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An example of a Schroedinger equation with almost periodic potential and nowhere dense spectrum*

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§1. Introduction

It is well known that the spectrum of a second order differential operator

$$L=-\left(\frac{d}{dx}\right)^2+q(x)$$

with a continuous periodic potential $q(x)=q(x+l)$ with period $l>0$ is given by an infinite or finite sequence of intervals extending to $+\infty$. In the latter case one of the intervals extends to infinity. Here we consider the unique selfadjoint extension of the above operator considered in $C''_{\text{comp}}(R)$, the space of twice continuously differential functions on the real line R with compact support. Such a selfadjoint extension, with dense domain in $L^2(R)$, is unique since this problem is in the "limit point case."

Little is known about the spectrum of such operators if $q(x)$ is a continuous almost periodic function, since one has too little information about the solutions of the differential equation $L\phi=\lambda\phi$ in this case. One knows examples of some hyperelliptic functions $q(x)$, which are almost periodic, even quasi-periodic for which the spectrum again consists of a finite number of intervals, one of which extends to $+\infty$. These hyperelliptic functions are, in fact, meromorphic on the Riemann surface constructed from two copies of the complex plane slit along the intervals which constitute the spectrum. For these considerations see Dubrovin, Novikov and Matveev [3, 9].

These examples are, however, in no way typical but rather exceptional, and generally one has to expect a much more complicated spectrum for almost periodic potentials. We want to mention a paper by Dynaburg and Sinai [4] in which it is shown that for quasi-periodic potentials with certain number theoretical restrictions on the frequency basis there exists a Cantor set extending to

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infinity which is contained in the spectrum. Recently Rüssmann [8] gave a simpler proof of this theorem and sharpened the result. However, these theorems only assert the containment of the Cantor set in the spectrum and it is not excluded that the spectrum is actually a halfline.

Here we want to construct a simple example of an almost periodic potential for which the spectrum is in fact a Cantor set. Actually, the potential is constructed as a limit periodic function, which is a function which can be uniformly approximated by periodic functions (see Besicovitch [1]). These form a special class of almost periodic functions for which the frequencies are rational multiples of one fixed positive number, which we could normalize to 1. Even for this simple kind of potential one finds such a complicated spectrum. We will show that *in the neighborhood of any periodic potential in $C(R)$-topology there are limit periodic potentials with nowhere dense spectrum.*

The idea of the construction is simply the following: We observe that for a periodic potential the gaps in the complement of the spectrum are given by the instability intervals, i.e. those λ-intervals for which the solutions are unbounded. If $M=M(\lambda)$ is the 2 by 2 matrix which takes

$$\begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} \quad \text{into} \quad \begin{pmatrix} y(l) \\ y'(l) \end{pmatrix}$$

then the eigenvalues μ, μ^{-1} of M, the so-called Floquet multipliers, are either real or on the unit circle. They are the solutions of the quadratic equation

$$\mu^2-\Delta\mu+1=0$$

where $\Delta=\Delta(\lambda)=\operatorname{tr} M$ is the so-called discriminant. The instability intervals are characterized as those real λ for which $|\mu|\neq 1$ or $\Delta^2>4$, and their boundaries by $\mu^2=1$ or $\Delta=\pm 2$. It is possible that such an instability interval collapses, which amounts to a double root of Δ^2-4. This situation occurs precisely if all solutions of the eigenvalue equation $L\phi=\lambda\phi$ are periodic of period $2l$; therefore one speaks of coexistence of periodic solutions, a case which has been frequently discussed in the literature [6, 7].

A potential $q(x)$ of period l has, of course, also $2l, 3l, \ldots$ etc. as periods, and considered as a function of period ml the matrix M has to be replaced by M^m and μ by μ^m, and Δ by $\Delta_m=\mu^m+\mu^{-m}$.

For $m=2, 3, \ldots$ the roots of the equation $\Delta_m^2=4$ correspond to roots of unity $\mu:\mu^{2m}=1$ and for these λ one has a periodic solution of period $2ml$. However, these roots, which are not also roots of $\Delta^2=4$, are all double roots of $\Delta_m^2=4$, i.e. we always have coexistence for all roots of $(\Delta_m^2-4)/(\Delta^2-4)$. This is simply due to the fact that for eigenvalues μ, $\mu^{-1}=\bar{\mu}$ which are roots of unity different from ± 1 the eigenvectors of M are linearly independent and are the initial values of the desired periodic solutions. This coexistence is due to the fact that the potential has the period l while the period of the solutions is $2ml$. However, if we perturb the given potential with a function of period ml the coexistence will in general be destroyed and an interval of instability will be formed. If we apply such a sequence of perturbations with longer and longer periods the instability intervals will become dense and the resulting potential a limit periodic function.

The fact that generically coexistence of periodic solutions is destroyed is not surprising. But it is important to observe that a double root of Δ_m^2-4 will not disappear under perturbation but generically split into two real roots. Assume we have $\Delta_m=+2$ at such a double root, then it turns out that after a small perturbation the local maximum of the corresponding discriminant is always $\geqq 2$ and generically >2. This fact can be derived from a closer investigation of the second variation of Δ_m (see Section 4).

Aside from proving the above statement we will discuss the Floquet theory from a geometrical point of view. Our discussion is based on the rotation number

$$\alpha=\lim_{x\to\infty}\frac{\arg(y(x)-iy'(x))}{x}$$

which exists for any periodic potential, and is the same for any nontrivial solution y of $Ly=\lambda y$. This defines a continuous monotone increasing function $\alpha=\alpha(\lambda)\geqq 0$ which is constant on the intervals of instability, but strictly increasing on the spectrum. In fact, in the interior of the spectrum one has

$$\frac{d(\alpha^2)}{d\lambda}\geqq 1$$

where equality can occur at any point only if q is a constant. The constant values of α on the instability interval are of the form $(\pi/l)j$, $j=1,2,3,\ldots$ In [3] the quantity $\alpha(\lambda)$ is referred to as "quasimomentum."

It is conceivably a generic phenomenon, in the sense of Baire category, that for a limit periodic, or even an almost periodic potential the spectrum is such a Cantor set. This will not be proven here.* In particular, it would require the existence and continuity of the above rotation number α. Actually this limit exists and depends continuously on λ for an arbitrary continuous almost periodic potential, as was proven recently by Russell Johnson, USC. Moreover, a spectral theoretical interpretation of $\alpha=\alpha(\lambda)$ can be given, showing that the set of constancy intervals of $\alpha(\lambda)$ agree with the complement of the spectrum.

* Added in proof: We learned that J. Avron and B. Simon just have proved such a result.

These results will be presented elsewhere and here we restrict ourselves to proving the underlined statement.

Our aim will be to construct a limit periodic potential $q(x)$ for which the function $\alpha = \alpha(\lambda)$ also exists and is a Cantor function: It is continuous and takes any rational value of the form $j/2^n$, $j \geqq 1$, $n \geqq 0$ both integers, on an interval of positive length. The complement of these intervals forms the spectrum of the operator $L = -(d/dx)^2 + q(x)$.

It is a known phenomenon that the rotation number $\alpha = \alpha(\lambda)$ of a circle mapping $M(\lambda)$ depending on a parameter generally takes all rational values on intervals of positive length. This is connected with the "lock in" phenomenon of nonlinear oscillation. A rational mapping of this type can be found in [5]. For the circle mapping induced by

$$\begin{pmatrix} y \\ y' \end{pmatrix} \to M(\lambda) \begin{pmatrix} y \\ y' \end{pmatrix}$$

on the rays through the origin this does not happen however, because of the linearity of this mapping. The point of the present note can be seen in the observation that the phenomenon of dense resonance can occur even in the linear case, if the coefficients are almost periodic.

