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On the existence of escaping geodesics

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In [5] Cohn-Vossen remarks that to his knowledge the following question is undecided

PROBLEM. Let M be a complete Riemannian manifold homeomorphic to the plane. Does there exist an escaping geodesic $c:\mathbf{R}\to M$ without self-intersections?

Here "escaping" means that c is a proper map, cf. Section 1.

Recently the same problem, though in weaker form, has been posed by Wojtkowski [9]. In [8] and [9] Wojtkowski applies symbolical dynamics to study the geodesic flow of certain complete surfaces. The existence of escaping geodesics is a crucial prerequisite for this application.

In this paper we solve Cohn-Vossen's problem in the affirmative. We have not been able to find a general method to construct escaping geodesics without self-intersections. Thus we use different methods according to different properties of the geodesic flow. Intuitively the most difficult case is when there exists a closed geodesic on M. In this case we use Lusternik–Schnirelmann theory on the space of curves with fixed end-points. When closed geodesics do not exist we apply results from [3]. We know that in this case bounded geodesic rays do not exist either. Furthermore the possible self-intersections of geodesics are of a very restricted type. This allows us to prove that through every point of a complete plane without closed geodesics there exists a geodesic without self-intersections. Unfortunately we are not able to exclude that this geodesic is oscillating.

Hence, in a final step, we prove that, in the absence of closed geodesics, the existence of an oscillating geodesic implies the existence of an escaping geodesic without self-intersections. We note that our arguments generalize to Finsler metrics, at least if one admits self-intersections of the escaping geodesic.

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1. Preliminaries

We consider a complete Riemannian manifold M homeomorphic to the plane $\mathbf{R}^2$. Such M is called a complete plane. The distance function on M is denoted by d. A curve on M is a piecewise C^1-map $\gamma: I \to M$ defined on some interval $I \subseteq \mathbf{R}$. If I is compact the energy $E(\gamma) := \int_I |\dot\gamma(t)|^2 \, dt$ and the length $L(\gamma) := \int_I |\dot\gamma(t)| \, dt$ of γ are finite. For $I = [a, b]$ length and energy of γ are related by $L^2(\gamma) \leqslant (b - a)E(\gamma)$ with equality if and only if γ is parametrized proportionally to arc-length. For $p, q \in M$ we denote by $\Gamma_{p,q}$ the space of curves $\gamma: [0, 1] \to M$ with $\gamma(0) = p$, $\gamma(1) = q$. $\Gamma_{p,q}$ is endowed with the compact-open topology. By $\tilde\Gamma_{p,q}$ we denote the subspace of $\Gamma_{p,q}$ consisting of curves without self-intersections. Finally, for $\kappa \geqslant 0$, we let $\Gamma^\kappa_{p,q}$ be the set of curves in $\Gamma_{p,q}$ of energy $\leqslant \kappa$ and $\tilde\Gamma^\kappa_{p,q} := \tilde\Gamma_{p,q} \cap \Gamma^\kappa_{p,q}$.

On a complete non-compact Riemannian manifold M the geodesic rays can be divided naturally into the following three classes, cf. [8]:

DEFINITION. A geodesic ray $c: [0, \infty) \to M$ is called
(i) bounded if its image $c([0, \infty))$ is a bounded set.
(ii) escaping if c is a proper map, i.e. if $c(t_i)$ diverges for every divergent sequence t_i in $[0, \infty)$.
(iii) oscillating if neither (i) nor (ii) is true.

A complete geodesic $c: \mathbf{R} \to M$ is bounded resp. escaping if both its geodesic rays are. In [6] escaping geodesic rays are called divergent. A geodesic $c: [a, b] \to M$ is called a geodesic loop if $c(a) = c(b)$. c is called simple if $c \mid [a, b)$ is one-to-one. A geodesic $c: \mathbf{R} \to M$ is simple closed if c is non-constant, periodic and if, for its smallest period ω, $c \mid [0, \omega)$ is one-to-one.

2. Escaping geodesics in the presence of closed geodesics

In this section we prove the existence of an escaping geodesic without self-intersections on a complete plane M containing a simple closed geodesic d. Examples for such M are complete planes of finite surface area or, more generally, complete planes containing a horn, cf. [2] and [8].

According to [3], Theorem 2, the existence of such a closed geodesic on M is equivalent to the existence of a bounded geodesic ray or to the existence of a compact, locally concave set D. This last property of M is essential for our proof. Here we can take as compact, locally concave set the disc D bounded by d. The following lemma is crucial:

LEMMA 1. *For* $p, q \in M$ *and some* $\kappa > 0$ *let there be given a continuous homotopy* $g : [0, 1] \to \tilde{\Gamma}^{\kappa}_{p,q}$ *such that* g_0 *and* g_1 *are curves in* $M - D$ *which are not homotopic in* $M - D$. *Then* $\tilde{\Gamma}^{\kappa}_{p,q}$ *contains a geodesic which intersects* D.

