Dirichlet regions in manifolds without conjugate points.

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Dirichlet regions in manifolds without conjugate points

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1. Introduction

The aim of this paper is to study Dirichlet tessellations of simply connected complete riemannian manifolds without conjugate points. In particular, we will describe the geometrical and topological structure of the boundary of a single Dirichlet region. Throughout this paper M will always denote an n-dimensional, simply connected complete riemannian manifold without conjugate points. Let $\langle\ ,\ \rangle$ denote the riemannian metric of M and d the distance function induced by $\langle\ ,\ \rangle$. Recall that M has no conjugate points if and only if every pair of distinct points of M can be joined by a unique geodesic segment (up to parametrization).

For a given discrete subset $D=\{p_i;\, i\in I\}$ of M, the *Dirichlet regions* F_i are defined by

$$F_i=\{p\in M;\, d(p, p_i)<d(p, p_k) \text{ for all } k\neq i\},$$

and the collection $T=\{F_i;\, i\in I\}$ is called the *Dirichlet tessellation* induced by D. For simplicity we state our main results for F_0.

THEOREM 3.6. $\operatorname{bd} F_0$ *is an* $(n-1)$-*dimensional topological submanifold of* M. *Moreover,* $\operatorname{bd} F_0$ *admits a differential structure and is thus triangulable.*

For a more detailed study of $\operatorname{bd} F_0$ we set

$$S_i=\{p\in M;\, d(p, p_0)=d(p, p_i)<d(p, p_k) \text{ for all } k\neq 0, i\}$$

and

$$B_i=\{p\in M;\, d(p, p_0)=d(p, p_i)\leq d(p, p_k) \text{ for all } k\neq 0, i\}.$$

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We call S_i a *side* of F_0 if S_i is nonempty. Except for the case that $\dim M = 2$ (*cf.* Eberlein [3]), the sides may fail to be connected and also $\mathrm{cl}\, S_i$ may be different from B_i. However we find

THEOREM 3.14. $\mathrm{bd}\, F_0 = \bigcup \mathrm{cl}\, S_i$.

Often the discrete set D is given as an orbit of a discrete group of isometries of M. Then the Dirichlet regions are all congruent and provide constructable fundamental regions for the group action. As an application of our study of the sides of a Dirichlet region we obtain

THEOREM 4.7. *Let $\Gamma = \{\psi_i;\, i \in I\}$ be a discrete group of isometries of M and let F_0 be the Dirichlet region based at p_0 with respect to $\{p_i = \psi_i(p_0);\, i \in I\}$ where p_0 is not a fixed point of Γ.*
Then the set

$$\Sigma = \{\psi_i \in \Gamma;\ S_i \text{ is a side of } F_0\}$$

generates Γ. In particular, if F_0 has only finitely many sides, then Γ is finitely generated.

The paper is organized as follows. Section 2 contains a summary of the properties of bisectors and half spaces needed in the sequel. In Section 3 we study the Dirichlet tessellation induced by an arbitrary discrete set $D \subset M$. First we develop along classical lines the elementary properties of Dirichlet tessellations (Prop. 3.3 and 3.4) which are well known in many particular cases. The main part of Section 3 deals with the boundary and the sides of a single Dirichlet region and contains the first two theorems stated above. Finally in Section 4 we investigate additional properties of the Dirichlet tessellation arising from the congruence of the Dirichlet regions when D is given as the orbit of a nonfixed point of a discrete group of isometries.

We would like to thank P. Eberlein for his help in connection with Theorem 3.14.

2. Bisectors and half spaces

We begin with some general properties of the bisectors and the half spaces determined by them. Throughout this section we fix three distinct points p, q, r of M.

DEFINITION 2.1.

(1) We define a function $h: M \to \mathbf{R}$ by $h(m) = d(m, p) - d(m, q)$.
(2) $M(p, q) = h^{-1}(0)$ is called the *bisector* of p and q.
(3) $H(p, q) = \{m \in M; h(m) < 0\}$ denotes the open *half space* determined by $M(p, q)$ containing p. Similarly $H(q, p)$ denotes the half space containing q.
(4) We set $M(p, q, r) = M(p, q) \cap M(p, r)$.

First we note

PROPOSITION 2.2.

(1) (Witt [12], Ozols [10], Prop. 2.4) *$M(p, q)$ is an $(n-1)$-dimensional differentiable submanifold of M.*
(2) (Ozols [10], Prop. 2.6) *$M(p, q)$ and $M(p, r)$ intersect transversely whenever they intersect at all.*
(3) *$M(p, q, r)$ is either empty or is an $(n-2)$-dimensional differentiable submanifold of M.*

Proof. (1) We have to show that grad h never vanishes on $M(p, q)$. Let d_p and d_q denote the distance functions given by $d_p(m) = d(m, p)$ and $d_q(m) = d(m, q)$. Now suppose $\operatorname{grad}_m h = 0$ for some $m \in M(p, q)$. This implies $\operatorname{grad}_m d_p = \operatorname{grad}_m d_q$ and thus $p = q$.

(2) Let d_p, d_q, d_r be defined as in (1). We have to show that $\operatorname{grad}_m d_p - \operatorname{grad}_m d_q$ and $\operatorname{grad}_m d_p - \operatorname{grad}_m d_r$ are linearly independent in T_mM, where m is any point of $M(p, q) \cap M(p, r)$. Suppose this is not the case. Then $\operatorname{grad}_m d_p$, $\operatorname{grad}_m d_q$, and $\operatorname{grad}_m d_r$ must lie on an affine line in T_mM. This is impossible for three distinct unit vectors. Clearly (2) implies (3).

A more detailed investigation (Im Hof [7], Prop. 2.6) shows that $M(p, q)$ is diffeomorphic to $\mathbf{R}^{n-1}$. Also, for dim $M = 2$, the sets $M(p, q, r)$ are either empty or consist of a single point (Eberlein [3], Prop. 2.8, Ehrlich-Im Hof [4], Cor. 2).