The paper [10] by G. Scharf is devoted to the study of the spectrum for almost periodic potentials. His result can be also used to effectively investigate the rotation number $\alpha(\lambda)$, and describe $\alpha(\lambda)$ as the boundary values of a positive harmonic function in the upper halfplane. However, the phenomenon of dense spectrum was not noted there. In some recent preliminary draft Joseph Avron and Barry Simon ("Cantor Sets, Almost Periodic Hill's Equations, and the Rings of Saturn," Cal. Inst. Technology) announced various statements about the spectrum of almost periodic potentials, as well as the appearance of Cantor sets as spectrum.

Finally, I should like to thank Russell Johnson for supplying me with a proof of the existence proof for the rotation number for almost periodic potential as well as for substantial advice. Also I am grateful to P. Deift and E. Zehnder for reading the manuscript and suggesting improvements.

§2. The rotation number

We consider the differential equation

$$y'' = Q(x)y \tag{2.1}$$

where $Q(x)$ is a real continuous function of period l. Later on we will take $Q(x)=q(x)-\lambda$ so that Q depends on the parameter λ. With this equation one can associate a flow on, a torus by identifying points $c\begin{pmatrix} y \\ y' \end{pmatrix}$ for $c \neq 0$ in the plane. Or introducing polar coordinates (Prüfer transformation, see [7])

$$y = r \sin \theta, \qquad y' = r \cos \theta \quad \text{for} \quad r > 0$$

one finds the differential equation

$$\frac{d\theta}{dx} = \cos^2 \theta - Q(x) \sin^2 \theta \tag{2.2}$$

which can be viewed as a vectorfield on the torus by identifying $\theta \bmod 2\pi$ and $x \bmod l$. For such a vectorfield the rotation number

$$\alpha = \lim_{x \to \infty} \frac{\theta(x)}{x}$$

exists and is independent of the choice of $\theta(0)$. Since $\arg(y - iy') = \theta - \pi/2$ $(\bmod 2\pi)$ it follows that for any non vanishing solution $y(x)$ of (2.1)

$$\alpha = \lim_{x \to \infty} \frac{\arg(y(x) - iy'(x))}{x}$$

exists and is independent of the particular solution. It also does not matter that the argument is defined only mod 2π since this constant drops out in the limit.

It is useful to express α in terms of the number $N(x)$ of zeroes of a nontrivial solution in the interval $(0, x)$. Since at zero ξ of $y(x)$ the angle θ is an integral multiple of π one sees from (2.2) that $d\theta/dx = 1 > 0$ at $x = \xi$, i.e. θ increases at such a value. From this fact it follows readily that

$$\theta(x) - \pi N(x)$$

is bounded for all x and hence

$$\alpha = \pi \lim_{x \to \infty} \frac{N(x)}{x}. \tag{2.3}$$

By Sturm's comparison theorem [6] one sees at once: If $Q(x) \leqq \tilde{Q}(x)$ are two periodic function and $\alpha, \tilde{\alpha}$ the corresponding rotation numbers then

$$\alpha \geqq \tilde{\alpha}, \tag{2.4}$$

for any choice of the corresponding nontrivial solutions $y, \tilde{y}$. This shows again the independence of α from the particular solution, if we take $\tilde{Q} = Q$. Moreover, by (2.3),

$$\alpha \geqq 0. \tag{2.5}$$

Finally we relate α to Floquet theory, in the case that the Floquet multipliers μ, μ^{-1} are not real, the stable case. In that case (2.1) admits a complex solution $w(x)$ satisfying

$$w(x+l) = \mu w(x), \tag{2.6}$$

or with $\mu = e^{i\beta}$, β being defined only mod 2π,

$$w(x) = e^{i\beta x/l} p(x)$$

where $p(x)$ has period l and does not vanish. Therefore $p(x)$ describes a closed curve with an integer winding number

$$\frac{1}{2\pi i} \int_0^l \frac{p'}{p}\, dx = j.$$

Replacing β by $\beta + 2\pi j$ we can achieve that this winding number is zero and this way define β completely and not only mod 2π. We can take $\beta \geqq 0$, otherwise we replace w by $\bar{w}$ and hence β by $-\beta$.

This solution can be written in the form

$$w(x) = e^{i(\beta x/l)} \rho(x) e^{is(x)} \tag{2.7}$$

where both $\rho(x) > 0$, $s(x)$ have period l. We show that

$$\beta = l\alpha. \tag{2.8}$$

For this purpose we consider the real solution

$$y = \operatorname{Re} w(x) = \rho(x) \cos\left(\frac{\beta x}{l} + s(x)\right)$$

and note that the number $N(x)$ of zeroes in $(0, x)$ satisfies

$$N(x)-\frac{\beta}{\pi}\frac{x}{l}=0(1)$$

so that (2.3) implies (2.8).

The Floquet solution (2.6) is determined only up to a complex factor $c\neq 0$. One usually normalized $|c|$ by requiring that the Wronskian

$$w\bar{w}'-\bar{w}w'=-2i.$$

One computes from (2.7) that this Wronskian has the value

$$-2i\rho^2\left(\frac{\beta}{l}+s'\right)$$

so that

$$\frac{\beta}{l}+s'=\rho^{-2}.$$

Inserting this into (2.7) we find the differential equation

$$\rho''=Q\rho+\rho^{-3}. \tag{2.9}$$

From the above relation we find

$$\alpha=\frac{\beta}{l}=\lim_{x\to\infty}\frac{1}{x}\int_0^x \rho^{-2}(t)\,dt.$$

On the other hand ρ being a positive function of period l we find

$$\alpha=\frac{1}{l}\int_0^l \rho^{-2}\,dx=\frac{1}{l}\int_0^l |w(x)|^{-2}\,dx \tag{2.10}$$

showing again $\alpha\geqq 0$, and even $\alpha>0$ in case of stability.

For an actual calculation of α (mod$(2\pi/l)$) we return to the Floquet matrix: Let $Y(x)$ be the two by two matrix satisfying

$$Y'=\begin{pmatrix}0 & 1\\ Q(x) & 0\end{pmatrix}Y, \qquad Y(0)=1. \tag{2.11}$$

Then the Floquet matrix M is given by $M = Y(l)$ and its eigenvalues μ by

$$\mu^2 - \Delta\mu + 1 = 0 \quad \text{where} \quad \Delta = \operatorname{tr} M.$$

Hence in the stable case, $\mu = e^{i\beta}$ we have

$$\Delta = \mu + \mu^{-1} = 2\cos\beta$$

or

$$\Delta = 2\cos l\alpha. \tag{2.12}$$

Now we take $Q(x) = q(x) - \lambda$, so that, for fixed $q(x)$, α becomes a function of λ. By (2.4) it is a monotone increasing which is, moreover continuous. This is a well known property of the rotation number, due to P. Bohl (see, e.g. [5]). For $\lambda < \min q(x)$ one has $Q(x) > 0$, and so any solution has at most one zero, i.e. $\alpha(\lambda) = 0$. On the other hand, again by Sturm comparison theorems $\alpha(\lambda) \to +\infty$ as $\lambda \to \infty$, so that α takes on all nonnegative values. In the stability intervals one has

$$\mu = e^{il\alpha}$$

for the Floquet multiplier. Because of the continuity and monotonicity one concludes that in the instability intervals α is a constant, an integer multiple of π/l.

