Cylinders in Riemann surfaces

Autor(en): **Randol, Burton**

Objekttyp: **Article**

Zeitschrift: **Commentarii Mathematici Helvetici**

Band (Jahr): **54 (1979)**

PDF erstellt am: **19.05.2024**

Persistenter Link: https://doi.org/10.5169/seals-41557

Comment. Math. Helvetici **54** (1979) 1–5 Birkhäuser Verlag, Basel

Cylinders in Riemann surfaces

BURTON RANDOL

Suppose S is a compact Riemann surface, with a metric of constant curvature -1. Let γ be a simple closed geodesic on S of length L_γ. Then for small d, the family of geodesic segments of length $2d$, perpendicular to and centered on γ, sweeps out a region homeomorphic to a cylinder. As d is enlarged, it must eventually happen that two endpoints from this family of segments intersect. Let $d_0 > 0$ be the smallest value of d for which this occurs. Setting $d = d_0$, and removing the envelope of the endpoints, we shall refer to the resulting set as the cylinder about γ, and denote it by C_γ. Note that C_γ is topologically an open cylinder, and has the property that any curve emanating from γ and of length less than d_0 must lie wholly within C_γ.

Now it can be shown (cf. [3]) that given $c > 0$, there exists $c' > 0$ such that $L_\gamma < c$ implies $\text{Area}\,(C_\gamma) > c'$. The techniques in [3] involve fine points in the theory of canonical fundamental domains and moduli for certain types of discontinuous groups, as well as substantial computation. Because of the interest of the result, it seems worthwhile to present an alternative, somewhat more geometric approach to the theorem, which at the same time leads to an improved result.

Remarks

1. It is possible to give examples to show that $\text{Area}\,(C_\gamma)$ may become arbitrarily small if L_γ becomes large (cf. the end of this paper).
2. (Cf. [3]). The result implies that d_0 is large when L_γ is small. In particular, if $\gamma' \neq \gamma$ is a simple closed geodesic intersecting γ, then a portion of γ' must pass through C_γ, so that L_γ and $L_{\gamma'}$ cannot simultaneously be small.
3. Substantial interest has recently centered on small eigenvalues for the Laplacian on S [1], [5], [6], [8], [9]. Using the fact that small L_γ implies large d_0, one can in some cases construct test functions for λ_1 on S for which the Dirichlet integral is small relative to the L^2 norm. A program of this sort is carried out in [9], in which it is shown, among other things, that if L_γ is small and γ separates S, then λ_1 must be small. We remark that the construction in [8] also leads to eigenvalues of arbitrarily small size.
4. It would be interesting to determine the extent to which results of this type hold for surfaces of variable negative curvature.

We will prove the following result:

THEOREM

Area $(C\gamma) \geq 2L_\gamma \operatorname{csch} \frac{L_\gamma}{2}$.

Proof. At $d = d_0$, one or both of the following events must take place:

1. The endpoints of two segments emanating from the same side of γ meet.
2. The endpoints of two segments emanating from opposite sides of γ meet.

Assume that #1 occurs. Let g be a path composed of two such segments. Now the two segments must join smoothly at their intersection, for if they did not, the point of intersection would look locally like Figure 1.

If we modify g near the intersection in the manner indicated in Figure 2, and call the result g^*, it is evident by the triangle inequality that the length of g^* is strictly less than that of g. It follows that g^* is wholly contained in C_γ, and together with one of the arcs of γ connecting its base-points, bounds a simply-connected quadrilateral. Thus, the attachment to this quadrilateral at its top of a small triangle congruent to the triangle in Figure 2 results in a geodesic triangle, the sum of whose angles is greater than π, which is impossible.

Denote by p and p^* the two points of intersection of g with γ. Lift γ into the hyperbolic disk, in such a way that p goes into the center of the disk, and γ corresponds to a centrally symmetric segment of the horizontal diameter of the disk.