In manifolds of constant sectional curvature, bisectors are totally geodesic and the half spaces determined by them are convex. These facts are widely used in the theory of fuchsian groups. However, in more general cases, the bisectors are no longer totally geodesic, nor are the half spaces they determine convex. Indeed, all bisectors $M(p, q)$ of M are totally geodesic if and only if M has constant sectional curvature (Busemann [2], Thm. 47.4, p. 331).

A property which is weaker than the convexity of the half spaces, but which is still very useful, is the starlikeness of the half spaces $H(p, q)$ with respect to p. To show that this starlikeness holds in our context (Prop. 2.6(2) below), we consider the behavior of geodesic rays through p with respect to $M(p, q)$. Let c_{pq} denote the unique geodesic with $c_{pq}(0) = p$ and $c_{pq}(d) = q$, where $d = (p, q)$.

PROPOSITION 2.3.

(1) *Let $c:[0, \infty) \to M$ be a geodesic ray with $c(0) = p$. If the ray c is not contained in c_{pq}, then $h(c(t))$ is strictly increasing for all $t \geqslant 0$. If $c = c_{pq} \| [0, \infty)$, then $h(c(t))$ is strictly increasing for $0 \leqslant t \leqslant d$ and $h(c(t)) = d$ for $t \geqslant d$. If $\dot{c}(0) = -\dot{c}_{pq}(0)$, then $h(c(t)) = -d$ for all $t \geqslant 0$.*

(2) *Let $c:[0, \infty) \to M$ be a geodesic ray with $c(0) = p$. Then c intersects $M(p, q)$ at most once and transversely.*

(3) *Let $c:\mathbf{R} \to M$ be a geodesic with $c(0) = p$. Then c intersects $M(p, q)$ at most once.*

Proof. (1) We may assume $\|\dot{c}\| = 1$. Consider first a geodesic ray c with $\dot{c}(0) \neq \pm \dot{c}_{pq}(0)$. Suppose $h(c(s)) \geqslant h(c(t))$ for $0 \leqslant s < t$. This implies

$$d(q, c(t)) \geqslant d(q, c(s)) + d(c(s), c(t)),$$

hence

$$d(q, c(t)) = d(q, c(s)) + d(c(s), c(t))$$

by the triangle inequality. But equality occurs only when the three points q, $c(s)$, $c(t)$ lie on a geodesic. This contradicts our assumption on c.

The behavior of h along the geodesic c_{pq} is obvious.

(2) Uniqueness of the intersections of c with $M(p, q)$ immediately follows from the monotonicity of $h \circ c$ proved in (1). Now we assume that a geodesic ray c intersects $M(p, q)$ at a point $m = c(t) \in M(p, q)$ and we show transversality. Suppose the intersection is not transverse, i.e., $\langle \dot{c}(t), \operatorname{grad}_m h \rangle = 0$. Using the same notation as in the proof of Proposition 2.2 we may write $\operatorname{grad} h = \operatorname{grad} d_p - \operatorname{grad} d_p$. Moreover, $\operatorname{grad}_m d_p = \dot{c}(t)$. Then our assumption implies $\langle \operatorname{grad}_m d_p, \operatorname{grad}_m d_q \rangle = \langle \operatorname{grad}_m d_p, \operatorname{grad}_m d_p \rangle = 1$, hence $\operatorname{grad}_m d_p = \operatorname{grad}_m d_q$. This is impossible, since $m \in M(p, q)$ and $p \neq q$.

(3) Suppose $c:\mathbf{R} \to M$ is a geodesic with $c(0) = p$ intersecting $M(p, q)$ in two points $c(s)$ and $c(t)$. By (2) we may assume $s < 0 < t$. Then we have

$$d(c(s), c(t)) = d(c(s), p) + d(p, c(t)) = d(c(s), q) + d(q, c(t)).$$

This equality is only possible if q lies on the geodesic segment between $c(s)$ and $c(t)$. This would imply $p = q$.

Remark. Uniqueness of the intersections of geodesic rays through p with $M(p, q)$ (or with bd F_0, see Prop. 3.3(3) below) is well known (Busemann [1], 2.12; Ozols [10], Prop. 3.2; for $n = 2$, Eberlein [3], Prop. 2.6). Transversality has first been proved in Im Hof [7], Lemma 2.5.

Now we define the concept of a strictly starlike subset of M. For this purpose we set

$$(m, m') = \{c(t);\ t \in (0, 1)\},$$

where $c:[0, 1] \to M$ is the geodesic segment joining m and m'.

DEFINITION 2.4. A subset F of M is *strictly starlike* with respect to a point $m \in F$ if for every point $m' \in \operatorname{cl} F$ the segment (m, m') is contained in int F.

Knowing that a set F is strictly starlike rather than just starlike with respect to a point has implications about the topology of F. Explicitly

LEMMA 2.5. *Let F and G be two subsets of M with $F \subset G$ and let m be a point of F. Suppose that for each $m' \in G$ the segment (m, m') is contained in F. Then $G \subset \operatorname{cl} F$ and* $\operatorname{int} G \subset F$.

Proof. Let m' be a point of G and let U be a neighborhood of m'. The segment (m, m') lies in F and contains points of U. Hence $m' \in \operatorname{cl} F$.

Now let m' be a point of int G and let U be a neighborhood of m' contained in G. The segment (m, m') can be extended to a segment (m, m'') such that $m'' \in U$ and $m' \in (m, m'')$. Since $m'' \in G$, the whole segment (m, m'') lies in F. In particular, $m' \in F$.

We are now able to show

PROPOSITION 2.6.

(1) $\operatorname{cl} H(p, q) = H(p, q) \cup M(p, q)$,
$\operatorname{int} \operatorname{cl} H(p, q) = H(p, q)$,
$\operatorname{bd} H(p, q) = M(p, q)$.
(2) *$H(p, q)$ is strictly starlike with respect to p.*
(3) *The half spaces $H(p, q)$ and $H(q, p)$ are the connected components of $M - M(p, q)$.*

Proof. Obviously $\operatorname{cl} H(p, q) \subset H(p, q) \cup M(p, q)$ and $H(p, q) \subset \operatorname{int} (H(p, q) \cup M(p, q))$. By Proposition 2.3(1), the hypothesis of Lemma 2.5 is satisfied for $H(p, q)$, $H(p, q) \cup M(p, q)$, and p. Thus we may conclude the proof of (1). Since $\operatorname{cl} H(p, q) = H(p, q) \cup M(p, q)$, Proposition 2.3(1) also implies (2) which then implies (3).

