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Spectra of manifolds with small handles

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To H. E. RAUCH, in memoriam

In this paper we consider a compact connected C^∞ Riemannian manifold M of dimension $n \geq 2$ and study the effect, on the spectrum of the associated Laplace–Beltrami operator Δ acting on functions, of adding a "small" handle to M.

The handles we consider are defined as follows: Fix two distinct points p_1, p_2 in M and for $\varepsilon > 0$ define

$B_\varepsilon \equiv:$ union of the open geodesic disks about p_1, p_2 of radius ε,
$\Omega_\varepsilon \equiv: M - \overline{B_\varepsilon}$,
$\Gamma_\varepsilon \equiv:$ common boundary of B_ε and Ω_ε,
$S_\varepsilon \equiv: (n-1)$-sphere in R^n of radius ε,
$S \equiv: S_1$.

For positive ε which is less than $\frac{1}{4}$ the injectivity radius of M and less than $\frac{1}{4}$ the distance from p_1 to p_2, let M_ε be a compact connected C^∞ Riemannian manifold with $\overline{\Omega_\varepsilon}$ isometrically imbedded in M_ε, and with a diffeomorphism

$$\Psi_\varepsilon : M_\varepsilon - \Omega_{2\varepsilon} \to [-1, 1] \times S$$

such that

$$C_\varepsilon \equiv: M_\varepsilon - \Omega_\varepsilon = \Psi_\varepsilon^{-1}[[-\tfrac{1}{2}, \tfrac{1}{2}] \times S].$$

We refer to such an M_ε as *obtained from M by adding the handle C_ε across Γ_ε.*

Denote the respective spectra of M, M_ε by

$$\text{spec}\,(M) \equiv: \{0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots\},$$

$$\text{spec}\,(M_\varepsilon) \equiv: \{0 = \sigma_0(\varepsilon) < \sigma_1(\varepsilon) \leq \sigma_2(\varepsilon) \leq \cdots\},$$

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where each distinct eigenvalue is repeated according to its multiplicity; and denote the associated theta functions by

$$\Theta(t) \equiv: \sum_{j=0}^{\infty} e^{-\lambda_j t}, \qquad \Theta_\varepsilon(t) \equiv: \sum_{j=0}^{\infty} e^{-\sigma_j(\varepsilon)t}.$$

Our interest in this paper is in determining whether the family of Riemannian manifolds M_ε can be chosen so that

$$\lim \sigma_j(\varepsilon) = \lambda_j \quad \text{as} \quad \varepsilon \downarrow 0 \tag{1}$$

for all $j = 1, 2, \ldots$.

Our first comment is that even if (1) is valid for all j, we do not expect that it be valid uniformly in j. In fact, when M is 2-dimensional the Minakshisundaram–Pleijel asymptotic expansion reads as [10, p. 45; 1, pp. 204–222]

$$\Theta(t) \sim \frac{A(M)}{4\pi t} + \frac{\chi(M)}{6} + O(t), \qquad \Theta_\varepsilon(t) \sim \frac{A(M_\varepsilon)}{4\pi t} + \frac{\chi(M_\varepsilon)}{6} + O(t), \tag{2}$$

as $t \downarrow 0$ (where $A(\cdot)$, $\chi(\cdot)$ denote area and Euler-characteristic, respectively). If (1) were valid uniformly in j then (2) would imply, by an easy argument, that $\chi(M_\varepsilon) = \chi(M)$ – an impossibility.

THEOREM A. *We always have*

$$\limsup \sigma_j(\varepsilon) \leq \lambda_j \quad \textit{as} \quad \varepsilon \downarrow 0 \tag{3}$$

for all $j = 1, 2, \ldots$. *A necessary condition that* (1) *be valid for all* j *is that* $\nu(\varepsilon)$, *the lowest eigenvalue of* C_ε *with Dirichlet data on* Γ_ε, *satisfy*

$$\lim \nu(\varepsilon) = +\infty \quad \textit{as} \quad \varepsilon \downarrow 0. \tag{4}$$

In particular, if for a fixed $l > 0$, *the ("long-thin") cylinder* $[-l/2, l/2] \times S_\varepsilon$ *is an isometrically imbedded open submanifold of* C_ε *for every* ε, *then* $\nu(\varepsilon) \leq \pi^2/l^2$ *and* (1) *cannot be satisfied for all* j.

To give a sufficient condition we require a definition,

DEFINITION 1. For any compact Riemannian manifold X of dimension

$n \geq 2$, we define the *isoperimetric constant* $c_1(X)$ by

$$c_1(X) = \inf_Y \frac{\{\mathrm{vol}_{n-1}(Y)\}^n}{\{\min(\mathrm{vol}_n(X_1), \mathrm{vol}_n(X_2))\}^{n-1}} \tag{5}$$

where $\mathrm{vol}_k(\cdot)$ denotes k-dimensional Riemannian measure, and Y ranges over all compact $(n-1)$-dimensional submanifolds of X which divide X into 2 open submanifolds X_1, X_2 each having boundary Y.

THEOREM B. *Assume there exists a constant* $c > 0$ *such that*

$$c_1(M_\varepsilon) \geq c > 0 \tag{6}$$

for all ε. *Then* (1) *is valid for all* $j = 1, 2, \ldots$.

That (6) is an indication of the "smallness" of C_ε is given by

LEMMA 1. *The sufficient condition* "(6) *for all* ε" *implies*

$$\mathrm{vol}_n(C_\varepsilon) = 0(\varepsilon^n), \tag{7}$$

$$\nu(\varepsilon) \geq \mathrm{const}/\varepsilon^2 \tag{8}$$

as $\varepsilon \downarrow 0$.

Indeed, one proves (7) by picking $Y = \Gamma_\varepsilon$, and $X_1 = C_\varepsilon$, $X_2 = \Omega_\varepsilon$.