Proof. To illustrate the idea we first forget about possible self-intersections and only construct a geodesic $c \in \Gamma := \Gamma^{\kappa}_{p,q}$ intersecting D. Our tool is the Birkhoff curve shortening process $\mathscr{D} : \Gamma \to \Gamma$, cf. [4], Section V, 7. To define $\mathscr{D}$ choose some big $k \in \mathbf{N}$. Then $\mathscr{D} := \mathscr{D}_{k-2} \circ \cdots \circ \mathscr{D}_0$ where $\mathscr{D}_i : \Gamma \to \Gamma$, $0 \leq i \leq k-2$, is defined as follows: For $\gamma \in \Gamma$ we let

$$\mathscr{D}_i\gamma \,\Big|\, \left[\frac{i}{k}, \frac{i+2}{k}\right]$$

be the shortest geodesic from $\gamma(i/k)$ to $\gamma((i+2)/k)$ while $\mathscr{D}_i\gamma$ coincides with γ on the rest of $[0, 1]$. If k is large enough each $\mathscr{D}_i$ is well-defined and continuous. We set $\mathscr{D}^n := \mathscr{D} \circ \mathscr{D}^{n-1}$, $\mathscr{D}^1 := \mathscr{D}$. Then the following is true:

(i) For every $\gamma \in \Gamma$ there exists a convergent subsequence of $\{\mathscr{D}^n\gamma\}_{n \in \mathbf{N}}$ and every limit curve of $\{\mathscr{D}^n\gamma\}_{n \in \mathbf{N}}$ is a geodesic in Γ.

(ii) If $\gamma \in \Gamma$ is contained in $M - D$ then so is $\mathscr{D}\gamma$. Furthermore γ and $\mathscr{D}\gamma$ are homotopic in $M - D$.

Here (i) is the standard property of the Birkhoff curve shortening process while (ii) is due to the fact that D is bounded by a geodesic. Now c is obtained as follows: By (ii) the curves $\mathscr{D}^n g_0$ and $\mathscr{D}^n g_1$ are not homotopic in $M - D$. Since $\mathscr{D}^n \circ g$ is a homotopy from $\mathscr{D}^n g_0$ to $\mathscr{D}^n g_1$ the set $I_n := \{t \in [0, 1] \mid \mathscr{D}^n g_t$ intersects $D\}$ is non-void. I_n is compact and, by (ii), we have $I_n \supseteq I_{n+1}$. Hence there exists $t \in [0, 1]$ such that $\mathscr{D}^n g_t$ intersects D for all $n \in \mathbf{N}$. Now every limit curve c of $\{\mathscr{D}^n g_t\}_{n \in \mathbf{N}}$ has the desired properties.

We now proceed to construct a geodesic $c \in \tilde{\Gamma} := \tilde{\Gamma}^{\kappa}_{p,q}$ intersecting D. This geodesic is produced by a continuous, energy-decreasing deformation $\mathscr{D}^{\sim}_h : \tilde{\Gamma} \to \tilde{\Gamma}$ which has "almost" properties (i) and (ii). To find such deformations is known to be an intricate problem. Here we use an adaptation to the present situation of the classical Lusternik–Schnirelmann deformation $\mathscr{D}^{\sim}$, cf. [7]. $\mathscr{D}^{\sim}$ has been described with great care in [1], and is actually only defined up to the choice of a constant. The following two problems arise:

(1) $\mathscr{D}^{\sim}$ is defined so as to map closed curves without self-intersections to closed curves without self-intersections. When $\mathscr{D}^{\sim}$ is applied to curves with end-points $\mathscr{D}^{\sim}$ may not leave the end-points fixed.

(2) $\mathscr{D}^{\sim}$ fails to have property (ii).

Difficulty (2) can easily be helped since $\mathcal{D}^{\sim}$ almost has property (ii): For $a>0$ set $D^a:=\{p\in M \mid d(p,q)\leq a$ for some $q\in D\}$. Then, given $\varepsilon>0$, we can define $\mathcal{D}^{\sim}$ in such a way that $\mathcal{D}^{\sim}\gamma$ does not intersect D^{ε} if γ does not intersect $D^{2\varepsilon}$. This amounts to an appropriate choice of the constant involved in the definition of $\mathcal{D}^{\sim}\gamma$, cf. [1].