If we define the clipped discriminant

$$\Delta_{cl}(\lambda) = \begin{cases} +2 & \text{if} \quad \Delta(\lambda) > 2 \\ \Delta(\lambda) & \text{if} \quad -2 \leqq \Delta(\lambda) \leqq +2 \\ -2 & \text{if} \quad \Delta(\lambda) < -2 \end{cases}$$

then we have from (2.12)

$$\Delta_{cl}(\lambda) = 2\cos l\alpha(\lambda).$$

Incidentally, although α is differentiable in the interior of the stability interval it does not even have a one-sided derivative at the boundary of an uncollapsed instability interval, since α behaves like $\sqrt{|\lambda - \lambda_0|}$ there.

§3. $\alpha(Q)$ as a functional

We saw that $\alpha(Q)$ is a monotone decreasing functional in the sense of (2.4). It is generally not differentiable but for a potential Q for which one has nonreal multipliers μ it has a functional derivative $\delta\alpha/\delta Q = -k(x)$, defined by

$$\frac{d}{d\varepsilon}\alpha(Q+\varepsilon\hat{q})\,|_{\varepsilon=0} = -\int_0^l k(x)\hat{q}(x)\,dx$$

for any continuous $\hat{q}(x)$ of period l. We insert the minus sign, since $k(x)$ will turn out to be positive. In particular, if $Q = q(x)-\lambda$ we set $\alpha(Q)=\alpha(\lambda, q)$ and get, with $\hat{q} = -1$,

$$\frac{\partial\alpha}{\partial\lambda} = \int_0^l k(x)\,dx. \tag{3.1}$$

Dubrovin [2] showed that

$$k(x) = \frac{\rho^2}{2l} = \frac{|w|^2}{2l} \tag{3.2}$$

where w is the Floquet solution (2.6) normalized by $w\bar{w}' - \bar{w}w' = -2i$. We will derive the relation (3.2) at the end of this section. Thus $k(x)$ is positive (if the Floquet multiplier is not real) which shows that $\alpha = \alpha(\lambda, q)$ satisfies the estimate

$$|\delta\alpha| < \sup_x |\delta q| \cdot \int_0^l k(x)\,dx = \sup_x |\delta q| \cdot \frac{\partial\alpha}{\partial\lambda}. \tag{3.3}$$

We also derive from (3.1) and (2.10)

$$\frac{d(\alpha^2)}{d\lambda} = 2\alpha\frac{\partial\alpha}{\partial\lambda} = \frac{1}{l^2}\int_0^l \rho^{-2}\,dx \int_0^l \rho^2\,dx \geqq 1$$

by Schwarz' inequality. Equality on the right occurs only if ρ, and hence q, is a constant. This proves a statement of the introduction.

Although the functional $\alpha = \alpha(Q)$ is not everywhere differentiable, the discriminant $\Delta = \operatorname{tr} Y(l)$ is.

By solving the differential equation

$$Z' = \begin{pmatrix} 0 & 1 \\ Q & 0 \end{pmatrix} Z + (\delta Q)EY; \qquad Z(0) = 0$$

for the first variation $Z = \delta Y$ of Y where $Y(x)$ is the fundamental solution of (2.11), we verify readily

$$\frac{\delta\Delta}{\delta Q} = \operatorname{tr}(Y(l)Y^{-1}(x)EY(x)), \qquad E = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{3.4}$$

In similar forms this formula can be found in [7], [9]. We rewrite it in several different forms.

If we write the fundamental matrix $Y(x)$ in the form

$$Y(x)=\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$$

then y_1, y_2 are linearly independent solutions of (2.1) normalized at $x=0$. In terms of these solutions (3.4) takes the form

$$\begin{gathered}\frac{\delta\Delta}{\delta Q}=Ay_1^2(x)+By_1(x)y_2(x)+Cy_2^2(x)\\ A=y_2(l), \qquad B=y_2'(l)-y_1(l), \qquad C=y_1'(l).\end{gathered} \tag{3.5}$$

The three products y_1^2, y_1y_2, y_2^2 are linearly independent functions since their Wronskian is

$$2(y_1y_2'-y_1'y_2)^3=2\neq 0.$$

In particular, we see from (3.5) that Δ has a critical point, i.e. $\delta\Delta/\delta Q=0$, if and only if $A=B=C=0$, i.e. if $Y(l)$ is a multiple of the identity. Since its determinant is 1 this means that $Y(l)=\pm 1$ which in turn means that all solutions of (2.1) satisfy

$$y(x+l)=\pm y(x), \tag{3.6}$$

or $y^2(x)$ has period l. We conclude: *The discriminant $\Delta(Q)$ has a critical point at Q if and only if $y^2(x)$ has period l for all solutions of* (2.1). In terms of the function $\Delta=\Delta(\lambda, q)$ the condition is equivalent for Δ^2-4 to have a double zero, as function of λ (see [7]).

The gradient $\delta\Delta/\delta q$ can be represented in another way as

$$\frac{\delta\Delta}{\delta Q}(\xi)=\frac{\phi(\xi+l)}{\phi'(\xi)} \tag{3.7}$$

where $\phi(x)$ is a nontrivial solution of (2.1) with $y(\xi)=0$, $y'(\xi)\neq 0$; of course, $\phi(x)$ depends also on ξ.

It suffices to prove this result for $\xi=0$, since the general result follows by translation. Setting $x=0$ in (3.5) we find

$$\frac{\delta\Delta}{\delta Q}(0)=y_2(l)$$

which proves (3.7) since $y_2(0)=0$, $y_2'(0)=1$. We conclude from (3.7) that $\delta\Delta/\delta Q$ has period l.

Finally, we express $\delta\Delta/\delta Q = K(x)$ in terms of the Floquet solution (2.7), assuming now that the Floquet multiplier of (2.1) is not real. Then on account of (3.5) $\delta\Delta/\delta Q$ can be expressed as a linear combination of the 3 products w^2, $w\bar{w}$, $\bar{w}^2$ and since it is of period l it must be a multiple of $w\bar{w}=\rho^2$. If we normalize w again by $w\bar{w}'-w'\bar{w}=-2i$ the constant is determined by

$$\frac{\delta\Delta}{\delta Q}=\rho^2 \sin l\alpha,$$

as we will show now. This will lead to a proof of (3.2) if we differentiate $\Delta = 2\cos l\alpha$ and use $\delta\alpha/\delta Q = -k$.

The determination of this constant could be done as follows: We use that the products $y_1 y_2$ of solutions of (2.1) as well as K satisfy the third order differential equation

$$K''' - 4QK' - 2Q'K = 0.$$

In the stable case K is a solution of period l of this equation, by which it is determined up to a constant. This equation possesses the integral

$$K(K''-2QK) - \tfrac{1}{2}K'^2$$

as one verifies immediately. Its value can be determined at $x=0$, since from (3.5) one finds $K(0)=y_2(l)$; $K'(0)=y_2'(l)-y_1(l)$, $K''(0)=2(y_2(l)Q(l)-y_1'(l))$ that the above integral has the value

$$\begin{aligned}
&= -2y_1'(l)y_2(l) - \tfrac{1}{2}(y_2'(l)-y_1(l))^2\\
&= -\tfrac{1}{2}(y_2'(l)+y_1(l))^2 + 2(y_1(l)y_2'(l)-y_1'(l)y_1(l))\\
&= -\tfrac{1}{2}(\operatorname{tr} Y(l))^2 + 2\\
&= \tfrac{1}{2}(4-\Delta^2) = -\tfrac{1}{2}(\mu-\mu^{-1})^2 = 2\sin^2 l\alpha.
\end{aligned}$$

On the other hand from the differential equation (2.9) we obtain for $R=\rho^2$

$$R(R''-2QR) - \tfrac{1}{2}R'^2 = 2$$

so that indeed

$$K = \pm \rho^2 \sin l\alpha.$$

To show that the sign is + we can use the monotonicity of α.