In order to prove (8) from (6) and (7) let us recall, a definition and Cheeger's inequality for manifolds with boundary [4; 14].

DEFINITION. Let M be a compact manifold with boundary ∂M. We define the constant $h(M)$ by

$$h(M) = \inf_Y \frac{\mathrm{vol}_{n-1}(Y)}{\mathrm{vol}_n(X)}$$

where Y ranges over all compact $(n-1)$ dimension submanifolds such that $\partial M \cap Y = \emptyset$, which divide M into X and X' where $\partial \bar{X} \cap \partial M = \emptyset$.

Cheeger's argument [4] shows that $\lambda_1(M) \geq h^2/4$ where $\lambda_1(M)$ is the first eigenvalue for the Laplacian with Dirichlet boundary data.

Let $M = C_\varepsilon$, $\partial M = \Gamma_\varepsilon$ and X and Y submanifolds of M as in the above definition then $\mathrm{vol}_{n-1}(Y) \geq c^{1/n} \mathrm{vol}_n(X)^{n-1/n}$ and

$$h(C_\varepsilon) \geq \inf_Y c^{1/n} \mathrm{vol}_n(X)^{-1/n} \geq k/\varepsilon$$

follow from (6) and (7). Therefore (8) follows from Cheeger's inequality.

We next remark that whereas the necessary condition for the validity of (1) for all j is a consequence of the max-min characterization of eigenvalues and thus best interpreted via vibration phenomena, the sufficient condition is obtained by working with the respective fundamental solutions of the heat equation on M, M_ε.

Most important is the interpretation of these fundamental solutions via Brownian motion, viz., if

$$p : M \times M \times (0, \infty) \to R$$

is the fundamental solution of the heat equation on M, then $p(x, y, t)$ is the probability density for a Brownian path in M starting at x at time 0 to be at y at time t. Of course one has a similar statement for

$$p_\varepsilon : M_\varepsilon \times M_\varepsilon \times (0, \infty) \to R,$$

the fundamental solution of the heat equation on M_ε. Similarly, if we let

$$q_\varepsilon : \Omega_\varepsilon \times \Omega_\varepsilon \times (0, \infty) \to R$$

denote the fundamental solution of the heat equation on Ω_ε with Dirichlet data on Γ_ε then $q_\varepsilon(x, y, t)$ is the probability density that a Brownian path starting at $x \in \Omega_\varepsilon$ at time 0 will be at $y \in \Omega_\varepsilon$ at time t *without having hit* Γ_ε *between time* 0 *and time* t. In particular, for x, y in

$$M_0 =: M - \{p_1, p_2\}$$

(we now think of q_ε as vanishing on the complement of $\Omega_\varepsilon \times \Omega_\varepsilon$) $q_\varepsilon(x, y, t)$ is a decreasing function in ε, and

$$q_\varepsilon \leq p, \qquad q_\varepsilon \leq p_\varepsilon \tag{9}$$

on $M \times M \times (0, \infty)$, $M_\varepsilon \times M_\varepsilon \times (0, \infty)$, cf. [7; 12] for the application and details in Euclidean space, and [9] for the construction on general Riemannian manifolds.

Our final concern is that we *can* construct manifolds M, M_ε for which (6) is satisfied for all ε.

MAIN THEOREM. *Let M be a compact 2-dimensional Riemannian manifold, $K: M \to R$ its Gaussian curvature and $\tilde{M} = \{M - K^{-1}[0]\} \cup \{\text{int } K^{-1}[0]\}$. Then $\tilde{M}$ is open and dense in M. Given any two distinct points p_1, p_2 in $\tilde{M}$ then M_ε may be constructed so that there exists $c > 0$ for which* (6) *is valid for all ε. Thus M_ε may be constructed so that $A(M_\varepsilon) \to A(M)$ as $\varepsilon \downarrow 0$ and so that* (1) *is valid for all $j = 1, 2, \ldots$.*

The theorem suggests that to the question "Can you hear the shape of a drum?" [7] one should answer "For a compact 2-manifold – not really." For to determine the Euler-characteristic, via (2), by actually listening to its tones (square roots of the eigenvalues) one would have to know *a priori* that what is heard in fact approximates *all* the tones with *uniform* accuracy. Anything less could lead the listener astray in determining the Euler-characteristic.

We wish to thank our colleagues S. Kaplan and B. Randol for many helpful discussions, and A. Heller for help with Lemma 5.

This paper is dedicated to the inspiring memory of H. E. Rauch, whom both authors knew and admired as a friend, teacher, and mathematician.

§1. Proof of Theorem A

Denote the spectrum of Ω_ε with Dirichlet boundary data (distinct eigenvalues are repeated according to multiplicity) by

$$\text{spec}\,(\Omega_\varepsilon) \equiv: \{0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \leq \lambda_3(\varepsilon) \leq \cdots\}.$$

Then the max-min characterizations of the eigenvalues [5, Chap. VI] imply that $\lambda_j(\varepsilon)$ is an increasing function of ε, and the validity of the inequalities

$$\lambda_j(\varepsilon) \geq \lambda_{j-1}, \qquad \lambda_j(\varepsilon) \geq \sigma_{j-1}(\varepsilon) \tag{10}$$

for $j = 1, 2, \ldots$. Moreover, in [3] it was shown (cf. [13] for the case of domains in Euclidean space) that

$$\lambda_j(\varepsilon) \to \lambda_{j-1} \quad \text{as} \quad \varepsilon \downarrow 0 \tag{11}$$

for all $j = 1, 2, \ldots$. Then (10), (11) imply (3).