We now consider the set of geodesic rays through p intersecting $M(p, q)$. Denote the unit tangent sphere at p by S. For each $v \in S$ we denote by $c_v : [0, \infty) \to M$ the geodesic ray with $c_v(0) = p$ and $\dot{c}_v(0) = v$. Since each ray c_v intersects $M(p, q)$ at most once, we may define $A = \{v \in S; c_v \text{ intersects } M(p, q)\}$ and two functions $\phi : A \to M(p, q)$ and $\varphi : A \to \mathbf{R}$ by $\phi(v) = c_v(0, \infty) \cap M(p, q)$ and $\varphi(v) = d(p, \phi(v))$. The transversality of the intersection of c_v with $M(p, q)$ implies

PROPOSITION 2.7.

(1) *A is open in S.*
(2) *$\phi : A \to M(p, q)$ is a diffeomorphism.*
(3) *$\varphi : A \to \mathbf{R}$ is differentiable.*

Proof. Let π denote the projection of $M - \{p\}$ onto S defined by $\pi(m) = v$, where $m = c_v(t)$ for suitable t. Since $M(p, q)$ is an $(n-1)$-dimensional submanifold transverse to the geodesic rays c_v, i.e., transverse to the fibers of π, the map $\pi | M(p, q) : M(p, q) \to S$ is a local diffeomorphism. As $A = \pi(M(p, q))$, property (1) holds. Now $\phi = (\pi | M(p, q))^{-1}$ and $\pi | M(p, q) : M(p, q) \to A$ is bijective. This implies (2) and (3).

3. The Dirichlet tessellation induced by a discrete set

In this section we consider an arbitrary discrete set $D \subset M$. By definition, D has no points of accumulation in M, or equivalently, D is locally finite, i.e., every compact set of M contains only finitely many points of D. In particular, D is at most countable and we may thus once and for all fix the notation

$$D = \{p_0, p_1, p_2, \ldots\} = \{p_i;\, i \in I\} \subset M,$$

where I is a suitable set of indices.

The discrete set D may be finite or infinite, but we will always assume that it contains at least two points.

DEFINITION 3.1. Let $D = \{p_i; i \in I\}$ be a discrete set of M.

(1) The *Dirichlet region* with basepoint p_i is the set

$$F_i = \{p \in M;\, d(p, p_i) < d(p, p_k) \text{ for all } k \neq i\}.$$

(2) The collection $T = \{F_i; i \in I\}$ is called the *Dirichlet tessellation* induced by D.

The purpose of this section is to investigate the properties of a single Dirichlet region as well as those of the whole tessellation with respect to a given discrete set D. Whenever we deal with a single region, we may restrict our attention to F_0. The following notation will be used throughout this section.

DEFINITION 3.2. For $i \in I$, $i \neq 0$ we set

(1) $M_i = M(p_0, p_i)$, the bisector of p_0 and p_i.
(2) $H_i = H(p_0, p_i)$, the half space determined by M_i containing p_0.
Using this notation we have $F_0 = \cap H_i$.

The basic properties of a single Dirichlet region are summarized in

PROPOSITION 3.3. *Let F_0 be the Dirichlet region with basepoint $p_0 \in D$. Then*

(1) *F_0 is nonempty and open in M.*
(2) $\mathrm{cl}\, F_0 = \{p \in M;\ d(p, p_0) \leq d(p, p_i)$ *for all* $i \neq 0\}$, *and* $\mathrm{int\,cl}\, F_0 = F_0$.
(3) *F_0 is strictly starlike with respect to p_0.*

Proof. Obviously $F_0 \neq \emptyset$. Now let p be any point of F_0 and denote by d_k the distance $d(p, p_k)$. Since $p \in F_0$ we have $d_0 < d_k$ for all $k \neq 0$. The discreteness of D implies that there exists an index $i \in I$ with $d_0 < d_i \leq d_k$ for all $k \neq 0, i$. Now we choose $\varepsilon = (d_i - d_0)/2$. By the triangle inequality the open ball $U(p, \varepsilon) = \{q \in M;\ d(q, p) < \varepsilon\}$ is contained in F_0. This proves (1).

For the proof of (2) let us write

$$G_0 = \{p \in M;\ d(p, p_0) \leq d(p, p_i) \text{ for all } i \neq 0\}.$$

Clearly $F_0 \subset G_0$. Since $G_0 = \cap\, \mathrm{cl}\, H_i$, G_0 is closed in M and we have $\mathrm{cl}\, F_0 \subset G_0$ as well as $F_0 \subset \mathrm{int}\, G_0$. Since all the half spaces H_i are strictly starlike with respect to p_0, the segment (p_0, p) is contained in F_0 for all $p \in G_0$. Using Lemma 2.5 we may then conclude the proof of (2).

Since $\mathrm{cl}\, F_0 = G_0$, we also obtain (3).

We now state the fundamental property of the Dirichlet tessellation.

PROPOSITION 3.4. *Let $T = \{F_i; i \in I\}$ be the Dirichlet tessellation induced by the discrete set D. Then*
(1) $F_i \cap \mathrm{cl}\, F_j = \emptyset$ *for* $i \neq j$.
(2) $M = \cup\, \mathrm{cl}\, F_i$, *and the covering is locally finite.*

Proof. (1) is obvious. For the proof of (2) let p be a point of M and denote by d_k the distance $d(p, p_k)$. Since D is discrete, there exists an index $i \in I$ such that $d_i \leq d_k$ for all $k \neq i$. This implies $p \in \mathrm{cl}\, F_i$.

Now let K be a compact set in M, e.g., the closed ball $K = B(p_0, R) = \{q \in M; d(q, p_0) \leq R\}$. Assume $p \in \mathrm{cl}\, F_i \cap K$. Then $d(p, p_i) \leq d(p, p_0) \leq R$ and thus $d(p_0, p_i) \leq d(p_0, p) + d(p, p_i) \leq 2R$. This is possible only for finitely many $i \in I$.