§4. The second variation of Δ

The collapse of an instability interval at a point λ^* corresponds to a double zero of Δ^2-4, which by the previous section is tantamount to the vanishing of the first variation $\delta\Delta/\delta Q$ at $Q=q(x)-\lambda^*$. At such a point we will need the second variation of Δ, which we now compute.

Therefore we assume that for the considered function $Q(x)$ all solutions of (2.1) satisfy (3.6). Replacing $Q(x)$ by $Q(x)+\varepsilon\hat{q}(x)$ we determine the corresponding fundamental matrix

$$Y(x)+\varepsilon Y_1+\varepsilon^2 Y_2+\cdots.$$

It satisfies the equations

$$Y'=FY$$

$$Y_1'=FY_1+\hat{q}EY$$

$$Y_2'=FY_2+\hat{q}EY_1$$

where

$$F=\begin{pmatrix}0 & 1\\ Q & 0\end{pmatrix};\qquad E=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix};\qquad Y_1(0)=Y_2(0)=0.$$

With

$$S(x)=Y^{-1}(x)EY(x)$$

one solves the equations in the form

$$Y_1(x)=Y(x)\int_0^x S(t)\hat{q}(t)\,dt$$

$$Y_2(x)=Y(x)\int_0^x\int_0^t S(t)S(s)\hat{q}(t)\hat{q}(s)\,dx\,dt.$$

From these relations one computes readily

$$\frac{1}{2}\frac{d^2\Delta}{d\varepsilon^2}\bigg|_{\varepsilon=0}=\operatorname{tr} Y_2(l)=\pm\frac{1}{2}\int_0^l\int_0^l \operatorname{tr}(S(t)S(s))\hat{q}(t)\hat{q}(s)\,ds\,dt$$

where the sign corresponds to that of $Y(l)=\pm 1$. Further we find

$$\operatorname{tr}(S(t)S(s)) = -(y_1(t)y_2(s)-y_1(s)y_2(t))^2.$$

This expression is the same for any basis of solutions $\phi_1(x)$, $\phi_2(x)$ with Wronskian $+1$, so we have

$$\begin{aligned}\pm\frac{1}{2}\frac{d^2\Delta}{d\varepsilon^2}\bigg|_{\varepsilon=0} &= -\frac{1}{2}\int_0^l\int_0^l (\phi_1(t)\phi_2(s)-\phi_1(s)\phi_2(t))^2\hat{q}(t)\hat{q}(s)\,ds\,dt\\ &= -\int_0^l \phi_1^2\hat{q}\,dx\int_0^l \phi_2^2\hat{q}\,dx + \left(\int_0^l \phi_1\phi_2\hat{q}\,dx\right)^2. \end{aligned} \tag{4.1}$$

Finally, if we introduce the 3 functions

$$\Phi_1=\tfrac{1}{2}(\phi_1^2+\phi_2^2), \qquad \Phi_2=\tfrac{1}{2}(\phi_1^2-\phi_2^2), \qquad \Phi_3=\phi_1\phi_2 \tag{4.2}$$

of period l we get

$$\pm\frac{1}{2}\frac{d^2\Delta}{d\varepsilon^2}\bigg|_{\varepsilon=0} = -\left(\int_0^l \Phi_1\hat{q}\,dx\right)^2+\left(\int_0^l \Phi_2\hat{q}\,dx\right)^2+\left(\int_0^l \Phi_3\hat{q}\,ds\right)^2. \tag{4.3}$$

We see from this formula that the second variation is a quadratic form of rank 3 and its null space N is the orthogonal complement of the space of Φ_1, Φ_2, Φ_3. Moreover, the quadratic form (4.3) has a 1-dimensional subspace on which it is negative definite and a two-dimensional subspace on which it is positive; it is of type (2, 1).

We obtain for $Q=q(x)-\lambda$, $\hat{q}=-1$ the second λ-derivative, and get from (4.1)

$$\pm\frac{1}{2}\frac{d^2\Delta}{d\lambda^2} = -\int_0^l \phi_1^2\,dx\int_0^l \phi_2^2\,dx+\left(\int_0^l \phi_1\phi_2\,dx\right)^2<0.$$

The negativity follows from Schwarz' inequality. Hence, at a point λ^* at which $\Delta(\lambda^*)=+2$, $\Delta'(\lambda^*)=0$ one has always a nondegenerate maximum, and similarly for $\Delta(\lambda^*)=-2$, $\Delta'(\lambda^*)=0$ a nondegenerate minimum.

PROPOSITION 1. *Consider the potential $q(x)+\varepsilon q_1(x)$ of period l where for λ^**

$$\Delta(\lambda^*, q)=\pm 2, \qquad \Delta'(\lambda^*, q)=0.$$

Unless q_1 lies in the space span $\{1, N\}$ (where N is the orthogonal complement of

Φ_1, Φ_2, Φ_3) *then for* $\varepsilon \neq 0$ *sufficiently small at some point* $\lambda^*(\varepsilon)$ *near* λ^* *we have*

$$|\Delta(\lambda^*(\varepsilon), q+\varepsilon q_1)| > 2.$$

This proposition expresses that generically a collapsed instability interval will "open up."

Proof. We give the proof in the case $\Delta(\lambda^*, q) = 2$. With some constant κ we set

$$Q(x, \varepsilon) = q(x) - \lambda^* + \varepsilon \hat{q}(x), \qquad \hat{q} = q_1(x) - \kappa.$$

If we determine κ so that

$$0 = \int_0^l \Phi_1 \hat{q}\, dx = \int_0^l \Phi_1 q_1\, dx - \kappa \int_0^l \Phi_1\, dx$$

– this is possible since $\Phi_1 > 0$ – then the right hand side of (4.3) is positive, unless $\hat{q}$ is orthogonal to Φ_2, Φ_3. By construction it is orthogonal to Φ_1, i.e. $\hat{q} = q_1 - \kappa \in N$, which would contradict our assumption. Hence for $\lambda^*(\varepsilon) = \lambda^* + \kappa\varepsilon$ we have

$$Q(x, \varepsilon) = q(x) - \lambda^*(\varepsilon) + \varepsilon q_1(x)$$

and from the Taylor expansion of $\Delta(Q)$

$$\Delta(\lambda^*(\varepsilon), q+\varepsilon q_1) > 2$$

if $\varepsilon \neq 0$ is sufficiently close to zero, as was to be proven. The same argument holds clearly for the case $\Delta(\lambda^*, q) = -2$.

In particular, we can always choose q_1 so, that the local maximum of $\Delta(\lambda, q+\varepsilon q_1)$ near λ is larger than 2.