With these preliminaries, establishing the necessary condition is done as follows: Let the union of the spectra of C_ε, Ω_ε with Dirichlet data on Γ_ε be denoted by

$$\operatorname{spec}(C_\varepsilon) \cup \operatorname{spec}(\Omega_\varepsilon) \equiv: \{0 < \mu_0(\varepsilon) \le \mu_1(\varepsilon) \le \cdots\}$$

where the eigenvalues have been re-listed in non-decreasing order and repeated according to multiplicity. Then a max-min argument [5, p. 408] implies

$$\sigma_j(\varepsilon) \le \mu_j(\varepsilon) \tag{12}$$

for all $j = 0, 1, 2, \ldots$. Assume

$$\alpha \equiv: \liminf \nu(\varepsilon) \quad \text{as} \quad \varepsilon \downarrow 0$$

is finite, and let λ_k be the first eigenvalue of M which is strictly greater than α (in particular, $\lambda_{k-1} < \lambda_k$). Then for any ε for which we have

$$\nu(\varepsilon) < \lambda_k$$

we also have $\nu(\varepsilon) < \lambda_k \le \lambda_{k+1}(\varepsilon)$, i.e.,

$$\nu(\varepsilon) \in \{\mu_0(\varepsilon), \ldots, \mu_k(\varepsilon)\},$$

which implies

$$\sigma_k(\varepsilon) \le \mu_k(\varepsilon) \le \max\{\nu(\varepsilon), \lambda_k(\varepsilon)\}.$$

Thus $\alpha < \lambda_k$ implies by (11) that

$$\liminf \sigma_k(\varepsilon) \le \liminf \max\{\nu(\varepsilon), \lambda_k(\varepsilon)\} = \max\{\alpha, \lambda_{k-1}\} < \lambda_k$$

as $\varepsilon \downarrow 0$. It is therefore impossible that $\delta_k(\varepsilon) \to \lambda_k$ as $\varepsilon \downarrow 0$.

COROLLARY 1. *If*

$$\liminf \nu(\varepsilon) = \alpha < +\infty \quad \textit{as} \quad \varepsilon \downarrow 0,$$

and λ_k *is the first eigenvalue of* M *greater than* α, *then*

$$\liminf \sigma_k(\varepsilon) < \lambda_k \quad \textit{as} \quad \varepsilon \downarrow 0.$$

Remark 1. We note that (4) is also a necessary condition that $\Theta_\varepsilon(t) \to \Theta(t)$, for any given $t>0$, $\varepsilon\downarrow 0$. Indeed, (12) implies that

$$\Theta_\varepsilon(t)\geq e^{-\nu(\varepsilon)t}+\sum_{j=1}^{\infty} e^{-\lambda_j(\varepsilon)t}.$$

But in [3] it was proved that the series on the right-hand side of the above inequality tends to $\Theta(t)$, uniformly on compact subsets of $(0,\infty)$, as $\varepsilon\downarrow 0$. That (4) is a consequence of $\Theta_\varepsilon(t)\to\Theta(t)$ is immediate.

§2. Proof of Theorem B

LEMMA 2. *Let dM, dM_ε denote the respective volume elements of M, M_ε (of course they agree on Ω_ε), and let f be any bounded measurable function compactly supported on M_0. Then*

$$\lim \int_{M_\varepsilon} p_\varepsilon(x,w,t)f(w)\,dM_\varepsilon=\int_M p(x,y,t)f(y)\,dM(y),$$

uniformly in $(x,t)\in$ compact subsets of $M_0\times(0,\infty)$, as $\varepsilon\downarrow 0$. In particular we have

$$\lim p_\varepsilon(x,y,t)=p(x,y,t)\quad \text{as}\quad \varepsilon\downarrow 0 \tag{13}$$

on $M_0\times M_0\times(0,\infty)$.

Proof. In [3] it was shown (cf. [13] for the case of domain in Euclidean space) that

$$\lim q_\varepsilon(x,y,t)=p(x,y,t)\quad \text{as}\quad \varepsilon\downarrow 0$$

uniformly on compact subsets of $M_0\times M_0\times(0,\infty)$. Let K be a compact subset of M_0 and pick ε sufficiently small so that Ω_ε contains K and the support of f. Then for $x\in K$, $t\in[a,b]\subseteq(0,\infty)$ we have

$$\begin{aligned}&\left|\int_{M_\varepsilon}\{p_\varepsilon(x,w,t)-q_\varepsilon(x,w,t)\}f(w)\,dM_\varepsilon(w)\right|\\&\leq\max|f|\int_{M_\varepsilon}\{p_\varepsilon(x,w,t)-q_\varepsilon(x,w,t)\}\,dM_\varepsilon(w)\\&=\max|f|\left\{1-\int_{\Omega_\varepsilon}q_\varepsilon(x,y,t)\,dM(y)\right\}=\max|f|\int_M\{p(x,y,t)-q_\varepsilon(x,y,t)\}\,dM(y),\end{aligned}$$

since

$$\int_{M_\varepsilon} p(x, w, t)\, dM_\varepsilon(w) = 1 = \int_M p(x, y, t)\, dM(y).$$

Thus

$$\left|\int_{M_\varepsilon} p_\varepsilon(x, w, t) f(w)\, dM_\varepsilon(w) - \int_M p(x, y, t) f(y)\, dM(y)\right|$$

$$\leq 2 \max |f| \int_M \{p(x, y, t) - q_\varepsilon(x, y, t)\}\, dM(y)$$

which goes to 0, uniformly in $(x, t) \in K \times [a, b]$, as $\varepsilon \downarrow 0$. Thus the lemma is proven.

To prove Theorem B we first reduce the problem to showing that for t bounded away from 0, $p_\varepsilon(z, w, t)$ is uniformly bounded above *independent of* ε.