To get around difficulty (1) we slow $\mathcal{D}^{\sim}$ down to the identity as soon as γ comes to lie close to its end-points. This means the following: Choose two small balls B_p and B_q about p and q and a function $h: M\to[0,1]$ which vanishes in a neighborhood of p and q and is identically one outside $B_p\cup B_q$. Choose some very big $k\in\mathbf{N}$. Using the Lusternik–Schnirelmann construction we define deformations $\mathcal{D}_i^{\sim}$, $(3\leq i\leq k-3)$, which replace

$$\gamma \left| \left[\frac{i}{k}, \frac{i}{k}+h\left(\gamma\left(\frac{i}{k}\right)\right)\frac{2}{k}\right]\right.$$

by the geodesic segment $c_{\gamma,i}$ from $\gamma(i/k)$ to

$$\gamma\left(\frac{i}{k}+h\left(\gamma\left(\frac{i}{k}\right)\right)\frac{2}{k}\right)$$

while the rest of γ is so deformed that self-intersections are avoided. This deformation takes place in some small neighborhood of $c_{\gamma,i}$ only. If k is big enough the end-points of γ will never lie in such a neighborhood. We define $\mathcal{D}_h^{\sim}$ by $\mathcal{D}_h^{\sim}=\mathcal{D}_{k-3}^{\sim}\circ\mathcal{D}_{k-4}^{\sim}\circ\cdots\circ\mathcal{D}_3^{\sim}$. Using similar arguments as before we can find $t\in[0,1]$ such that $(\mathcal{D}_h^{\sim})^n g_t$ intersects D^{ε_n} for $\varepsilon_n=\varepsilon\cdot 2^{-n}$. When we restrict a limit curve of $\{(\mathcal{D}_h^{\sim})^n g_t\}_{n\in\mathbf{N}}$ to an appropriate subinterval $[s_0,s_1]\subseteq[0,1]$ we obtain a geodesic without self-intersections which starts in B_p, ends in B_q, intersects D and has energy $\leq\kappa$. If we let the radii of B_p and B_q converge to zero we obtain a geodesic $c\in\tilde{\Gamma}$ intersecting D.

Now we can prove

THEOREM 1. *Let M be a complete plane containing a simple closed geodesic d. Then there exists an escaping geodesic $c:\mathbf{R}\to M$ without self-intersections.*

Proof. c will be obtained as a limit of geodesics constructed by means of Lemma 1. We first construct a sequence of homotopies g^n which satisfy the assumptions of Lemma 1. Take any smooth disc D' containing D in its interior where D is the set bounded by d. Then there exists $\kappa>0$ such that for all $p,q\in\partial D'$ there is a homotopy $h:[0,1]\to\tilde{\Gamma}^{\kappa}_{p,q}$ such that h_0 and h_1 are contained and not homotopic in $M-D$ and such that all the curves h_t are contained in D'. Obviously we can assume that the curves h_t are parametrized proportionally to arc-length. Now choose points $p_n \neq q_n$ at distance $n \in \mathbf{N}$ from D'. There exist geodesics c_n, d_n of length n such that c_n joins p_n to D' while d_n joins D' to q_n. We define a homotopy $g^n : [0, 1] \to \tilde{\Gamma}_{p_n, q_n}$ by requiring that g^n_t be the composition of c_n, h_t and d_n, parametrized proportionally to arc-length. Then $E(g^n_t) \leq (2n + \sqrt{\kappa})^2$ for all $t \in [0, 1]$ and g^n satisfies the assumptions of Lemma 1. Thus there exists a geodesic $\gamma'_n \in \tilde{\Gamma}_{p_n, q_n}$ which intersects D and has length $L(\gamma'_n) = (E(\gamma'_n))^{1/2} \leq 2n + \sqrt{\kappa}$. Let $\gamma_n : [a_n, b_n] \to M$ be a unit speed parametrization of γ'_n such that $a_n < 0 < b_n$ and $\gamma_n(0) \in D$. Then $b_n - a_n = L(\gamma_n) \leq 2n + \sqrt{\kappa}$. Since p_n and q_n are points at distance n from $D' \supseteq D$ we have $-a_n \leq n + \sqrt{\kappa}$ and $b_n \leq n + \sqrt{\kappa}$. Let v be a limit vector of the sequence $\{\dot{\gamma}_n(0)\}$ and define $c(t) = \exp(tv)$. Then $c(t) = \lim \gamma_n(t)$ and c does not have self-intersections. We have

$$d(\gamma_n(t), D) \geq d(\gamma_n(b_n), D) - d(\gamma_n(b_n), \gamma_n(t)) \geq n - (b_n - t) \geq t - \sqrt{\kappa}.$$

Now $d(c(t), D) \geq t - \sqrt{\kappa}$ implies that $c \mid [0, \infty)$ is escaping. An analogous argument shows that $c \mid (-\infty, 0]$ escapes as well.