This can be made more explicit if one makes use of the fact that the basis ϕ_1, ϕ_2 can be replaced by $(a\phi_1 + b\phi_2, c\phi_1 + d\phi_2)$ where $ad - bc = 1$ by which Φ_1, Φ_2, Φ_3 are replaced by

$$\sum_{k=1}^{3} c_{jk} \Phi_k$$

where the matrix $C=(c_{jk})$ is given by

$$C=\begin{pmatrix} \frac{1}{2}(a^2+b^2+c^2+d^2) & \frac{1}{2}(a^2-b^2+c^2-d^2) & ab+cd \\ \frac{1}{2}(a^2+b^2-c^2-d^2) & \frac{1}{2}(a^2-b^2-c^2+d^2) & ab-cd \\ ac+bd & ac-bd & ad+bc \end{pmatrix}.$$

PROPOSITION 2. *It is always possible to find a basis* ϕ_1, ϕ_2 *of* (1.2) *with Wronskian* $=1$ *such that the three functions* Φ_1, Φ_2, Φ_3 *of* (4.2) *satisfy*

$$\int_0^l \Phi_j \Phi_k \, dx = 0 \quad \textit{for} \quad j \neq k.$$

In other words, for this basis we have

$$\int_0^l \phi_1^4 \, dx = \int_0^l \phi_2^4 \, dx; \qquad \int_0^l \phi_1^3 \phi_2 \, dx = \int_0^l \phi_1 \phi_2^3 \, dx = 0.$$

We indicate the proof. The main observation is that the mapping $x \to Cx$; $x \in R^3$ preserves the quadratic form $x_1^2 - x_2^2 - x_3^2$, so that $C \in SO(1,2)$. In fact the mapping

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to C$$

defines a homomorphism of $Sl(2)$ into $SO(1,2)$. This mapping is not onto, but only onto the component of $SO(1,2)$ containing the identity, which is characterized by $c_{11} > 0$.

Now we apply the theorem of linear algebra that the two quadratic forms

$$\int_0^l \left(\sum_{j=1}^3 x_j \Phi_j \right)^2 dx \quad \text{and} \quad x_1^2 - x_2^2 - x_3^2$$

can be simultaneously diagonalized, i.e. brought into the form

$$\kappa_1 x_1^2 + \kappa_2 x_2^2 + \kappa_3 x_3^2, \qquad x_1^2 - x_2^2 - x_3^2$$

by a linear transformation. Such a transformation belongs to $SO(1,2)$ since the second form is preserved. Since we can still replace x_j by $\pm x_j$ we can assume that the linear transformation belongs to the component of $SO(1,2)$ containing the identity. Therefore this transformation can be realized by a change of basis $(\phi_1, \phi_2) \to (a\phi_1 + b\phi_2, c\phi_1 + d\phi_2)$.

If we write $\hat{q}$ in terms of such a basis

$$\hat{q} = \sum_{j=1}^{3} \xi_j \Phi_j + n, \qquad n \in N$$

then (4.3) takes the form

$$\pm \frac{1}{2} \frac{d^2 \Delta}{d\varepsilon^2}\bigg|_{\varepsilon=0} = -A_1 \xi_1^2 + A_2 \xi_2^2 + A_3 \xi_3^2 \tag{4.4}$$

where

$$A_j = \int_0^l \Phi_j^2 \, dx.$$

We mention that the above basis ϕ_1, ϕ_2 is unique up to the obvious changes

$$(\phi_1, \phi_2) \to \pm(\phi_1, \phi_2), \qquad \pm(\phi_2, -\phi_1)$$

if $A_2 \neq A_3$, i.e. if

$$\int_0^l (\phi_1^4 + \phi_2^4) \, dx \neq 6 \int_0^l \phi_1^2 \phi_2^2 \, dx.$$

If this condition is violated the basis is fixed only up to a rotation.

The representation (4.4) makes it clear that for $\hat{q} = \xi_2 \Phi_2 + \xi_3 \Phi_3 \neq 0$ the maximum of $\Delta(\lambda, q + \varepsilon\hat{q})$ near λ^* is increased.

§5. Construction of the example

We normalize the basic period to $l_0 = \pi$ and consider an arbitrary continuous function $q_0(x)$ of period l_0. In any neighborhood of q_0 we will construct a limit periodic function $q(x)$ with a Fourier series

$$q(x) = a_0 + \sum_{j,s=1}^{\infty} (a_{js} \cos s2^{-j}x + b_{js} \sin s2^{-j}x)$$

for which the rotation number $\alpha(\lambda, q) = \alpha(\lambda)$ takes on every rational number of the form $s \cdot 2^{-t}$, $s \geqq 0$, $t \geqq 0$ on an interval of positive length. More precisely, we will prove the

THEOREM. *Given $\eta > 0$; $q_0 \in C(R)$ of period π there exists a limit periodic continuous function q with basic frequencies 2^{-j} $(j = 0, 1 \cdots)$ in $\|q - q_0\| < \eta$ for which* (i) *the rotation number $\alpha(\lambda) = \alpha(\lambda, q)$ exists and is continuous,* (ii) *$\alpha(\lambda)$ takes on each rational numbers of the form $s \cdot 2^{-t}$, $s, t \geqq 0$ on an interval of positive length and* (iii) *the union*

$$E = \bigcup_{s,t \geqq 0} \operatorname{int} \alpha^{-1}(s \cdot 2^{-t})$$

of these intervals is dense on R. E is contained in the complement of the spectrum $\sigma(L)$ of $L = -D^2 + q$; hence $\sigma(L)$ is nowhere dense.

Remark. According to some recent result by R. Johnson the rotation number exists and is continuous for every almost periodic continuous function $q(x)$, so that (i) follows. For completeness we give a proof of (i) on our simpler situation.

Proof. (a) We write the rational numbers of the form $s \cdot 2^{-t} s \geqq 1$, $t \geqq 0$ as a sequence r_j. We choose the labelling in the following way: $r_1 = \frac{1}{2}$, $r_2 = 1$, then $r_3, r_4, \ldots, r_8$ denote all rationals $\leqq 2$ with denominator $2^2 = 4$ not listed before, and more general, $r_{\nu(N-1)+1} < \cdots < r_{\nu(N)} = N$ denote all rationals of denominator 2^N in $0 < r \leqq N$ not listed before. Thus $r_1, r_2, \ldots, r_{\nu(N)}$ is a list of all positive rationals with denominators $1, 2, 2^2, \ldots, 2^N$ in the interval $0 < r \leqq N$. One verifies that $\nu(N) = N \cdot 2^N$.

We set also $l_j = \pi 2^N$ for $\nu(N-1) + 1 \leqq j \leqq \nu(N)$, $N \geqq 0$ so that

$$r_j = \frac{s_j}{l_j} \pi = \frac{s_j}{2^N} \tag{5.1}$$

with some integer s_j.

We will construct $q(x)$ as the limit of continuous functions $q_\nu(x)$, $\nu \geqq 0$ of period l_ν for which the rotation number $\alpha_\nu(\lambda) = \alpha(\lambda, q_\nu)$ takes on the value r_ν on an interval of positive length.

In the following we will constantly use the fact mentioned already above, that for any periodic function p the rotation number $\alpha(\lambda, p)$ depends continuously on λ, and also on p in the class of continuous functions of fixed period. Moreover, the spectrum of $-D^2 + p$ agrees with the set of λ where $\alpha(\lambda, p)$ is strictly increasing. These facts, not obvious, for a limit periodic function will be forced for the function q by the construction of q_ν for which $q = \lim_{\nu \to \infty} q_\nu$.