Assume that this has in fact been accomplished. Then one has by the Sturm–Liouville expansion (cf. below) of p_ε, p that for any fixed $t > 0$,

$$\begin{aligned}
\Theta_\varepsilon(t) - \Theta(t) &= \int_{M_\varepsilon} p_\varepsilon(z, z, t)\, dM_\varepsilon(z) - \int_M p(x, x, t)\, dM(x) \\
&= \int_{C_\varepsilon} p_\varepsilon(z, z, t)\, dM_\varepsilon(z) - \int_{B_\varepsilon} p(x, x, t)\, dM(x) \\
&\quad + \int_{\Omega_\varepsilon} \{p_\varepsilon(x, x, t) - p(x, x, t)\}\, dM(x) \to 0 \quad \text{as} \quad \varepsilon \downarrow 0.
\end{aligned}$$

Indeed, the first two integrands are bounded and the volumes of C_ε, B_ε tend to 0. The convergence of the third integral follows from (13) and Lebesgue's dominated convergence theorem. Thus p_ε uniformly bounded independent of ε implies for $t > 0$,

$$\lim \Theta_\varepsilon(t) = \Theta(t) \quad \text{as} \quad \varepsilon \downarrow 0. \tag{14}$$

Finally assume there exists $k \geq 1$ such that $\liminf \sigma_k(\varepsilon) < \lambda_k$ as $\varepsilon \to 0$. Let ε_l be a sequence going to 0, with $\sigma_k(\varepsilon_l) \to \sigma_k < \lambda_k$ as $l \to \infty$. Then by (14), (3) and Fatou's lemma we have

$$\begin{aligned}
\Theta(t) = \lim \Theta_{\varepsilon_l}(t) &\geq \sum_{j=0}^{\infty} \liminf \exp(-\sigma_j(\varepsilon_l)t) = \sum_{j=0}^{\infty} \exp(-\limsup \sigma_j(\varepsilon_l)t) \\
&= e^{-\sigma_k t} + \sum_{j \neq k} \exp(-\limsup \sigma_j(\varepsilon_l)t) > \Theta(t)
\end{aligned}$$

which implies a contradiction.

So to prove Theorem B we must bound p_ε above independently of ε. To do so we require some estimates of P. Li [8].

DEFINITION 2. Given a compact Riemannian manifold X of dimension $n \geq 2$ we define the *Sobolev constant of* X, $c_0(X)$, by

$$c_0(X) =: \inf_f \left[\left\{ \int_M |\nabla f| \right\}^n \Big/ \inf_{\beta \in \mathbf{R}} \left\{ \int_M |f-\beta|^{n/(n-1)} \right\}^{n-1} \right]$$

where f ranges over the Sobolev space of functions with L^1-derivatives.

LEMMA (P. Li) 3. *Let $v = \mathrm{vol}_n(X)$, $c_0 = c_0(X)$. Then there exist constants depending only on n such that*

for any eigenfunction f with eigenvalue $\tau \neq 0$ we have

$$\|f\|_\infty^2 \leq \mathrm{const} \begin{cases} \|f\|_2^2(\tau^{n/2}/c_0) \exp\{\mathrm{const}\,(c_0/v)^{2/n}/\tau\}, & n \geq 3 \\ \|f\|_2^2(\tau^2 v/c_0^2) \exp\{\mathrm{const}\; c_0/\tau v\}, & n = 2 \end{cases} \tag{15}$$

for the k^{th} eigenvalue τ_k of X we have

$$k \leq \mathrm{const} \begin{cases} \{\tau_k(v/c_0)^{2/n}\}^{n-1} & n \geq 3 \\ \{\tau_k v/c_0\}^2 & n = 2 \end{cases} \tag{16}$$

for all $k = 1, 2, \ldots$.

Before turning to the proof of Theorem B we remark (as in [8]) that the argument of [2, Section 3], when applied to compact X without boundary, yields

$$c_1(X) \leq c_0(X) \leq 2c_1(X). \tag{17}$$

Also, by considering arbitrarily small geodesic disks, one has under *all* circumstances

$$c_1(X) \leq n^{n-1}\,\mathrm{vol}_{n-1}(S). \tag{18}$$

We now prove Theorem B; recall that we must establish an upper bound on p_ε which is independent of ε. Assume (6). Then Lemma 1 implies

$$\lim \mathrm{vol}_n(M_\varepsilon) = \mathrm{vol}_n(M) \quad \text{as} \quad \varepsilon \downarrow 0. \tag{19}$$

Also, $\sigma_1(\varepsilon)$ is bounded away from zero, by either using Cheeger's inequality [4] with (19) or by using (17), (19), and (16) for $k=1$. Thus we have that $\{\sigma_1(\varepsilon), \mathrm{vol}_n(M_\varepsilon), c_0(M_\varepsilon)\}$ are all restricted to a compact subset of $(0,\infty)$. Then there exist constants *independent of* ε for which Li's estimates now read as

$$\|f\|_\infty^2 \le \text{const}\,\|f\|_2^2 \begin{cases} \tau^{n/2} & n \ge 3 \\ \tau^2 & n=2 \end{cases} \tag{15'}$$

$$\tau_k \ge \text{const} \begin{cases} k^{1/(n-1)} & n \ge 3 \\ k^{1/2} & n=2. \end{cases} \tag{16'}$$

Now fix $t>0$ and let $\{\Phi_j(\varepsilon)\}$ be an orthonormal basis of $L^2(M_\varepsilon)$ consisting of eigenfunctions corresponding respectively to $\{\sigma_j(\varepsilon)\}$. Then the eigenfunction expansion of p_ε is given by, and satisfies,

$$\begin{aligned} p_\varepsilon(z,w,t) &= \sum_{j=0}^{\infty} e^{-\sigma_j(\varepsilon)t}\Phi_j(\varepsilon)(z)\Phi_j(\varepsilon)(w) \\ &\le \sum_{j=0}^{\infty} e^{-\sigma_j(\varepsilon)t}\|\Phi_j(\varepsilon)\|_\infty^2. \end{aligned}$$