Remark. Our geodesic c even has the following property which is stronger than being escaping: Both its rays $c \mid [0, \infty)$ and $c \mid (-\infty, 0]$ are almost minimizing in the sense of [6], Definition 7.1; i.e. there exists $A > 0$ such that $d(c(t), c(0)) \geq |t| - A$.

3. Escaping geodesics in the absence of closed geodesics

On a complete plane M the absence of simple closed geodesics has surprisingly strong implications on the geodesic flow, cf. [3], Section 2. As far as self-intersections are concerned the geodesics on M have similar properties as geodesics on complete planes of positive Gaussian curvature, cf. [5]. Hence it is not surprising that [5], Satz 11 has an analogue in our situation:

THEOREM 2. *Let M be a complete plane without simple closed geodesics. Then through every point of M there exists a geodesic $c : \mathbf{R} \to M$ without self-intersections.*

Remark. By [3], Theorem 2, the geodesic rays of c are not bounded.

Proof. According to [3], Corollary 2 a non-constant geodesic $c : \mathbf{R} \to M$ either has no self-intersections or c determines uniquely a simple geodesic loop $c \mid [t_1, t_2]$. Hence we can repeat Cohn-Vossen's proof of [5], Satz 11: Assume that all geodesics through $p \in M$ have self-intersections. For every unit vector $v \in T_pM$ let c_v denote the simple geodesic loop determined by the geodesic $c(t) = \exp(tv)$.

Obviously c_v and c_{-v} bound the same disc but with the opposite orientation. Since this orientation depends on v continuously we obtain a contradiction.

We are not able to conclude that the geodesic c from Theorem 2 is escaping. This is actually part of the following open problem: Can oscillating geodesic rays exist on complete planes without simple closed geodesics?

Regardless of the answer to this question the following proposition completes the proof for the existence of an escaping geodesic without self-intersections on every complete plane:

PROPOSITION. *Let M be a complete plane without simple closed geodesics. Suppose there exists an oscillating geodesic ray $d:[0,\infty)\to M$. Then there exists an escaping geodesic without self-intersections.*

Proof. We first recall that a totally convex set $C \subseteq M$ is a closed set such that every geodesic which joins two points of C is contained in C. The totally convex hull A^c of a set $A \subseteq M$ is the intersection of all totally convex sets containing A. Using this concept and a result from [3] we will find a sequence s_i diverging to ∞ such that $\dot{d}(s_i)$ converges to the initial vector of an escaping geodesic without self-intersections.

Choose a sequence t_i diverging to ∞ such that $\lim d(t_i) =: p$ exists. Let B be a compact ball about p and, for $n \in \mathbf{N}$, let K_n denote the closure of $B \cap d([n,\infty))$. Then the totally convex hull K_n^c of K_n contains $d([t,\infty))$ for some $t>0$. According to [3], Corollary 1, there exist points $q_n \in K_n \cap \partial K_n^c$. Let q be a limit point of the sequence q_n. Since K_n^c is a decreasing sequence of closed sets we have $q \in \bigcap K_n^c =: C$. Now $q_n \in \partial K_n^c$ implies $q \in \partial C$. Since $q_n \in K_n$ we can find a sequence s_i diverging to ∞ such that $\lim \dot{d}(s_i) =: w$ exists and $w \in T_qM$. Then the geodesic $c:\mathbf{R}\to M$ with initial vector $\dot{c}(0)=w$ is contained in C since $c(t)=\lim d(s_i+t)$ for all $t\in\mathbf{R}$. Actually c is contained in the boundary ∂C of the totally convex set C since $q=c(0)\in\partial C$. Hence c is either a simple closed geodesic or an escaping geodesic without self-intersections. The first case is excluded by assumption.

Finally we note that an escaping geodesic without self-intersections can easily be constructed on all complete non-compact Riemannian manifolds which contain a non-separating compact hypersurface or which have at least two ends. In particular the answer to our problem is yes for all complete non-compact surfaces M which are not planes. For such surfaces one can even obtain geodesics without self-intersections which "start" and "end" in the same end of M unless $M \simeq S^1 \times \mathbf{R}$.

Obviously one can ask many interesting questions in this context, even in the 2-dimensional case. We just mention two of them:

1. Are there infinitely many escaping geodesics on every complete plane M, or even one through every point of M?

2. What is the minimal number of escaping geodesics without self-intersections on complete planes?

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