(b) Construction of q_ν: We assume that $q_j(x)$ of period l_j $j=1,2,\ldots,\nu$ are constructed already so that $\alpha_j(\lambda)=\alpha(\lambda, q_j)$ takes on the values $r_1, r_2, \ldots, r_j$ on intervals of positive length. Moreover, $\varepsilon_\nu>0$ is any positive number which may depend on $q_0, q_1, \ldots, q_\nu$, and $\eta>0$; then we construct a continuous function $q_{\nu+1}(x)$ of period $l_{\nu+1}$ such that

$$\|q_{\nu+1}-q_\nu\|<\varepsilon_\nu \tag{5.2}$$

and such that $\alpha_{\nu+1}(\lambda)=\alpha(\lambda, q_{\nu+1})$ takes on $r_1, r_2, \ldots, r_{\nu+1}$ in intervals of positive length.

Since $\alpha_\nu(\lambda)$ is a continuous monotone increasing function taking on all positive values there exists a $\lambda^*=\lambda_\nu^*$ such that

$$\alpha_\nu(\lambda^*)=r_{\nu+1}. \tag{5.3}$$

If α_ν takes this value on an interval of positive length, then we set $q_{\nu+1}=q_\nu$. If, however, λ^* is uniquely determined by (5.3) then the discriminant $\Delta(Q)$ (with respect to the period $l_{\nu+1}$) has a critical point at $Q=q_\nu-\lambda^*$. In particular, this will be the case if $l_{\nu+1}>l_\nu$. According to §3 all solutions of $y''=(q_\nu(x)-\lambda^*)y$ satisfy (3.6) with $l=l_{\nu+1}$. Therefore, if ϕ_1, ϕ_2 is a basis with Wronskian 1, then the functions Φ_1, Φ_2, Φ_3 defined by (4.2) have period $l_{\nu+1}$. If Φ_1, Φ_2 are normalized so that the Φ_j are orthogonal then we set

$$q_{\nu+1}=q_\nu+\varepsilon\Phi_3. \tag{5.4}$$

Then $\Delta(\lambda, q_{\nu+1})$ has the value $r_{\nu+1}$ are a noncritical value, if $\varepsilon>0$ is near zero. It also has the period $l_{\nu+1}$ since l_ν divides $l_{\nu+1}$. Moreover, if ε is chosen small enough the noncritical values $r_1, r_2, \ldots, r_\nu$ are also noncritical for $q_{\nu+1}$ and (5.2) will be satisfied. This completes the construction of the sequence of periodic potentials q_ν of period l_ν.

(c) Smallness condition: Now we will select the numbers ε_ν recursively so that

(i) q_ν converges uniformly to a limit periodic function q in $\|q-q_0\|<\eta$.

(ii) $\alpha_\nu(\lambda)=\alpha(\lambda, q_\nu)$ converges uniformly on compact intervals to a continuous function $\alpha(\lambda)$, having the properties given in the theorem.

(iii) The rotation number $\alpha(\lambda, q)$ exists and agrees with $\alpha(\lambda)$.

(iv) The interior of the intervals of constancy of $\alpha(\lambda)$ form a dense set E on the real axis.

(v) The spectrum of $-D^2+q$ is contained in the complement of E, hence is nowhere dense.

In the following we will prove these five claims which imply the above theorem.

In order to define the ε_ν we introduce the resolvents

$$R_\nu(\lambda)=(-D^2+q_\nu-\lambda)^{-1}; \qquad R(\lambda)=(-D^2+q-\lambda)^{-1}. \tag{5.5}$$

By construction the interval $I_{j,\nu}=\alpha_\nu^{-1}(r_j)$ is for $j=0,1,2,\ldots,\nu$ an interval of positive length whose interior belongs to the resolvent set of $-D^2+q_\nu$. For $j\geqq 1$ these intervals are bounded and for $j=0$ it extends to $-\infty$. Let $\tau_\nu>0$ be a monotone decreasing sequence tending to 0 and pick for $j\geqq 1$ a closed subinterval $I'_{j,\nu}\subset \operatorname{int} I_{j,\nu}$, with the same center as $I_{j,\nu}$ and so that the measure

$$m(I_{j,\nu}-I'_{j,\nu})<2\tau_\nu. \tag{5.6}$$

For $j=0$ we take $I'_{0,\nu}$ as half infinite interval with

$$m(I_{0,\nu}-I'_{0,\nu})<\tau_\nu.$$

Then $R_\nu(\lambda)$ is bounded in $I'_{j,\nu}$ and we set

$$B_\nu=\max_{0\leqq j\leqq \nu}\sup_{\lambda\in I'_{j,\nu}}|R_\nu(\lambda)|.$$

Let $\eta>0$ be the number given in the theorem and δ_ν a positive sequence satisfying

$$\sum_{\nu=1}^{\infty}\delta_\nu=1.$$

Then we set $\varepsilon_0=\eta\delta_1$ and for $\nu\geqq 1$

$$\varepsilon_\nu=\min\left\{\eta\delta_{\nu+1},\frac{\delta_\nu}{2B_1},\frac{\delta_{\nu-1}}{2B_2},\ldots,\frac{\delta_1}{2B_\nu}\right\}. \tag{5.7}$$

Then from (5.2), (5.7) we have, in particular,

$$\|q_{\nu+1}-q_\nu\|<\eta\delta_{\nu+1}.$$

Therefore the sequence q_ν converges uniformly to a limit periodic function $q(x)$ with

$$\|q-q_0\|<\eta\sum_{\nu=1}^{\infty}\delta_\nu=\eta$$

verifying (i) of our claim.

From the other restrictions of (5.7) follows that for any fixed ν and $p = 0, 1, \ldots$

$$\|q_{\nu+p+1} - q_{\nu+p}\| \leqq \frac{\delta_{p+1}}{2B_\nu}$$

and hence

$$\|q_{\nu+p} - q_\nu\| \leqq \frac{1}{2B_\nu}(\delta_1 + \delta_2 + \cdots) = \frac{1}{2B_\nu}. \tag{5.8}$$

From this we conclude that $I'_{j,\nu}$ belongs to the resolvent set of $-D^2 + q_{\nu+p}$ for any $p \geqq 0$. Indeed, from the resolvent identity

$$R_{\nu+p}(\lambda) - R_\nu(\lambda) = R_{\nu+p}(q_\nu - q_{\nu+p})R_\nu$$

we obtain the estimate

$$|R_{\nu+p}(\lambda)| \leqq |R_\nu(\lambda)| \, (1 - \|q_{\nu+p} - q_\nu\| \, |R_\nu(\lambda)|)^{-1}.$$

For $\lambda \in I'_{j,\nu}$ we have by (5.8)

$$\|q_{\nu+p} - q_\nu\| \, |R_\nu(\lambda)| \leqq \frac{1}{2B_\nu} |R_\nu(\lambda)| \leqq \tfrac{1}{2}$$

and hence

$$|R_{\nu+p}(\lambda)| \leqq 2B_\nu < \infty$$

which proves that $I'_{j,\nu}$ belongs to the resolvent set of $-D^2 + q_{\nu+p}$.