Now we begin a more detailed study of the boundary of a single Dirichlet region. Again we restrict our attention to F_0. By Proposition 3.3(2) we have $\mathrm{bd}\, F_0 = \mathrm{cl}\, F_0 - F_0 = \bigcup B_i$, where

$$B_i = \mathrm{cl}\, F_0 \cap M_i = \{p \in M; d(p, p_0) = d(p, p_i) \leq d(p, p_k) \text{ for all } k \neq 0, i\}.$$

Our investigation of $\mathrm{bd}\, F_0$ is based on a study of the geodesic rays emanating from p_0. Let S denote the unit tangent sphere at p_0. For $v \in S$ let $c_v : [0, \infty) \to M$ be the geodesic ray with $c_v(0) = p_0$ and $\dot{c}_v(0) = v$. According to Proposition 3.3(3), each ray c_v intersects $\mathrm{bd}\, F_0$ at most once. Thus we may define $A = \{v \in S; c_v$ intersects $\mathrm{bd}\, F_0\}$ and two functions $\phi : A \to \mathrm{bd}\, F_0$ and $\varphi : A \to \mathbf{R}$ by $\phi(v) = c_v(0, \infty) \cap \mathrm{bd}\, F_0$ and $\varphi(v) = d(p_0, \phi(v))$.

PROPOSITION 3.5.
(1) *A is open in S.*
(2) *$\varphi : A \to \mathbf{R}$ is continuous.*
(3) *$\phi : A \to \mathrm{bd}\, F_0$ is a homeomorphism with respect to the induced topology on* $\mathrm{bd}\, F_0 \subset M$.

Proof. First we recall the analogous construction for a single bisector. Let A_i be the set $\{v \in S; c_v$ intersects $M_i\}$ and define $\phi_i : A_i \to M_i$ and $\varphi_i : A_i \to \mathbf{R}$ by $\phi_i(v) = c_v(0, \infty) \cap M_i$ and $\varphi_i(v) = d(p_0, \phi_i(v))$. According to Proposition 2.7, A_i is open in S, φ_i is differentiable, and ϕ_i is a diffeomorphism.

Now consider $v \in A$. Since $\phi(v) \in \mathrm{bd}\, F_0 = \bigcup B_i$, there is some $i \in I$ such that $\phi(v) \in M_i$. Hence $v \in A_i$, $\phi(v) = \phi_i(v)$, and $\varphi(v) = \varphi_i(v)$. Conversely, if $v \in A_i$, then $\phi_i(v) \in M_i$. Thus $\phi_i(v) \notin F_0$ and so the geodesic segment from p_0 to $\phi_i(v)$ must intersect $\mathrm{bd}\, F_0$. Therefore $v \in A$ and $\varphi(v) \leq \varphi_i(v)$. These arguments show that $A = \bigcup A_i$ and $\varphi = \min \varphi_i$. This implies at once that A is open in S and that φ is upper-semicontinuous.

In order to show that φ is continuous, we choose an element $v \in A$ and a sequence $\{v_\alpha\} \subset A$ converging to v. Since φ is upper-semicontinuous, the sequence $\{\varphi(v_\alpha)\}$ is bounded. Thus we may assume that $\{\varphi(v_\alpha)\}$ converges to a number $d \in \mathbf{R}$. It remains to show that $\varphi(v) = d$.

Since $\phi(v_\alpha) = c_{v_\alpha}(\varphi(v_\alpha)) \in \operatorname{bd} F_0$ and $\operatorname{bd} F_0$ is closed, the sequence $\{\phi(v_\alpha)\}$ converges to $c_v(d) \in \operatorname{bd} F_0$. This implies that $\phi(v) = c_v(d)$, hence $\varphi(v) = d$.

Using geodesic polar coordinates at p_0, we may regard $\phi: A \to \operatorname{bd} F_0$ as the graph of the continuous function φ. Therefore ϕ is a homeomorphism.

As an immediate consequence of Proposition 3.5 we have

THEOREM 3.6. $\operatorname{bd} F_0$ *is an* $(n-1)$*-dimensional topological submanifold of* M. *Moreover, the homeomorphism* $\phi: A \to \operatorname{bd} F_0$ *induces a differential structure on* $\operatorname{bd} F_0$. *Hence* $\operatorname{bd} F_0$ *is triangulable.*

Recall the presentation $\operatorname{bd} F_0 = \bigcup B_i$. Our aim is to simplify this presentation by omitting those members of the collection $\{B_i\}$ which are redundant. Whenever B_i contains a point p which is not contained in any B_k for $k \neq i$, then certainly B_i may not be omitted. This motivates the following

DEFINITION 3.7.
(1) $S_i = \{p \in M;\ d(p, p_0) = d(p, p_i) < d(p, p_k)$ for all $k \neq 0, i\}$.
(2) If $S_i \neq \varnothing$, we call S_i a *side* of F_0.

Remark. This definition of sides is identical with Ozols' definition of faces (*cf.* Ozols [10], p. 225) and essentially equivalent to Eberlein's definition of bounding sides (*cf.* Eberlein [3], Def. 2.2). If $\dim M = 2$, arguments of Eberlein (*loc. cit.*, Lemma 2.11, p. 38) show that the sides are connected. An essential ingredient in his proof is the fact noted above that for $\dim M = 2$, the sets $M(p, q, r)$ are either empty or consist of a single point. However, examples show that the sides may fail to be connected for $\dim M \geq 3$.

The next lemma is crucial for the fitting together of neighboring Dirichlet regions.

LEMMA 3.8.
(1) *If* S_i *is a side of* F_0, *then it is also a side of* F_i.
(2) $F_0 \cup S_i \cup F_i$ *is open in* M.

Proof. (1) is an obvious consequence of the definition of sides. For the proof of (2) it suffices to show that each point of S_i has a neighborhood contained in $F_0 \cup S_i \cup F_i$. Let E_i be the Dirichlet region based at p_0 with respect to $D - \{p_i\}$. Then E_i is open, and it is easily verified that $S_i \subset E_i \subset F_0 \cup S_i \cup F_i$.

