(\mathbf{Q})/\mathrm{SO}(b)(\mathbf{Q})) = \mathrm{vol}_\omega\,(\mathrm{SO}(b)(\mathbf{R}) \times \prod_p x_{i,p} \cdot \Gamma_p(b) \cdot x_{i,p}^{-1}/\mathrm{SO}(b)(\mathbf{Q}) \cap g_i \Gamma g_i^{-1})$. From the natural fibration

$$\begin{aligned} \mathrm{SO}(b)(\mathbf{R}) \times \prod_p x_{i,p} \cdot \Gamma_p(b) \cdot x_{i,p}^{-1}/\mathrm{SO}(b)(\mathbf{Q}) \cap g_i \Gamma g_i^{-1} \\ \to \mathrm{SO}(b)(\mathbf{R})/\mathrm{SO}(b)(\mathbf{Q}) \cap g_i \Gamma g_i^{-1} \end{aligned}$$

and the invariance of the factors ω_v of ω we get

$$\mathrm{vol}_\omega\left(\prod_v \Gamma_v(b)x_i^{-1}\,\mathrm{SO}\,(b)(\mathbf{Q})/\mathrm{SO}\,(b)(\mathbf{Q})\right)=\mathrm{vol}_{\omega_\infty}(\mathrm{SO}\,(b)(\mathbf{R})/\mathrm{SO}\,(b)(\mathbf{Q})\cap g_i\Gamma g_i^{-1})\cdot\prod_p \mathrm{vol}_{\omega_p}(\Gamma_p(b)).$$

Now $\mathrm{vol}_\omega\,(\mathrm{SO}\,(b)(\mathbf{A})/\mathrm{SO}\,(b)(\mathbf{Q}))=2$ and we obtain finally

$$\sum_{i=1}^{n(b)}\mathrm{vol}_{\omega_\infty}(\mathrm{SO}\,(b)(\mathbf{R})/\mathrm{SO}\,(b)(\mathbf{Q})\cap g_i\Gamma g_i^{-1})=2\prod_p \mathrm{vol}_{\omega_p}(\Gamma_p(b))^{-1}.$$

5.3. We suppose now that $\Gamma=\Gamma(m)$, $m\geq 3$, is a full congruence subgroup mod m of $\mathrm{Sl}_n\,(\mathbf{Z})$. From 5.1, 5.2 and 4.4 we see that

$$\sum_{\substack{\gamma\in H^1(\mathfrak{g},\Gamma)\\ h(\gamma)=h(b)}}\chi(F(\gamma))=2(-1)^{d(b)}\binom{l}{s(b)/2}\mathrm{vol}\,(\mathrm{SO}_n\,(\mathbf{R}))^{-1}\prod_p \mathrm{vol}\,(\Gamma_p(b))^{-1}.$$

Here $(r(b), s(b))$ is the signature of b and $d(b)=r(b)\cdot s(b)/2$. The right hand side of the equation depends only on the image of b in $H^1(\mathfrak{g},\mathrm{Sl}_n\,(\mathbf{R}))\times\prod_p' H^1(\mathfrak{g},\mathrm{Sl}_n\,(\mathbf{Z}_p))$ and we know from §3 that only the places 2 and ∞ of $\mathbf{Q}$ contribute. Therefore we write

$$L(\tau,\Gamma)=\sum_{r_2(\gamma)=E}\chi(F(\gamma))+\sum_{r_2(\gamma)=L}\chi(F(\gamma))+\sum_{r_2(\gamma)=S}\chi(F(\gamma))$$

Here $r_2: H^1(\mathfrak{g},\Gamma_2)\to H^1(\mathfrak{g},\mathrm{Sl}_n\,(\mathbf{Z}_2))$ is the map induced by the inclusion $\Gamma_2\hookrightarrow\mathrm{Sl}_n\,(\mathbf{Z}_2)$.