The same holds true for any potential

$$q^t = tq_\nu + (1-t)q_{\nu+p}, \qquad 0 \leqq t \leqq 1,$$

and hence the value of its rotation number is rational for $\lambda \in I'_{j,\nu}$. Because of the continuous dependence the value of $\alpha(\lambda, q^t)$ is independent of t, and hence

$$\alpha_{\nu+p}(\lambda) = \alpha_\nu(\lambda) = r_j \quad \text{for} \quad \lambda \in I'_{j\nu} \text{ and } j \leqq \nu \leqq \nu + p.$$

Therefore

$$I'_{j,\nu} \subset I_{j,\nu+p} \quad \text{for} \quad j \leqq \nu \leqq \nu + p. \tag{5.9}$$

In particular, if λ_j is the midpoint of $I_{j,j}$ we have $\lambda_j \in I'_{j,j}$ and

$$\alpha_\nu(\lambda_j) = \alpha_j(\lambda_j) = r_j \quad \text{for} \quad \nu \geqq j. \tag{5.10}$$

This proves that for $\lambda = \lambda_j$ the sequence of monotone functions $\alpha_\nu(\lambda)$ converges for $\nu \to \infty$. We claim that it converges for all real λ. Indeed, given any λ, $\varepsilon > 0$ we can find λ_j, λ_k so that

$$\lambda_j < \lambda < \lambda_k, \qquad r_j = \alpha_j(\lambda_j) \leqq \alpha_\nu(\lambda) \leqq \alpha_\nu(\lambda_k) = r_k < r_j + \varepsilon$$

for all $\nu \geqq j, k$. This is possible since all numbers r_j – which form a dense set – are taken on by $\alpha_\nu(\lambda)$. Hence for any $p > 0$

$$|\alpha_\nu(\lambda) - \alpha_{\nu+p}(\lambda')| < 2\varepsilon \quad \text{if} \quad \lambda_j < \lambda, \ \lambda' < \lambda_k.$$

In particular, for $\lambda' = \lambda$ we see that $\alpha_\nu(\lambda) \to \alpha(\lambda)$ and

$$|\alpha(\lambda) - \alpha(\lambda')| < 2\varepsilon \quad \text{for} \quad \lambda_j < \lambda, \ \lambda' < \lambda_k.$$

This shows that $\alpha_\nu(\lambda)$ converges to a continuous monotone increasing function $\alpha(\lambda)$, taking on all values $\geqq 0$. Moreover, by (5.10),

$$\alpha(\lambda_j) = r_j.$$

Similarly, we see that the convergence $\alpha_\nu \to \alpha$ is uniform on compact sets. Indeed, otherwise there would exist a sequence $\lambda^{(n)} \to \lambda$ for which $|\alpha_{\nu_n}(\lambda^{(n)}) - \alpha(\lambda)| > \delta > 0$. But if $\varepsilon = \frac{1}{2}\delta$ and if we construct λ_j, λ_k as before such that $\lambda_j < \lambda < \lambda_k$, $0 < r_k < r_j + \varepsilon$ then for any sufficiently large n we have $\lambda_j < \lambda^{(n)} < \lambda_k$ and hence

$$|\alpha_{\nu_n}(\lambda^n) - \alpha(\lambda)| < 2\varepsilon = \delta,$$

which gives the desired contradiction.

Next we show that $\alpha(\lambda)$ is indeed the rotation number for $y'' = (q(x) - \lambda)y$: Let

$$\alpha^* = \pi \varlimsup_{x \to \infty} \frac{N(x, \lambda)}{x}, \qquad \alpha_* = \pi \varliminf_{x \to \infty} \frac{N(x, \lambda)}{x}$$

for any nontrivial solution $y(x)$ of our differential equation, say one with $y(0) = 0$, $y'(0) = 1$. Set

$$\eta_\nu = \|q - q_\nu\|$$

and

$$\underline{q_\nu} = q_\nu - \eta_\nu, \qquad \overline{q_\nu} = q_\nu + \eta_\nu$$

so that

$$\underline{q_\nu}(x) \leqq q(x) \leqq \overline{q_\nu}(x)$$

and $\underline{q_\nu}$, $\overline{q_\nu}$ are periodic functions of period l_ν. By the comparison argument we have for the number of zeros $N(x, \lambda, p)$ of a nontrivial solution of $y'' = (p-\lambda)y$ with $y(0) = 0$

$$N(x, \lambda, \underline{q_\nu}) \geqq N(x, \lambda, q) \geqq N(x, \lambda, \overline{q_\nu})$$

and hence

$$\begin{aligned}\alpha_\nu(\lambda+\eta_\nu) = \alpha(\lambda+\eta_\nu, q_\nu) = \alpha(\lambda, \underline{q_\nu}) \geqq \alpha^*(\lambda) \geqq \alpha_*(\lambda) \geqq \alpha(\lambda, \overline{q_\nu}) &= \alpha(\lambda-\eta_\nu, q_\nu) \\ &= \alpha_\nu(\lambda-\eta_\nu).\end{aligned}$$

Because of the uniform convergence on compact sets of $\alpha_\nu \to \alpha$ we conclude that

$$\alpha(\lambda) \geqq \alpha^*(\lambda) \geqq \alpha_*(\lambda) \geqq \alpha(\lambda),$$

i.e. we have equality in all places. This proves (iii) of the claim.

Next we show that the set

$$E = \bigcup_{j=0}^{\infty} \operatorname{int} \alpha^{-1}(r_j) \tag{5.11}$$

is dense on the real axis. For this purpose we make use of the special labelling of the rational numbers r_j, which was chosen so that $r_1, r_2, \ldots, r_{\nu(N)}$ is a complete list of all rationals with denominators 2^N in the half open interval $(0, N]$. We set

$$Q_N(x) = q_{\nu(N)}(x), \qquad A_N(\lambda) = \alpha(\lambda, Q_N)$$

and analogue to $I_{j,\nu}$ we define the intervals

$$J_{j,N} = A_N^{-1}(r_j).$$

We pick a closed subinterval $J'_{j,N} \subset \operatorname{int} J_{j,N}$, $j \geqq 1$ with the same center as $J_{j,N}$ and so that the measure

$$m(J_{j,N}-J'_{j,N})<2t_N \tag{5.12}$$

where

$$t_N=\tau_{\nu(N)}.$$

The half interval $J'_{0,N}$ is defined similarly as $I'_{0,\nu}$. We show more than (5.11) if we establish that the set

$$E'=\bigcup_{N=1}^{\infty}\bigcup_{j=0}^{\nu(N)}J'_{j,N}\subset E$$

is dense.

For this purpose it is obviously sufficient to show that for any open interval $\Delta\subset R^1$

$$\Delta\cap\bigcup_{j=0}^{\nu(N)}J'_{j,N}\neq\varnothing \tag{5.13}$$

if N is large enough.

We chose a compact interval $[\Lambda_1,\Lambda_2]\supset\Delta$ with $\Lambda_2>0$, $\Lambda_1<\min_x(q_0(x)-\eta)$ so that $A_N(\lambda)=0$ for $\lambda\leqq\Lambda_1$. Moreover, we pick N so large that

$$A_N(\lambda)\geqq r_{\nu(N)}=N \quad\text{for}\quad \lambda\geqq\Lambda_2. \tag{5.14}$$

This is possible, since by (2.2), for $\lambda>0$

$$\frac{d\theta}{dx}\leqq 1+\|Q_N\|+\lambda<1+\|q_0\|+\eta+\lambda$$

hence

$$A_N(\lambda)\leqq 1+\|q_0\|+\eta+\lambda$$

and (5.14) follows if we choose

$$N\geqq 1+\|q_0\|+\eta+\Lambda_2.$$

By construction A_N takes on all rationals $r_1, r_2, \ldots, r_{\nu(N)}$ on intervals of positive length, i.e. none of those intervals in $[\Lambda_1,\Lambda_2]$ collapse to a point.