The following result is a topological characterization of the sides of $\operatorname{bd} F_0$.

PROPOSITION 3.9. $S_i = \mathrm{int}_{\mathrm{bd}\, F_0} B_i = \mathrm{int}_{M_i} B_i$.

Proof. Recall $S_i \subset B_i = \mathrm{bd}\, F_0 \cap M_i$. We first show that $\mathrm{int}_{\mathrm{bd}\, F_0} B_i = \mathrm{int}_{M_i} B_i$. It suffices to prove that a set $U \subset B_i$ which is open in $\mathrm{bd}\, F_0$ is also open in M_i, and vice-versa. Let $\phi : A \to \mathrm{bd}\, F_0$ and $\phi_i : A_i \to M_i$ be the maps studied in Proposition 3.5. Given $U \subset B_i$ we set $V = \phi^{-1}(U) \subset A$ and $V_i = \phi_i^{-1}(U) \subset A_i$. Since $U \subset B_i = \mathrm{bd}\, F_0 \cap M_i$, the sets V and V_i coincide, and so do the maps $\phi \mid V$ and $\phi_i \mid V_i$. Now it is clear that U is open in $\mathrm{bd}\, F_0$ if and only if it is open in M_i. From now on we will write $\mathrm{int}\, B_i$ instead of $\mathrm{int}_{\mathrm{bd}\, F_0} B_i$ (or $\mathrm{int}_{M_i} B_i$).

Next we show that $S_i = \mathrm{int}\, B_i$. Let p be a point of S_i. According to Lemma 3.8(2), there is a neighborhood U of p in M which is contained in $F_0 \cup S_i \cup F_i$. Then $U \cap M_i = U \cap S_i$ and $V = U \cap M_i$ is an M_i-neighborhood of p contained in S_i. Hence p is an interior point of B_i.

Conversely, let p be a point of $B_i - S_i$. We claim that p cannot be an interior point of B_i (with respect to M_i). Let U be any M_i-neighborhood of p. We will construct a point $q \in U$ which does not belong to B_i. Since $p \in B_i - S_i$, there is an index $k \neq i$ such that $p \in B_i \cap B_k \subset M_i \cap M_k$. Let h_k denote the function defined by $h_k(p) = d(p, p_0) - d(p, p_k)$. Since M_i and M_k intersect transversely, $\mathrm{grad}\,(h_k \mid M_i)$ does not vanish on $M_i \cap M_k$. Let μ be an integral curve of $\mathrm{grad}\,(h_k \mid M_i)$ through p. Then for sufficiently small $\varepsilon > 0$ the point $q = \mu(\varepsilon)$ lies in U, but $h_k(q) > 0$, so q lies outside $\mathrm{cl}\, F_0$.

As a consequence of Propositions 3.9 and 2.7(2) we have

COROLLARY 3.10. *S_i is an open submanifold of* $\mathrm{bd}\, F_0$ *and of M_i. The differential structures of* $\mathrm{bd}\, F_0$ *(as given by Theorem* 3.6) *and of M_i (as a submanifold of M) coincide on S_i.*

DEFINITION 3.11.

(1) A point $p \in \mathrm{bd}\, F_0$ is called a *regular boundary point* of F_0 if it belongs to a side of F_0. The set of regular boundary points of F_0 is denoted by $\mathrm{reg\,bd}\, F_0$.

(2) The complement $\mathrm{bd}\, F_0 - \mathrm{reg\,bd}\, F_0$ is called the set of *singular boundary points* of F_0 and denoted by $\mathrm{sing\,bd}\, F_0$.

PROPOSITION 3.12.

(1) $\mathrm{sing\,bd}\, F_0$ *is closed in* $\mathrm{bd}\, F_0$.

(2) *The topological dimension of* $\mathrm{sing\,bd}\, F_0$ *does not exceed* $n-2$.

Proof. We have the presentation $\mathrm{sing\,bd}\, F_0 = \bigcup (\mathrm{bd}\, F_0 \cap M_i \cap M_j)$, where the union is taken over all indices i, j with $0 \neq i \neq j \neq 0$. Each of the sets $\mathrm{bd}\, F_0 \cap M_i \cap M_j$ is closed in $\mathrm{bd}\, F_0$, and their union is locally finite. Thus $\mathrm{sing\,bd}\, F_0$ is closed in $\mathrm{bd}\, F_0$.

By Proposition 2.2(3), $M_i \cap M_j$ is either empty or is an $(n-2)$-dimensional submanifold of M. Hence $\mathrm{bd}\, F_0 \cap M_i \cap M_j$ has topological dimension at most $n-2$ (*cf.* Hurewicz-Wallman [6], Thm. III.1, p. 26). As a countable union of such spaces, $\mathrm{bd}\, F_0$ itself has topological dimension at most $n-2$ (*cf.* Hurewicz-Wallman [6], Thm. III.2, p. 30).

COROLLARY 3.13. $\mathrm{reg}\,\mathrm{bd}\, F_0$ *is open and dense in* $\mathrm{bd}\, F_0$.

Proof. It suffices to observe that $\mathrm{sing}\,\mathrm{bd}\, F_0$ cannot contain an open subset of $\mathrm{bd}\, F_0$, since $\dim(\mathrm{bd}\, F_0) = n-1$, whereas $\dim(\mathrm{sing}\,\mathrm{bd}\, F_0) \leq n-2$.

Remark. For Dirichlet regions induced by a discrete group acting freely and isometrically on M, Corollary 3.13 is a consequence of Sugahara [11], Theorem A. A proof for general discrete groups using the Baire Category Theorem was communicated to us by Eberlein (personal communication).

The closures of S_i with respect to $\mathrm{bd}\, F_0$, M_i, or M, all coincide. In the following theorem we may therefore use the notation $\mathrm{cl}\, S_i$ without ambiguity.

THEOREM 3.14. $\mathrm{bd}\, F_0 = \bigcup \mathrm{cl}\, S_i$.