If $r_2(\gamma)=E$ then the Hasse–Witt invariants are $\varepsilon_2(\gamma)=1$ and $\varepsilon_\infty(\gamma)=(-1)^{s(\gamma)(s(\gamma)-1)/2}=1$. Therefore $s(\gamma)\equiv 0 \bmod 4$ and $(-1)^{d(\gamma)}=1$. We have

$$\sum_\gamma\binom{l}{s(\gamma)/2}=2^{l-1}$$

for $\gamma\in H^1(\mathfrak{g},\mathrm{Sl}_n\,(\mathbf{R}))$ and $\varepsilon_\infty(\gamma)=1$. We get

$$\sum_{r_2(\gamma)=E}\chi(F(\gamma))=2^l\,\mathrm{vol}\,(\mathrm{SO}_n\,(\mathbf{R}))^{-1}\prod_p \mathrm{vol}\,(\Gamma_p(E))^{-1}\,|r_2^{-1}(\{E\})|.$$

If $r_2(\gamma)=L$ then the Hasse–Witt invariants are $\varepsilon_2(\gamma)=-1$ and $\varepsilon_\infty(\gamma)=(-1)^{s(\gamma)(s(\gamma)-1)/2}=-1$. Therefore $s(\gamma)\equiv 2 \bmod 4$ and $(-1)^{d(\gamma)}=(-1)^n$. We have

$\sum_\gamma \binom{l}{s(\gamma)/2} = 2^{l-1}$ for $\gamma \in H^1(\mathfrak{g}, \mathrm{Sl}_n(\mathbf{R}))$ and $\varepsilon_\infty(\gamma) = -1$. We get

$$\sum_{r_2(\gamma)=L} \chi(F(\gamma)) = (-1)^n 2^l \operatorname{vol}(\mathrm{SO}_n(\mathbf{R}))^{-1} \prod_p \operatorname{vol}(\Gamma_p(L))^{-1} |r_2^{-1}(\{L\})|.$$

If $r_2(\gamma) = S$ then $n \equiv 0 \bmod 4$ and $\Gamma_2 = \mathrm{Sl}_n(\mathbf{Z}_2)$. We have $\varepsilon_2(\gamma) = 1$, $(-1)^{d(\gamma)} = 1$ and $s(\gamma) \equiv 0 \bmod 4$. Therefore we get

$$\sum_{r_2(\gamma)=S} \chi(F(\gamma)) = 2^l \operatorname{vol}(\mathrm{SO}_n(\mathbf{R}))^{-1} \prod_p \operatorname{vol}(\Gamma_p(S))^{-1}.$$

5.4. Using the local results of §3 we see that the number $L(\tau, \Gamma(m))$ has the following form. It is a product of a rational number depending elementary on n and m with some power of π and with factors of the form $\zeta(2j)$, $j \in N$, and a factor $L(\chi, l)$ if $n \equiv 2 \bmod 4$. Here $\zeta(2j)$ is the Riemann zeta function at $2j$ and $L(\chi, l)$ is the value of the L-function belonging to the non trivial character of $\mathbf{Q}(i)/\mathbf{Q}$ at l. In order to avoid irrationalities in the final formula for $L(\tau, \Gamma(m))$ we use the functional equations

$$\zeta(s) = \frac{(2\pi)^s \zeta(1-s)}{2\Gamma(s)\cos(\pi s/2)} \quad \text{and} \quad \mathrm{L}(\chi, s) = \frac{(2\pi)^s\, \mathrm{L}(\chi, 1-s)}{2^{2s}\Gamma(s)\sin(\pi s/2)}$$

for $s \in \mathbf{C}$. Here $\Gamma(\ )$ denotes the gamma function [11: §1]. I hope no confusion with the notation of congruence subgroups will arise.

5.5. To shorten the notation we set $l = [n/2]$ and introduce the following abbreviations. For all $2 \le n \in \mathbf{N}$, $n \not\equiv 2 \bmod 4$ and $3 \le m \in \mathbf{N}$ we define an integer $u(n, m)$.

$$u(n,m) = \begin{cases} 1 & \text{if } 2 \mid m \text{ and } 4 \nmid m \text{ and } n \equiv 1 \bmod 2 \text{ or} \\ & \text{if } (2,m) = 1 \text{ and } n \equiv 1 \bmod 2 \\ (-1)^{l(l+1)/2} 2^l & \text{if } 4 \mid m \text{ and } n \equiv 1 \bmod 2 \\ 2^{2l-1} - 2^{l-1} + 1 & \text{if } (2,m) = 1 \text{ and } n \equiv 0 \bmod 4 \\ 2^{n-1} & \text{if } 2 \mid m \text{ and } n \equiv 0 \bmod 4 \end{cases}$$