We estimate the distance between two neighboring instability intervals

$$[\xi_N, \eta_N] = J_N = A_N^{-1}(r), \qquad [\xi_N^+, \eta_N^+] = J_N^+ = A_N^{-1}(r^+) \tag{5.15}$$

where $r, r^+ = r + 2^{-N}$ are two neighboring numbers of the set $r_0, r_1, \ldots, r_{\nu(N)}$, so that

$$\xi_N < \eta_N < \xi_N^+ < \eta_N^+.$$

Then the distance between J_N and J_N^+ can be estimated by

$$0 < \xi_N^+ - \eta_N < 3N2^{-N}. \tag{5.16}$$

For the proof we recall that $A_N(\lambda)$ is strictly monotone increasing in $\eta_N < \lambda < \xi_N^+$ and by the inequality below (3.3) satisfies there the inequality

$$\frac{d}{d\lambda}(A_N^2) \geqq 1.$$

Hence, with $r = s2^{-N}$,

$$\xi_N^+ - \eta_N \leqq \int \frac{d}{d\lambda}(A_N^2)\, d\lambda = (r^+)^2 - r^2 = \frac{2s+1}{4^N} \leqq 3N2^{-N}$$

since $1 \leqq s \leqq N \cdot 2^N$, which proves our estimate (5.16). This shows that the gaps between adjacent instability intervals tends to zero. Because of the construction of the $J'_{j,N}$ the distance of these intervals intersecting $[\Lambda_1, \Lambda_2]$ will be at most $2t_N = 2\tau_{\nu(N)}$ bigger, i.e.

$$\leqq 3N \cdot 2^{-N} + 2t_N.$$

Since this sequence tends to zero this number will be smaller than the length of Δ if N is large enough. This proves (5.13), and hence the claim (iv).

Finally we observe that by construction the intervals $J'_{j,N}$, $J = 0, 1, \ldots, \nu(N)$ belong to the resolvent set of $L = -D^2 + q$. Therefore, since E' is dense, the spectrum of L is nowhere dense, and all our claims of (c) are proven.

The theorem at the beginning of this section is clearly a consequence of these claims.

Remark 1. We showed that the spectrum of $L = -D^2 + q$ is contained in the complement of the set E, defined by (5.11). Actually this spectrum agrees with the complement of E, as we want to verify now.

For this purpose we first show that the intervals

$$J_N = A_N^{-1}(r) = [\xi_N, \eta_N]$$

tend for $N \to \infty$ to the interval

$$\alpha^{-1}(r) = [\xi, \eta]$$

for any $r = r_j$.

By our construction the interval J_N' belongs to the resolvent set of $-D^2 + q_M$ for every $M \geqq N$. This implies

$$A_M^{-1}(r) = J_M \supset J_N' = [\xi_N', \eta_N'] \quad \text{for} \quad M \geqq N. \tag{5.17}$$

On the other hand, the right endpoint η_M of J_M stays below the interval $(J_N^+)'$ contained in

$$J_N^+ = A_N^{-1}(r^+), \qquad r^+ = r + 2^{-N}.$$

Using our estimate (5.16) for the gap between J_N and J_N^+ we find

$$\eta_M \leqq \xi_N^+ + t_N \leqq \eta_N + 3N2^{-N} + t_N$$

and because of (5.17)

$$\eta_M \geqq \eta_N' > \eta_N - t_N.$$

for all $M \geqq N$. These relations show that η_N is a Cauchy sequence, and let $\eta^* = \lim_{N\to\infty} \eta_N$. Similarly $\xi_N \to \xi^*$.

From the convergence of $A_N(\lambda)$ to $\alpha(\lambda)$ it follows that

$$[\xi^*, \eta^*] \subset [\xi, \eta] = \alpha^{-1}(r),$$

hence $\eta^* \leqq \eta$. If, however, $\eta^* < \eta$ then also

$$\xi_N^+ \leqq \eta_N + 3N2^{-N} < \eta$$

for large N which implies

$$r < r^{+} = A_N(\xi_N^{+}) \leqq A_N(\eta).$$

This is a contradiction since the right hand side tends to $\alpha(\eta) = r$. Hence $\eta^* = \eta$ and similarly one proves $\xi^* = \xi$, i.e.

$$[\xi^*, \eta^*] = \alpha^{-1}(r).$$

Finally we show that any point $\zeta \notin E$ belongs to the spectrum of $L = -D^2 + q$. Since E is dense there exists a sequence $\zeta_\nu \in E$ converging to E. Moreover, we may take ζ_ν as the left endpoints ξ_{N_ν} of J_{N_ν} which belong to the continuous spectrum of $L_{N_\nu} = -D^2 + q_{N_\nu}$.

It is a standard result that the cluster spectrum of L_N converges to that of L since

$$|L - L_N| = \|q - q_N\| \to 0.$$

Hence ζ belongs to the spectrum $\sigma(L)$ of L, hence $E^c \subset \sigma(L)$. Since we know already that $\sigma(L) \subset E^c$ these two sets agree.

Remark 2. The spectrum $\sigma(L)$ is a Cantor set, in particular a nowhere dense, perfect set. It remains to show that every point $\lambda^* \in \sigma(L)$ is the cluster point of $\sigma(L)$. For this purpose we set $\alpha(\lambda^*) = a$ and choose a sequence $a_\nu \to a$, where a_ν are different irrational numbers. Then the $\alpha^{-1}(a_\nu) = \lambda_\nu$ belong to $\sigma(L)$ and, because of the continuity of $\alpha(\lambda)$, λ^* is the only cluster-point of $\{\lambda_\nu\}$. This proves our claim, since $\lambda_\nu \in E^c = \sigma(\lambda)$.

Remark 3. By the same argument one can construct such limit periodic potentials which are real analytic and not only continuous. One would pick q_0 real analytic and replace the norm $\|\ \|$ by

$$\|q\| = \sup_{|\mathrm{Im}\, x| < \rho} |q(x)|$$

for some fixed $\rho > 0$. The above argument goes through and yields a real analytic potential $q(x)$ with the above properties.

Remark 4. We mention without proof that the functional $\alpha = \alpha(q(x) - \lambda)$ for fixed period l is Hölder continuous in the sense that

$$|\alpha(q_1 - \lambda) - \alpha(q_2 - \lambda)| \leqq c\, \|q_1 - q_2\|^{1/2} \quad \text{if} \quad \|q_1 - q_2\| < 1$$

where $c = c(l, M)$ depends on l and a bound for

$$\|q_1\|, \|q_2\| \leqq M.$$

However, c grows exponentially with l, like $\exp 3lM$, and one can not hope to have Hölder continuity for limit periodic potentials.

Finally we remark that the existence and continuity of $\alpha(\lambda, q)$ can actually be established for any almost periodic function $q(x)$, as was shown by R. Johnson. Moreover, one can relate this rotation number to the spectral resolution. In particular, it is a constant in any interval contained in the resolvent set of $-D^2+q$. In our case $2\alpha(\lambda, q)$ takes the values

$$\frac{s}{2^j}$$

in the resolvent set which are precisely the nonnegative number of the frequency mod 1. It is interesting that this phenomenon generalizes to quasi-periodic potentials, i.e. $2\alpha(\lambda, q)$ takes on values of the frequency module when λ is in the resolvent set. These and other properties of the rotation number will be published elsewhere.

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