Consider the laterally symmetric quadrilateral figure illustrated in Figure 3, where segments are labelled by length. Figure 3 is drawn as though the top geodesic does not intersect the two geodesic sides within the disk. It is, in fact,

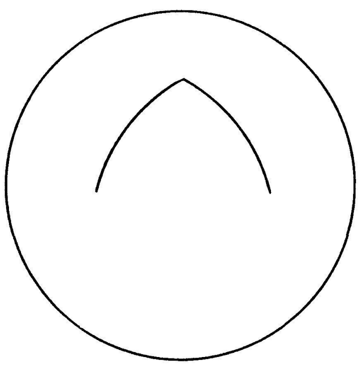

Figure 1

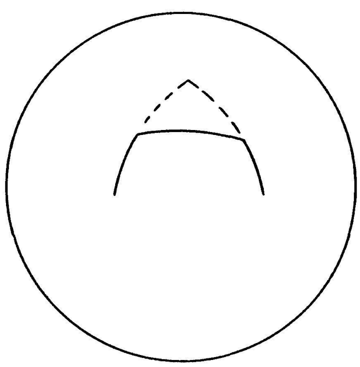

Figure 2

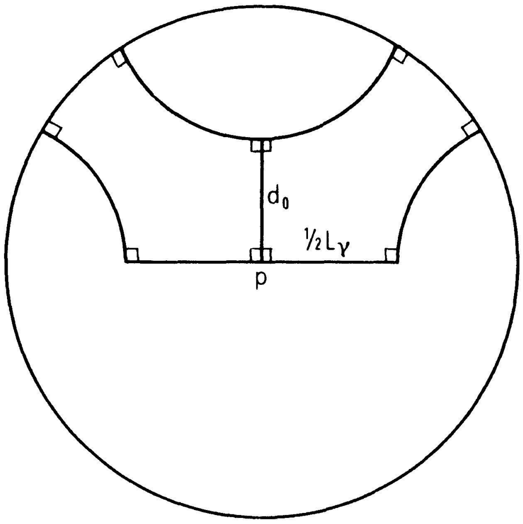

Figure 3

impossible for such intersections to occur. For suppose this happened. Now points of the two geodesic sides having equal distance from the base are the same on S itself, since γ, regarded as an isometry of the disk, identifies the two sides in this manner. It follows that the segment of the top geodesic between the two sides, oriented left to right, would correspond to a closed oriented curve in S, which would, in an obvious way, be freely homotopic to γ itself, with a left to right orientation in the model of Figure 3. Note that the top segment has an intrinsic description in S as the geodesic segment of a certain computable length perpendicular to and centered at the point of g which is d_0 units up from p, and oriented left to right through g when g is traced out from p to p^*. Now if we again consider the configuration of Figure 3, with the difference that the center corresponds to p^* rather than p, and note that in this picture the top segment must be oriented right to left in order to correspond to the top segment of the original figure, we obtain a free homotopy between γ and γ^{-1}, which is impossible. I.e., the top segment in Figure 3 does not intersect the sides.

Now by standard hyperbolic trigonometry ([7], p. 87), the condition for the top segment not to meet the sides is $(\cosh d_0)(\tanh L_\gamma/2) \geq 1$. On the other hand, the area of the cylinder of length $2d_0$ centered on γ is $2L_\gamma \sinh d_0$, since in Fermi coordinates (x, y), the area element is $\cosh y \, dx \, dy$. Thus, since $\cosh d_0 \geq (\tanh L_\gamma/2)^{-1}$, and $\sinh^2 d_0 = \cosh^2 d_0 - 1$, we find that

$$\text{Area}\,(C_\gamma) \geq 2L_\gamma \operatorname{csch} \frac{L_\gamma}{2},$$

which proves the theorem in case #1.

We next deal with case #2. i.e., assume that the two segments of length d_0

emanating perpendicularly from γ in opposite directions meet at their endpoints. As before, the intersection must be smooth, for if not, we could by the previous procedure find two broken geodesics, each of length less than d_0, emanating from γ in opposite directions, and meeting at their endpoints. This is impossible, since the endpoints must be two distinct points in a topological cylinder.

Now cut S along γ, which gives a bordered surface whose boundary consists of two closed geodesics γ_1 and γ_2, both isometric to γ. Let p and p^* be the points on γ_1 and γ_2, respectively, which correspond to the base points of the two segments of length d_0. We again consider the configuration of Figure 3, and assume that the top segment intersects the two sides within the disk. By the previous procedure, this would give rise to a free homotopy between γ_1 and γ_2, which is impossible, since S is of genus greater than 1. Continuing the previous arguments, we conclude that in this case as before,

$$\text{Area}\,(C_\gamma) \geq 2L_\gamma \operatorname{csch} \frac{L_\gamma}{2},$$

which proves the theorem.