We proceed with estimate for the case $n=2$ as this is the situation in which we will construct our explicit examples (the argument for $n>2$ is similar). From (15′) we have

$$p_\varepsilon(z,w,t) \le \text{const}\left\{1+\sum_{j=1}^{\infty} \sigma_j^2(\varepsilon)e^{-\sigma_j(\varepsilon)t}\right\}$$

with the constant independent of ε. Now (16′) implies the existence of a positive integer J, independent of ε, such that for all $j \ge J$ we have

$$\sigma_j^5(\varepsilon)e^{-\sigma_j(\varepsilon)t} \le 1.$$

Then (16′) implies that

$$\sum_{j=1}^{\infty} \sigma_j^2(\varepsilon)e^{-\sigma_j(\varepsilon)t} \le \sum_{j<J} \sigma_j^2(\varepsilon) + \text{const}\sum_{j \ge J} j^{-3/2}$$

which is bounded above, independently of ε, by (3).

This concludes the proof of Theorem B.

§3. The construction of M_ε for the main theorem

Let M be 2-dimensional and $p \in \tilde{M}$, i.e., either $K(p) \neq 0$ or K vanishes identically on some neighborhood of p. To p we associate a number $\alpha(p)$ with the following list of properties:

(i) α will be less than the convexity radius of M (in particular, it is less than $\frac{1}{2}$ the injectivity radius of M). If K, the Gauss curvature of M, has maximum equal to κ then α will be chosen so that it is also less than $\pi/2\sqrt{\kappa}$.

(ii) Set

$$B(p; r) \equiv: \text{metric disk about } p \text{ of radius } r.$$

Then we require that K either vanishes identically on $B(p; \alpha)$ or never vanishes on $B(p; \alpha)$.

Should K never vanish on $B(p; \alpha)$ then α will be sufficiently small so that

$$\inf |K(q)| > (\tfrac{2}{3}) \sup |K(q)| \tag{20}$$

where q ranges over $B(p; \alpha)$.

(iii) Let dA denote the Riemannian element of area and, as in the introduction, $A(\cdot)$ denote the area. Then we require that

$$A(B(p; \alpha)) < A(M)/8 \tag{21}$$

$$\iint_{B(p;\alpha)} K\, dA < \pi/2. \tag{22}$$

In particular for

$$K^+ \equiv: \max\{K, 0\}$$

we have

$$\iint_{B(p;\alpha)} K^+\, dA < \pi/2. \tag{23}$$

Remark 2. In a moment we shall change the Riemannian metric in a compact subset of $B(p; \alpha)$, when $K(p) \neq 0$, such that the new Gauss curvature does not change sign. The Gauss–Bonnet formula implies that the left-hand side of (22) does *not* change, hence (23) remains valid in the new Riemannian metric.

If $K(p) \neq 0$, then for every $\varepsilon \in (0, \alpha/2)$ we introduce a new Riemannian metric on $B(p; \varepsilon)$. The details will be given for $K(p) > 0$, as the case $K < 0$ is similar. Let

$$k_1 \equiv: \inf K, \qquad k_2 \equiv: \sup K$$

on $B(p; \alpha)$. Then $k_1/k_2 > \frac{2}{3}$ implies that for all r satisfying $0 < r < \alpha$ we have

$$\cos \sqrt{k_1}\, r < 2\left\{\frac{\sin \sqrt{k_2}\, r}{\sqrt{k_2}\, r} - \frac{1}{2}\right\}.$$

Introduce geodesic polar coordinates (r, θ) about p and write the given Riemannian metric as

$$ds^2 = dr^2 + \eta^2(r, \theta)\, d\theta^2.$$

Then η satisfies Jacobi's equation

$$\frac{\partial^2}{\partial r^2}\eta + K\eta = 0$$

with initial data

$$\eta(0, \theta) = 0, \qquad (\partial\eta/\partial r)(0, \theta) = 1.$$

Standard Sturmian arguments imply

$$\frac{\partial\eta}{\partial r}(r, \theta) \le \cos\sqrt{k_1}\, r < 2\left\{\frac{\sin\sqrt{k_2}\, r}{\sqrt{k_2}\, r} - \frac{1}{2}\right\}$$

$$\le 2\left\{\frac{\eta(r, \theta)}{r} - \frac{1}{2}\right\} = \frac{\eta(r, \theta) - r/2}{r/2}$$

i.e.,

$$\frac{\partial\eta}{\partial r} \le \frac{\eta - r/2}{r/2}. \tag{24}$$

Geometrically, (24) implies that for each fixed θ, the tangent line to the curve $y = \eta(x, \theta)$ (in the (x, y)-plane) at $x = r$ intersects the line $y = x$ for some $x(r, \theta)$ satisfying $r/2 < x(r, \theta) < r$.

Given ε satisfying $0 < \varepsilon < \alpha/2$, set $x_1 \equiv: x(3\varepsilon/4, \theta)$ and replace $y = \eta(x, \theta)$ for

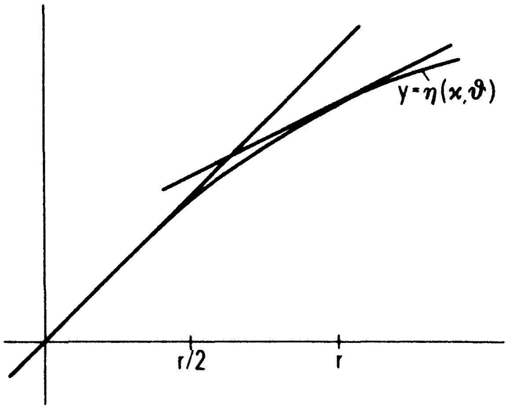

Figure 1

$0 \leq x \leq \varepsilon$ by $y = \tilde{\eta}(x, \theta)$ where

$$\tilde{\eta}(x, \theta) = x \quad \text{for} \quad 0 \leq x \leq x_1,$$

$\tilde{\eta}(x, \theta)$ is given by the tangent line to $y = \eta(x, \theta)$ at $x = 3\varepsilon/4$ for $x_1 \leq x \leq 3\varepsilon/4$, $\tilde{\eta}(x, \theta) = \eta(x, \theta)$ for $3\varepsilon/4 \leq x \leq \alpha$.