Proof. Let p be a point of $\mathrm{bd}\, F_0$. If p is a regular boundary point, then $p \in S_i$ for some i. In general, choose a basis $\{U_\alpha\}$ of $\mathrm{bd}\, F_0$-neighborhoods with $p \in U_{\alpha+1} \subset U_\alpha$. According to Corollary 3.13, each U_α contains a regular boundary point q_α belonging to some side S_{i_α}. The sequence $\{q_\alpha\}$ converges to p.

We may assume that all U_α lie in some compact ball $B(p_0, R)$. Then $q_\alpha \in B(p_0, R)$ implies $p_{i_\alpha} \in B(p_0, 2R)$. Therefore only finitely many different indices can occur in the sequence $\{i_\alpha\}$. Hence there exists an index i that occurs infinitely many times. This determines a subsequence of $\{q_\alpha\}$ contained in S_i and converging to p. Thus $p \in \mathrm{cl}\, S_i$.

Remark. Obviously $\mathrm{cl}\, S_i \subset B_i$, but examples show that $S_i \neq \varnothing$ and $\mathrm{cl}\, S_i \neq B_i$ is possible. Thus the presentation $\mathrm{bd}\, F_0 = \bigcup \mathrm{cl}\, S_i$ may be finer than the presentation $\mathrm{bd}\, F_0 = \bigcup \{B_i;\, S_i \text{ is a side of } F_0\}$. The latter was obtained by Eberlein (personal communication) for Dirichlet regions induced by a discrete group (see Section 4) using the density of $\mathrm{reg}\,\mathrm{bd}\, F_0$ in $\mathrm{bd}\, F_0$. For $\dim M = 2$, see also Eberlein [3], Proposition 2.9.

So far we have only considered the boundary of a single Dirichlet region, but it is clear that the classification into regular and singular boundary points applies to all regions of the Dirichlet tessellation. More precisely, we have

LEMMA 3.15. *Let* $N(p)$ *denote the cardinality of the set* $\{i \in I;\, p \in \mathrm{cl}\, F_i\}$. *Then* $1 \leq N(p) < \infty$ *and the following hold*:

(1) $N(p)=1$ *if and only if* p *belongs to* F_i *for some* $i \in I$.
(2) $N(p)=2$ *if and only if* p *belongs to a side of some* F_i.
(3) $N(p) \geqslant 3$ *if and only if* p *is a singular boundary point of some* F_i.

Now we define

DEFINITION 3.16.
(1) $\operatorname{reg} T = \{p \in M; N(p) \leqslant 2\}$,
(2) $\operatorname{sing} T = \{p \in M; N(p) \geqslant 3\}$.

PROPOSITION 3.17.
(1) $\operatorname{sing} T$ *is closed in* M.
(2) *The topological dimension of* $\operatorname{sing} T$ *does not exceed* $n-2$.

Proof. We have the presentation $\operatorname{sing} T = \bigcup (\operatorname{cl} F_i \cap \operatorname{cl} F_j \cap \operatorname{cl} F_k)$, where the union is taken over all triples of pairwise distinct indices. Since this union is locally finite and countable, the rest of the proof is similar to that of Proposition 3.12.

COROLLARY 3.18.
(1) $\operatorname{reg} T$ *is open in* M.
(2) $\operatorname{reg} T$ *is connected.*

Proof. It suffices to observe that a subspace of topological dimension at most $n-2$ cannot disconnect M (*cf.* Hurewicz-Wallman [6], Thm. IV.4, p. 48).

Remark. The properties of the sets $\{p \in M; N(p)=1\}$ and $\{p \in M; N(p)=2\}$, which by Lemma 3.15 follow from our study of F_0 and its sides, might suggest that a stratification of M could be obtained using $\{p \in M; N(p)=k\}$ as strata. However, these sets are not necessarily manifolds if $k \geqslant 3$, because intersections of the form $M(p, q, r) \cap M(p, s)$ need not be transverse.

4. The Dirichlet tessellation induced by a discrete group

In this section we come to the most important application of the Dirichlet tessellation. Here we no longer begin with an arbitrary discrete set $D \subset M$, but with a particular discrete set obtained from the action of a discrete group of isometries of M.

Let $I(M)$ be the full group of isometries of M. A subgroup Γ of $I(M)$ is called *discrete* if it is a discrete subset of $I(M)$ with respect to the compact-open topology of $I(M)$.

Throughout this section we consider a fixed discrete subgroup Γ of $I(M)$. The fact that we are dealing with isometries of a riemannian manifold has some important consequences.

LEMMA 4.1. *Let Γ be a discrete subgroup of $I(M)$.*
(1) *For all $p \in M$ the isotropy group $\Gamma_p = \{\psi \in \Gamma;\ \psi(p) = p\}$ is finite. Denote its order by $i(p)$.*
(2) *For all $p \in M$ the orbit $\Gamma(p) = \{\psi(p) \in M;\ \psi \in \Gamma\}$ is a discrete subset of M.*
(3) *The canonical map from Γ to $\Gamma(p)$ given by $\psi \mapsto \psi(p)$ is $i(p)$-to-one.*
(4) *Γ is countable.*

Proof. (1) The full isotropy group $I_p(M) = \{\psi \in I(M);\ \psi(p) = p\}$ is compact (Kobayashi-Nomizu [9], vol. I, p. 239), and hence $\Gamma_p = \Gamma \cap I_p(M)$ is finite.

(2) Suppose that $\Gamma(p)$ is not discrete. Then there exists a sequence $\{\psi_\alpha(p)\} \subset \Gamma(p)$ of pairwise distinct points converging to some point of M. This implies the existence of a subsequence of $\{\psi_\alpha\} \subset \Gamma$ which converges to an element of $I(M)$ (Kobayashi-Nomizu [9], vol. I, p. 47–48). Since the elements of $\{\psi_\alpha\}$ are pairwise distinct, this contradicts the discreteness of Γ.

(3) By definition the canonical map $\Gamma \to \Gamma(p)$ is $i(p)$-to-one at the point $p \in \Gamma(p)$. For any other point $\psi(p) \in \Gamma(p)$ it suffices to observe that $\Gamma_{\psi(p)} = \psi \circ \Gamma_p \circ \psi^{-1}$, hence $i(\psi(p)) = i(p)$.