Finally, in order to justify Remark #1, we select numbers d_0 and L_γ such that

$$(\cosh d_0)\left(\tanh \frac{L_\gamma}{2}\right) = 1,$$

and construct the geodesic quadrilateral illustrated in Figure 4, in which the vertex angles are zero. Then the area of the rectangle R swept out by segments of length $2d_0$ perpendicular to and centered on the central segment of length L_γ is $2L_\gamma \operatorname{csch} L_\gamma/2$.

If, now, we increase d_0 very slightly, the area of the corresponding R will change very little, but the four geodesic sides will no longer meet. If we draw perpendicular geodesics as illustrated in Figure 5, we obtain a geodesic octagon

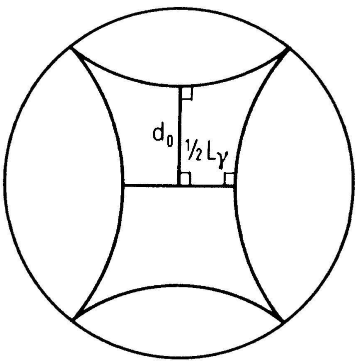

Figure 4

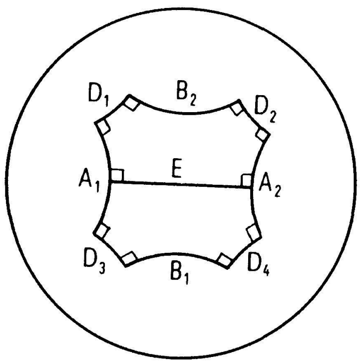

Figure 5

with central segment E. Take another copy of this figure, with corresponding sides A'_1, A'_2, B'_1, B'_2, D'_1, D'_2, D'_3, D'_4, and central segment E', and attach the two copies along (A_1, A'_1), (A_2, A'_2), (B_1, B'_1), and (B_2, B'_2), while making $D_1 \cup D'_1$, $D_2 \cup D'_2$, $D_3 \cup D'_3$, $D_4 \cup D'_4$, and $E \cup E'$ into closed geodesics. The result is a 4-holed sphere for which the cylinder based on the closed geodesic $E \cup E'$ has area as close to $4L_\gamma \operatorname{csch} L_\gamma/2$ as we wish. Since the last quantity tends to zero as $L_\gamma \to \infty$, we can, by the process of attaching two such figures along the holes, obtain a sequence of closed surfaces having the desired property.

We conclude with some comments on [3]. It is stated in [3] that Area $(C_\gamma) \geq 8/\sqrt{5}$, regardless of the length of γ. As the last example shows, this is not the case. The reason for this is that the computations in [3] ignore terms which are infinitesimal only when L_γ is infinitesimal. Accordingly, the result in [3] should be regarded as applying only to the case of infinitesimal L_γ. Additionally, the statement in [3] that even with this provision, the constant $8/\sqrt{5}$ is best possible requires emendation, since in the case of infinitesimal L_γ, the result of the present paper becomes Area $(C_\gamma) \geq 4 > 8/\sqrt{5}$.

REFERENCES

[1] BUSER, P. *Riemannsche Flächen mit Eigenwerten in* $(0, \frac{1}{4})$, Comment. Math. Helv. *52* (1977), 25–34.
[2] FRICKE, R. and KLEIN, F., *Vorlesungen über die Theorie der automorphen Funktionen*, B. G. Teubner, 1965.
[3] KEEN, L., *Collars on Riemenn surfaces*, Annals of Mathematics Studies 79, Princeton University Press, Princeton, New Jersey, 1974.
——, *Intrinsic moduli on Riemann surfaces*, Ann. of Math. *84* (1966), 404–420.
[5] HUBER, H., *Über den ersten Eigenwert des Laplace-Operators auf kompakten Riemannschen Flächen*, Comment. Math. Helv. *49* (1974), 251–259.
[6] ——, *Über die Eigenwerte des Laplace-Operators auf kompakten Riemannschen Flächen*, Comment. Math. Helv. *51* (1976), 215–231.
[7] MESCHKOWSKI, H., *Noneuclidean geometry*, Academic Press, 1964.
[8] RANDOL, B., *Small eigenvalues of the Laplace operator on compact Riemann surfaces*, Bull. Amer. Math. Soc. *80* (1974), 996–1,000.
[9] SCHOEN, R., WOLPERT, S., and YAU, S. T., *On the first eigenvalue of a compact Riemann surface*, preprint.

CUNY-Graduate Center
33 West 42 Street
New York, New York 10036

Received September 12, 1977