Now smooth $\tilde{\eta}$ to η_ε which is a C^∞ function in (r, θ) and which satisfies

$$\begin{aligned} &\eta_\varepsilon(r, \theta) = r && 0 \leq r \leq 5\varepsilon/16 \\ &\eta_\varepsilon(r, \theta) = \eta(r, \theta) && \varepsilon \leq x \leq \alpha \\ &\frac{\partial \eta_\varepsilon}{\partial r} > 0, \quad \frac{\partial^2 \eta_\varepsilon}{\partial r^2} \leq 0 && 0 \leq x \leq \alpha. \end{aligned} \tag{25}$$

Finally replace

$$ds^2 = dr^2 + \eta^2(r, \theta)\, d\theta^2$$

on $B(p; \alpha)$, by

$$ds_\varepsilon^2 = dr^2 + \eta_\varepsilon^2(r, \theta)\, d\theta^2.$$

Then the new metric is flat on the geodesic disk of radius $5\varepsilon/16$, has non-negative Gauss curvature on $B(p; \alpha)$, and agrees with the original metric on $B(p; \alpha) \cap \overline{\Omega_\varepsilon}$. One sees easily that the smoothing may be chosen so that (21) can be replaced by

$$A(B(p; \alpha)) < A(M)/6. \tag{21'}$$

A similar argument can be carried out when $K(p_1) < 0$.

We are now ready to define M_ε. Given $p_1, p_2 \in \tilde{M}$ fix α to be less than $\alpha(p_1)$, $\alpha(p_2)$ and less than $\frac{1}{2}$ the distance from p_1 to p_2. For each $i=1,2$, if $K(p_i)=0$ leave well enough alone. If $K(p_i)\neq 0$ then for every $\varepsilon<\alpha/2$ introduce the change of metric in $B(p_i;\varepsilon)$ just described. Call the new Riemannian manifold M^*_ε.

Now attach the tube

$$T\equiv:[-\varepsilon/4,\ \varepsilon/4]\times S_{\varepsilon/4}$$

to M^*_ε "at right angles" to the geodesic circles about p_1, p_2 of radius $\varepsilon/4$, identifying $\{-\varepsilon/4\}\times S_{\varepsilon/4}$ with the geodesic circle about p_1, and $\{\varepsilon/4\}\times S_{\varepsilon/4}$ with that about p_2, of radius $\varepsilon/4$. Put differently, in the polar coordinate system about each p_i, replace

$$\eta_\varepsilon(r,\theta)=r \qquad 0\leq r\leq 5\varepsilon/16$$

with

$$\zeta_\varepsilon(r,\theta)=\begin{cases}\varepsilon/4 & 0\leq r\leq \varepsilon/4\\ r & \varepsilon/4\leq r\leq 5\varepsilon/16\end{cases}$$

and identify the two circles $\{0\}\times S_{\varepsilon/4}$.

The resulting manifold with "creased" Riemannian metric will be our M_ε and we shall estimate $C_1(M_\varepsilon)$ for this "creased" metric from below. Once we, in fact, verify the existence of c for which (6) is valid for all ε, it is easy to smooth the "crease," with non-positive curvature near the crease, such that (6) is valid with c replaced by $c/2$ for all ε.

§ 4. Estimating the isoperimetric constant

LEMMA (F. Fiala [6, p. 336; 11, p. 12]) 3. *Let M be a complete Riemannian surface with Riemannian measure dA, Gauss curvature K, $K^+=\max\{K,0\}$, and D a simply connected domain in M of area A with smooth boundary of length L. Then*

$$\iint_D K^+\,dA<2\pi$$

implies

$$L^2-4\pi A+2A\iint_D K^+\,dA\geq 0.$$

COROLLARY 1. *In the above,*

$$\iint_D K^+ \, dA < \pi$$

implies

$$L^2 \geq 2\pi A.$$

LEMMA (S. T. Yau [4, p. 489]) 4. *For a compact n-dimensional Riemannian manifold M, to evaluate $c_1(M)$ it suffices to let Y range over those compact $(n-1)$-manifolds which separate M into connected open submanifolds X_1, X_2.*

Remark 3. In [14] Yau proves this fact for Cheeger's isoperimetric constant. However his induction argument and the Minkowski inequality (Triangle inequality if $\dim M = 2$) immediately yields the above lemma.

In what follows $L_\varepsilon(\cdot)$, dA_ε, $A_\varepsilon(\cdot)$, K_ε will denote length, area element, area, and Gauss curvature of M_ε. We also write $K_\varepsilon^+ = \max\{K_\varepsilon, 0\}$.