We define

$$(d_1, \ldots, d_l) = \begin{cases} (2, 4, \ldots, 2l) & \text{if } n = 1 + 2l \\ (2, 4, \ldots, 2l-2, l) & \text{if } n = 2l. \end{cases}$$

By $\chi:\mathbf{Z}\to\{\pm1,0\}$ we denote the number theoretical character with $\chi(n)=0$ if n is even and $\chi(n)=(-1)^{(n-1)/2}$ if n is odd. We define

$$c(n,m)=\begin{cases} m^{n(n-1)/2}\prod_{p\mid m}\prod_{i=1}^{l}(1-p^{-d_i}) & \text{if} \quad n\not\equiv 2 \bmod 4\\ m^{n(n-1)/2}\prod_{p\mid m}(1-\chi(p)p^{-1})\prod_{i=1}^{l-1}(1-p^{-d_i}) & \text{if} \quad n\equiv 2 \bmod 4. \end{cases}$$

THEOREM 5.6. *If $n\geq 2$ and $m\geq 3$ and if $\Gamma(m)$ is the full congruence subgroup mod m of $\mathrm{Sl}_n(\mathbf{Z})$ then the following hold*

$$L(\tau,\Gamma(m))=c(n,m)\mathrm{L}(\chi,1-d_l)\prod_{i=1}^{l-1}\zeta(1-d_i) \quad \textit{if} \quad n\equiv 2 \bmod 4$$

resp.

$$L(\tau,\Gamma(m))=u(n,m)c(n,m)\prod_{i=1}^{l}\zeta(1-d_i) \quad \textit{if} \quad n\not\equiv 2 \bmod 4.$$

The right hand side of the formula for $L(\tau,\Gamma(m))$ is computable in terms of Bernoulli numbers if $n\not\equiv 2 \bmod 4$ and in terms of generalized Bernoulli numbers if $n\equiv 2 \bmod 4$ [11: §2]. We give a list of the first few values of $L(\tau,\Gamma(m))$.

n	$L(\tau,\Gamma(3))$	$L(\tau,\Gamma(4))$
2	2	2
3	-2	2^3
4	$7\cdot 2^2$	2^7
5	$-3^2\cdot 2^2$	2^{11}
6	$2^3\cdot 3^4\cdot 7$	2^{18}
7	$2^3\cdot 3^5\cdot 13$	2^{26}
8	$2^4\cdot 3^7\cdot 11^2\cdot 13$	2^{37}
9	$2^4\cdot 3^{11}\cdot 13\cdot 41$	$2^{45}\cdot 17$
10	$2^5\cdot 3^{15}\cdot 5\cdot 13\cdot 41\cdot 61$	$2^{58}\cdot 5\cdot 17$

Concluding remark. Obviously we have an estimation $\dim_{\mathbf{Q}} H^*(\Gamma,\mathbf{Q})\geq |L(\tau,\Gamma)|$. Together with our formula for the Lefschetz number this gives some idea how big $H^*(\Gamma,\mathbf{Q})$ is.

Using information on the cohomology of the boundary $\partial(\bar{X}/\Gamma)$ of the Borel–Serre compactification $\bar{X}/\Gamma$ of X/Γ one can try to do more. If the

Lefschetz number of τ acting on the image of the natural map $H^*(\bar{X}/\Gamma, \mathbf{Q}) \xrightarrow{\text{res}} H^*(\partial(\bar{X}/\Gamma)), \mathbf{Q})$ is known – here one has to use Schwermer's methods [20] – one can compute the Lefschetz number of τ acting on the image $H_!^*(X/\Gamma, \mathbf{Q})$ of the cohomology with compact supports $H_c^*(X/\Gamma, \mathbf{Q})$ in $H^*(X/\Gamma, \mathbf{Q})$. Now B. Speh has determined all the irreducible representations of $\mathrm{Sl}_n(\mathbf{R})$ having non trivial cohomology [B. Speh: Bonn 1980, private communication]. Thus hopefully one has a chance to get information on multiplicities of cuspidal representations having non trivial cohomology. Naturally Langland's lifting principle is connected with all of this.

For $\mathrm{Sl}_3/\mathbf{Q}$ R. Lee and J. Schwermer are pursuing these questions.

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Received March 20, 1980/November 29, 1980