(4) Since $\Gamma(p)$ is discrete in M, it is countable. Together with (3) this implies that Γ is countable.

Remark. By the same type of arguments as above one shows that the following properties are equivalent.
(1) Γ is a discrete subgroup of $I(M)$.
(2) Γ acts discontinuously at a point of M.
(3) Γ acts discontinuously on M.
(4) Γ acts properly discontinuously on M.
Hence we will not use the concepts of discontinuity.

Since Γ is countable we may once and for all fix the notation

$$\Gamma = \{\psi_0, \psi_1, \psi_2, \ldots\} = \{\psi_i;\ i \in I\} \subset I(M),$$

where I is a suitable set of nonnegative indices. We will also use the notation $\psi_0 = \mathrm{id} \in \Gamma$ and $\psi_{-i} = \psi_i^{-1}$ for $i \neq 0$.

LEMMA 4.2. *Let* $\mathrm{Fix}\,(\Gamma)$ *denote the set* $\{p \in M;\ \psi_i(p) = p \text{ for some } i \neq 0\}$. *Then* $M - \mathrm{Fix}\,(\Gamma)$ *is dense in* M.

Proof. Recall that $\mathrm{Fix}(\psi_i)=\{p\in M;\ \psi_i(p)=p\}$ is a submanifold of dimension at most $n-1$ (Kobayashi [8], p. 59). As $\mathrm{Fix}(\Gamma)=\bigcup \mathrm{Fix}(\psi_i)$, the topological dimension of $\mathrm{Fix}(\Gamma)$ is at most $n-1$. Therefore $M-\mathrm{Fix}(\Gamma)$ is dense in M.

Now we choose a point $p_0\notin \mathrm{Fix}(\Gamma)$. According to Lemma 4.1, the orbit $\Gamma(p_0)$ is a discrete set of M, and since $p_0\notin\mathrm{Fix}(\Gamma)$, the canonical map from Γ to $\Gamma(p_0)$ is one-to-one. In particular, the points $p_i=\psi_i(p_0)$ are pairwise distinct and $\Gamma(p_0)$ is given as $\{p_i;\ i\in I\}$. We will fix this notation for the rest of this section.

In the preceeding section the Dirichlet tessellation has been defined with respect to an arbitrary discrete set. Now we apply this construction to the orbit $\Gamma(p_0)$. Let us denote by F_i the Dirichlet region based at $p_i=\psi_i(p_0)$ with respect to $\Gamma(p_0)$ and by $T=\{F_i;\ i\in I\}$ the Dirichlet tessellation so obtained. Clearly all the results of Section 3 apply to the present situation. Here we consider the additional properties of T which result from taking the discrete set to be the orbit of a nonfixed point of a discrete group of isometries. The most important new property is the congruence of all of the regions F_i. More precisely,

LEMMA 4.3. *Consider the Dirichlet tessellation $T=\{F_i;\ i\in I\}$ with respect to $\Gamma(p_0)$. Then $F_i=\psi_i(F_0)$, and a similar statement holds for* $\mathrm{cl}\,F_i$, $\mathrm{bd}\,F_i$, $\mathrm{reg\,bd}\,F_i$, $\mathrm{sing\,bd}\,F_i$, *respectively. In particular, if S is a side of F_0, then $\psi_i(S)$ is a side of F_i.*

Proof. We only prove $F_i=\psi_i(F_0)$. Assume $p\in F_0$. Then $d(\psi_i(p),p_i)=d(p,p_0)<d(p,p_k)=d(\psi_i(p),\psi_i(p_k))$ for all $k\neq 0$. Now observe that if k runs over all indices different from 0, then $\psi_i\circ\psi_k$ runs over all elements of Γ different from ψ_i. Thus $\psi_i(p_k)=\psi_i\circ\psi_k(p_0)$ runs over all points of $\Gamma(p_0)$ different from p_i. Therefore the inequalities for $\psi_i(p)$ imply that $\psi_i(p)\in F_i$.

Conversely, if $q\in F_i$, then the same argument shows that $\psi_{-i}(q)\in F_0$, hence $q\in\psi_i(F_0)$.

Together with Lemma 4.3, the basic property of the Dirichlet tessellation (Proposition 3.4) translates into the well known fact that any single Dirichlet region provides a fundamental region for the action of Γ on M.

In Lemma 3.15 we have classified the points of M with respect to a discrete set by the number

$$N(p)=\#\{i\in I;\ p\in\mathrm{cl}\,F_i\}.$$

Here we may equivalently define

$$N(p)=\#\{i\in I;\ \psi_i(p)\in\mathrm{cl}\,F_0\}.$$

Let us define in addition $n(p) = \#(\Gamma(p) \cap \mathrm{cl}\, F_0)$. Then Lemma 4.1(3) implies $N(p) = i(p) \cdot n(p)$, where $i(p)$ is the order of the isotropy group Γ_p.

PROPOSITION 4.4. *Let p be a point of M.*

(1) *If* $N(p) = 1$, *then* $p \in F_i$ *for some i, p is not a point of* $\mathrm{Fix}(\Gamma)$, *and p has no point equivalent under* Γ *in* $\mathrm{cl}\, F_i$.
(2) *If* $N(p) = 2$, *then* $p \in \mathrm{reg\,bd}\, F_i$ *for some i, and one of the following holds.*
   (a) $p \notin \mathrm{Fix}(\Gamma)$, *and p has exactly one point equivalent under* Γ *in* $\mathrm{cl}\, F_i$ *(actually in* $\mathrm{reg\,bd}\, F_i$).
   (b) *p is a fixed point of an isometry* $\psi \in \Gamma$ *with* $\psi^2 = \mathrm{id}$, *and p has no point equivalent under* Γ *in* $\mathrm{cl}\, F_i$.
(3) $N(p) \geq 3$, *then* $p \in \mathrm{sing\,bd}\, F_i$ *for some i.*

Proof. Recall Lemma 3.15. For the additional assertions, if suffices to observe that $N(p) = 1$ implies $i(p) = n(p) = 1$, whereas $N(p) = 2$ implies either $i(p) = 1$ and $n(p) = 2$, or $i(p) = 2$ and $n(p) = 1$.