To start with our estimate of $c_1(M_\varepsilon)$ from below, let γ be a compact 1-manifold imbedded in Ω_ε, separating M_ε into M_1, M_2, and assume $C_\varepsilon \subseteq M_1$. Set $M_1^* = (M_1 - C_\varepsilon) \cup B_\varepsilon$, where B_ε is, as originally defined, the union of the open geodesic disks of radius ε in M about p_1, p_2. One easily checks that $A_\varepsilon(C_\varepsilon)/A(B_\varepsilon)$ is bounded away from $0, +\infty$ independently of ε. Thus

$$\begin{aligned} L_\varepsilon^2(\gamma)/\min(A_\varepsilon(M_1), A_\varepsilon(M_2)) &\geq \text{const}\, L^2(\gamma)/\min(A(M_1^*), A(M_2)) \\ &\geq \text{const}\, c_1(M) > 0 \end{aligned}$$

which is independent of ε.

Also if γ is any compact imbedded 1-manifold of length $\geq \alpha$ separating M_ε into M_1, M_2, then

$$L_\varepsilon^2(\gamma)/\min(A_\varepsilon(M_1), A_\varepsilon(M_2)) \geq \alpha^2/A_\varepsilon(M_\varepsilon) \geq \text{const} > 0$$

by (19).

Thus our task is to estimate

$$L_\varepsilon^2(\gamma)/\min(A_\varepsilon(M_1), A_\varepsilon(M_2))$$

where γ ranges over compact imbedded 1-manifolds separating M_ε into connected M_1, M_2 and such that

$$L_\varepsilon(\gamma)<\alpha, \qquad \gamma\cap\operatorname{int}(C_\varepsilon)\neq\emptyset.$$

Since $\varepsilon<\alpha/2$ we immediately have

$$\gamma\subseteq M_\varepsilon-\overline{\Omega_\alpha}.$$

We will always have one of the domains, say M_1, contained in $M_\varepsilon-\overline{\Omega_\alpha}$ and (19), (21′) allow us to assume $A_\varepsilon(M_1)<A_\varepsilon(M_\varepsilon)/2$; so we are always estimating

$$L_\varepsilon^2(\gamma)/A_\varepsilon(M_1)$$

from below.

Since $M_\varepsilon-\overline{\Omega_\alpha}$ is topologically a cylinder, we have that γ is an imbedded circle bounding a disk $M_1\subseteq M_\varepsilon-\overline{\Omega_\alpha}$ or γ is a pair of imbedded circles bounding a cylinder $M_1\subseteq M_\varepsilon-\overline{\Omega_\alpha}$. In the first case, then by smoothing the "crease" with non-positive curvature we would have (from Remark 2)

$$\int_{M_\varepsilon-\bar{\Omega}_\alpha} K_\varepsilon^+\,dA_\varepsilon<\pi$$

and therefore

$$L_\varepsilon^2(\gamma)\geq 2\pi A_\varepsilon(M_1)$$

in the approximating metric, which implies the inequality for our "creased" metric. So we need only consider the second case.

First let $d(,)$ denote distance in M_ε^* and define for $i=1,2$

$$\Gamma_i(t)\equiv:\{q\in M_\varepsilon^*: d(q,p_i)=t\}$$
$$W_i\equiv:\{q\in M_\varepsilon^*: \varepsilon/4\leq d(q,p_i)<\alpha\}.$$

We are now considering $\gamma=\gamma_1\cup\gamma_2$ where γ_1,γ_2 are imbedded circles bounding a cylinder M_1 in $M_\varepsilon-\overline{\Omega_\alpha}$. We think of γ_i as "closest" to $\Gamma_i(\alpha)$. First we shall assume that γ_i does not cross $\Gamma_i(\varepsilon/4)$ for either $i=1,2$, and then reduce the general case (viz., where at least one γ_i crosses $\Gamma_i(\varepsilon/4)$) to this one. The assumption that neither γ_i crosses $\Gamma_i(\varepsilon/4)$ involves three essential possibilities: (i) $\gamma_1, \gamma_2\subseteq T$, (ii) $\gamma_1\subseteq W_1$, $\gamma_2\subseteq W_1\cup T$, (iii) $\gamma_1\subseteq W_1$, $\gamma_2\subseteq W_2$.

(i) If γ_1, $\gamma_2 \subseteq T$ then $M_1 \subseteq T$, and $L_\varepsilon(\gamma_i) \geq \pi\varepsilon/2$, $A_\varepsilon(M_1) \leq \pi\varepsilon^2/4$, which implies

$$L_\varepsilon^2(\gamma) \geq 4\pi A_\varepsilon(M_1).$$

(ii) If $\gamma_1 \subseteq W_1$, $\gamma_2 \subseteq W_1 \cup T$ then $M_1 \subseteq T \cup (M_1 \cap W_1)$, which implies

$$A_\varepsilon(M_1) \leq \pi\varepsilon^2/4 + A_\varepsilon(M_1 \cap W_1).$$

Now think of γ_1 as bounding a disk $D \subseteq B(p_1; \alpha) \subseteq M_\varepsilon^*$. Then

$$A_\varepsilon(D) = \pi(\varepsilon/4)^2 + A_\varepsilon(M_1 \cap W_1) \geq A_\varepsilon(M_1)/4,$$

and by Remark 2 and Fiala's inequality we therefore have

$$L_\varepsilon^2(\gamma) \geq L_\varepsilon^2(\gamma_1) \geq 2\pi A_\varepsilon(D) \geq (\pi/2) A_\varepsilon(M_1).$$

This argument, of course, covers the possibility: $\gamma_2 \subseteq W_2$, $\gamma_1 \subseteq W_2 \cup T$.

(iii) If $\gamma_1 \subseteq W_1$, $\gamma_2 \subseteq W_2$ then our argument is similar to the one just given. Let D_i correspond to γ_i as D corresponded to γ_1 in (ii). Then we have

$$\begin{aligned} L_\varepsilon^2(\gamma) \sum_i L_\varepsilon^2(\gamma_i) &\geq \sum_i 2\pi A_\varepsilon(D_i) \\ &= 2\pi \sum_i \{\pi\varepsilon^2/16 + A_\varepsilon(M_1 \cap W_i)\} \\ &> \sum_i \{\pi\varepsilon^2/8 + A_\varepsilon(M_1 \cap W_i)\} = \pi A_\varepsilon(M_1). \end{aligned}$$

In summary, when neither γ_i crosses $\Gamma_i(\varepsilon/4)$ we have

$$L_\varepsilon^2(\gamma) \geq (\pi/2) A_\varepsilon(M_1). \tag{26}$$

For the general case, we shall assume γ_1 crosses $\Gamma_1(\varepsilon/4)$ transversally with an even number of intersections, and show that we can replace γ_1 with $\tilde{\gamma}_1$ having fewer intersections (therefore, ultimately no intersections), shorter length, and enclosing with γ_2 larger area. Thus (26) will remain valid in the general case.

So we now have γ_1 crossing $\Gamma_1(\varepsilon/4)$ transversally. We shall think of $M_\varepsilon - \overline{\Omega_\alpha}$ as the cylinder $R^2 - \{0\}$, with γ_1 and $\Gamma_1(\varepsilon/4)$ winding about 0 once, let

$$D \equiv: \text{component of } R^2 - \{\gamma_1\} \text{ not containing } M_1,$$

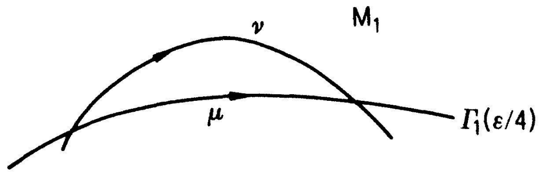

Figure 2

assume for convenience that D contains 0, and let

$$\{D_1, \ldots, D_k\} \equiv: \text{components of } D - \Gamma_1(\varepsilon/4).$$

We say that a component D_l is *simple* if its boundary consists of the union of 2 smooth arcs ν and μ, with ν part of γ_1 and μ part of $\Gamma_1(\varepsilon/4)$.

Given a simple D_l we have that either $D_l \subseteq W_1$ or $D_l \subseteq W_2 \cup T$, with $\nu \subseteq W_1$ or $\nu \subseteq W_2 \cup T$ respectively. However, in either case

$$L(\nu) > L(\mu)$$

– note that we are using here the hypothesis that α was picked to be less than the convexity radius of M. Then replace γ_1 by $\tilde{\gamma}_1$ by first replacing ν by μ, then sliding μ a drop to the side of $\Gamma_1(\varepsilon/4)$ opposite to D_l, and, finally, smoothing the corners. Define $\tilde{\gamma} = \tilde{\gamma}_1 \cup \gamma_2$. Then $L_\varepsilon(\tilde{\gamma}) \leq L_\varepsilon(\gamma)$, and $\tilde{M}_1$, the domain in $M_\varepsilon - \overline{\Omega_\alpha}$ bounded by $\tilde{\gamma}$, contains $M_1 \cup D_l$ which implies $A_\varepsilon(\tilde{M}_1) \geq A_\varepsilon(M_1)$. Thus

$$L_\varepsilon^2(\gamma)/A_\varepsilon(M_1) \geq L_\varepsilon^2(\tilde{\gamma})/A_\varepsilon(\tilde{M}_1).$$

The curve $\tilde{\gamma}$ has the same properties as γ, so we may repeat the argument just given until we are left with a curve not intersecting $\Gamma_1(\varepsilon/4)$. In this last case we already have the estimate (26). Thus the last thing for us to verify is that given γ, a simple D_l exists.

LEMMA 5. *Let Γ, Λ be two simply closed smooth curves in $R^2 - \{0\}$ which wind about the origin once and which meet each other transversally. Let G be the component of $R^2 - \Lambda$ containing 0, and let $\{G_1, \ldots, G_r\}$ be the components of $G - \Gamma$. Then there exists a component G_s whose boundary consists of two smooth arcs; one a part of Γ and one a part of Λ.*

Proof. We can assume Γ is the unit circle which Ω divides into a finite number of arcs alternately belonging to G and $R^2 - (G \cup \Lambda)$.

Let G_0 be the component of $G - \Gamma$ containing 0, and consider the segments

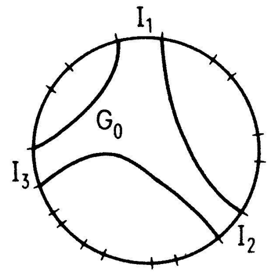

Figure 3

$I_1, \dots, I_{l_1}$ of $\overline{G_0} \cap \Gamma$. (Of course if $l_1 = 1$ then G_0 is simple. So assume otherwise.) *To each* I_{j_1}, $j_1 = 1, \dots, l_1$ associate the component G_{j_1} of $G - \Gamma$ distinct from G_0 having I_{j_1} as part of its boundary. $G_1, \dots, G_{l_1}$ are all distinct for otherwise Λ would not be simple by the Jordan curve theorem.

If there exists G_{j_1} having only I_{j_1} for the intersection of ∂G_{j_1} with Γ then we are done, for then G_{j_1} is then simple. So assume the opposite. Then to each $j_1 = 1, \dots, l_1$ associate the segments $I_{j_1 1}, \dots, J_{j_1 l_2}$ of $\overline{G_{j_1}} \cap \Gamma - I_{j_1}$ and to each segment $I_{j_1 j_2}$ associate the component of $G - \Gamma$ distinct from G_{j_1}, $G_{j_1 j_2}$, having $I_{j_1 j_2}$ as part of its boundary. Again, the collection $\{G_{j_1 j_2}\}$ are all distinct.

By continuing this process, if necessary, we exhaust all the arcs of $G - \Gamma$ and the process stops. But then every component of the last step is simple.

This concludes the proof of Lemma 5 and, with it, the proof of the main theorem.

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