Now we turn our attention to the sides of a given Dirichlet region.

DEFINITION 4.5. Two sides S_i and S_j of F_0 are called *conjugate sides* if there is an element $\psi \in \Gamma$, $\psi \neq \mathrm{id}$, with the property $\psi(S_i) = S_j$. Such an isometry is called a *conjugating isometry*.

PROPOSITION 4.6. *Let* S_i *be a side of* F_0. *Then there exists a unique conjugate side, namely* $S_{-i} = \psi_{-i}(S_i)$, *and the conjugating isometry* ψ_{-i} *is uniquely determined. Moreover, if* S_i *is self-conjugate, then* $\psi_i^2 = \mathrm{id}$.

Proof. Let S_i be a side of F_0. Then S_i is also a side of F_i by Lemma 3.8 (1). Thus $S_{-i} = \psi_{-i}(S_i)$ is a side of $\psi_{-i}(F_i) = F_0$ (*cf.* Lemma 4.3). Therefore S_{-i} is a conjugate side of S_i with conjugating isometry ψ_{-i}.

Set $\Sigma_i = \{\psi \in \Gamma;\ \psi(S_i) \subset \mathrm{cl}\, F_0\}$. Since

$$\{\psi_0, \psi_{-i}\} \subset \Sigma_i \subset \{\psi \in \Gamma;\ \psi(p) \in \mathrm{cl}\, F_0\}$$

for any $p \in S_i$ and $N(p) = 2$ for all $p \in S_i$, we have $\Sigma_i = \{\psi_0, \psi_{-i}\}$. This proves both uniqueness assertions.

If $\psi_{-i}(S_i) = S_i$, then $\psi_{-i}^2(S_i) = S_i$. Hence $\psi_{-i}^2 \in \Sigma_i$. Since $\psi_{-i} \neq \mathrm{id}$, this implies $\psi_{-i}^2 = \mathrm{id}$. Hence $\psi_i^2 = \mathrm{id}$.

We now come to the main result of Section 4.

THEOREM 4.7. *Let Γ be a discrete subgroup of $I(M)$ and let F_0 be the Dirichlet region based at $p_0 \notin \operatorname{Fix}(\Gamma)$ with respect to $\Gamma(p_0)$. Denote by Σ the set of isometries*

$$\Sigma = \{\psi_i \in \Gamma;\ S_i \text{ is a side of } F_0\}.$$

Then Σ generates Γ. In particular, if F_0 has only finitely many sides, then Γ is finitely generated.

Proof. Let ψ_j be an arbitrary element of Γ. By Corollary 3.18 (2), we may join p_0 and p_j by a path $c:[0,1] \to \operatorname{reg} T$. The compact set $c([0,1])$ meets only finitely many members of the tessellation $T = \{F_i;\ i \in I\}$.

Set $t_0 = \sup\{t \in [0,1];\ c(t) \in \operatorname{cl} F_0\}$. Since $\operatorname{cl} F_0$ is closed, we actually have $c(t_0) \in \operatorname{cl} F_0$. If $t_0 = 1$, then $\psi_j = \mathrm{id}$ and there is nothing to prove. On the other hand, $t_0 > 0$, because $c(0) = p_0$ lies in the open set F_0. By our choice of the path c and the definition of t_0, the point $c(t_0)$ is a regular boundary point of F_0. Hence there is a well defined index $i_1 \in I$ such that $c(t_0) \in S_{i_1} \subset \operatorname{cl} F_0 \cap \psi_{i_1}(\operatorname{cl} F_0)$. By definition of Σ the isometry ψ_{i_1} is contained in Σ.

Now we set $t_1 = \sup\ \{t \in [0,1];\ c(t) \in \operatorname{cl} F_{i_1}\}$. If $t_1 = 1$ the process stops and $\psi_j = \psi_{i_1}$. Otherwise $t_1 < 1$. We claim that $t_1 > t_0$. It is clear that $t_1 \geqslant t_o$. According to Lemma 3.8(2), there is a neighborhood of $c(t_0)$ which is contained in $F_0 \cup S_{i_1} \cup F_{i_1}$, and since c must leave cl F_0 for $t > t_0$, it has to stay in F_{i_1} for some $t > t_0$. This implies that $t_1 > t_0$.

Again $c(t_1)$ is a regular boundary point and there is a well defined $i_2 \in I$ such that $c(t_1) \in \operatorname{cl} F_{i_1} \cap \operatorname{cl} F_{i_2}$. Moreover, $i_2 \neq 0, i_1$. Since F_{i_1} and F_{i_2} have a common side, the isometry $\psi_{i_2} \circ \psi_{-i_1}$ belongs to Σ. Therefore ψ_{i_2} can be written as a product of elements of Σ.

After finitely many steps this process ends with $\psi_{i_k} = \psi_j$. Thus ψ_j can be written as

$$\psi_j = (\psi_{i_k} \circ \psi_{-i_{k-1}}) \circ \cdots \circ (\psi_{i_2} \circ \psi_{-i_1}) \circ \psi_{i_1}.$$

Remarks. (1) The proof of Theorem 4.7 follows the classical scheme (*cf.* Busemann [1], Thm. 2.10 for general spaces). The new ingredient is Corollary 3.18, which enables us to choose a path in reg T. Thus we get $\Sigma = \{\psi_i \in \Gamma;$ S_i is a side of $F_0\}$ as a set of generators, rather than the larger set $\{\psi_i \in \Gamma;$ $\operatorname{cl} F_0 \cap \psi_i(\operatorname{cl} F_0) \neq \emptyset\}$.

(2) As has been noted by L. Danzer, the group generated by the translations $z \mapsto z+1$ and $z \mapsto z + e^{i\pi/3}$, $z \in \mathbf{C}$, shows that the set of generators given by Theorem 4.7 is not necessarily minimal.

(3) For $n = 2$ and Γ acting freely, Eberlein has proved that Γ is finitely

generated if and only if one (and hence all) Dirichlet regions have finitely many sides (Eberlein [3], Thm. A). For $n \geq 3$ this is no longer true (*cf.* Greenberg [5], Thm. 2).

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