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# Dirichlet regions in manifolds without conjugate points

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### **1. Introduction**

The aim of this paper is to study Dirichlet tessellations of simply connected complete riemannian manifolds without conjugate points. In particular, we will describe the geometrical and topological structure of the boundary of a single Dirichlet region. Throughout this paper M will always denote an *n*-dimensional, simply connected complete riemannian manifold without conjugate points. Let  $\langle , \rangle$  denote the riemannian metric of M and d the distance function induced by  $\langle , \rangle$ . Recall that M has no conjugate points if and only if every pair of distinct points of M can be joined by a unique geodesic segment (up to parametrization).

For a given discrete subset  $D = \{p_i; i \in I\}$  of M, the Dirichlet regions  $F_i$  are defined by

$$F_i = \{p \in M; d(p, p_i) < d(p, p_k) \text{ for all } k \neq i\},\$$

and the collection  $T = \{F_i; i \in I\}$  is called the *Dirichlet tessellation* induced by D. For simplicity we state our main results for  $F_0$ .

THEOREM 3.6. bd  $F_0$  is an (n-1)-dimensional topological submanifold of M. Moreover, bd  $F_0$  admits a differential structure and is thus triangulable.

For a more detailed study of  $bd F_0$  we set

$$S_i = \{p \in M; d(p, p_0) = d(p, p_i) < d(p, p_k) \text{ for all } k \neq 0, i\}$$

and

 $B_i = \{p \in M; d(p, p_0) = d(p, p_i) \le d(p, p_k) \text{ for all } k \ne 0, i\}.$ 

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We call  $S_i$  a side of  $F_0$  if  $S_i$  is nonempty. Except for the case that dim M = 2 (cf. Eberlein [3]), the sides may fail to be connected and also cl  $S_i$  may be different from  $B_i$ . However we find

THEOREM 3.14. bd  $F_0 = \bigcup \operatorname{cl} S_i$ .

Often the discrete set D is given as an orbit of a discrete group of isometries of M. Then the Dirichlet regions are all congruent and provide constructable fundamental regions for the group action. As an application of our study of the sides of a Dirichlet region we obtain

THEOREM 4.7. Let  $\Gamma = \{\psi_i; i \in I\}$  be a discrete group of isometries of M and let  $F_0$  be the Dirichlet region based at  $p_0$  with respect to  $\{p_i = \psi_i(p_0); i \in I\}$  where  $p_0$ is not a fixed point of  $\Gamma$ . Then the set

 $\Sigma = \{\psi_i \in \Gamma; S_i \text{ is a side of } F_0\}$ 

generates  $\Gamma$ . In particular, if  $F_0$  has only finitely many sides, then  $\Gamma$  is finitely generated.

The paper is organized as follows. Section 2 contains a summary of the properties of bisectors and half spaces needed in the sequel. In Section 3 we study the Dirichlet tessellation induced by an arbitrary discrete set  $D \subset M$ . First we develop along classical lines the elementary properties of Dirichlet tessellations (Prop. 3.3 and 3.4) which are well known in many particular cases. The main part of Section 3 deals with the boundary and the sides of a single Dirichlet region and contains the first two theorems stated above. Finally in Section 4 we investigate additional properties of the Dirichlet tessellation arising from the congruence of the Dirichlet regions when D is given as the orbit of a nonfixed point of a discrete group of isometries.

We would like to thank P. Eberlein for his help in connection with Theorem 3.14.

## 2. Bisectors and half spaces

We begin with some general properties of the bisectors and the half spaces determined by them. Throughout this section we fix three distinct points p, q, r of M.

**DEFINITION 2.1.** 

- (1) We define a function  $h: M \to \mathbb{R}$  by h(m) = d(m, p) d(m, q).
- (2)  $M(p,q) = h^{-1}(0)$  is called the *bisector* of p and q.
- (3)  $H(p,q) = \{m \in M; h(m) < 0\}$  denotes the open half space determined by M(p,q) containing p. Similarly H(q,p) denotes the half space containing q.
- (4) We set  $M(p, q, r) = M(p, q) \cap M(p, r)$ .

First we note

**PROPOSITION 2.2.** 

- (1) (Witt [12], Ozols [10], Prop. 2.4) M(p, q) is an (n-1)-dimensional differentiable submanifold of M.
- (2) (Ozols [10], Prop. 2.6) M(p, q) and M(p, r) intersect transversely whenever they intersect at all.
- (3) M(p, q, r) is either empty or is an (n-2)-dimensional differentiable submanifold of M.

*Proof.* (1) We have to show that grad h never vanishes on M(p, q). Let  $d_p$  and  $d_q$  denote the distance functions given by  $d_p(m) = d(m, p)$  and  $d_q(m) = d(m, q)$ . Now suppose grad<sub>m</sub> h = 0 for some  $m \in M(p, q)$ . This implies grad<sub>m</sub>  $d_p = \text{grad}_m d_q$  and thus p = q.

(2) Let  $d_p$ ,  $d_q$ ,  $d_r$  be defined as in (1). We have to show that  $\operatorname{grad}_m d_p - \operatorname{grad}_m d_q$  and  $\operatorname{grad}_m d_p - \operatorname{grad}_m d_r$  are linearly independent in  $T_m M$ , where *m* is any point of  $M(p, q) \cap M(p, r)$ . Suppose this is not the case. Then  $\operatorname{grad}_m d_p$ ,  $\operatorname{grad}_m d_q$ , and  $\operatorname{grad}_m d_r$  must lie on an affine line in  $T_m M$ . This is impossible for three distinct unit vectors. Clearly (2) implies (3).

A more detailed investigation (Im Hof [7], Prop. 2.6) shows that M(p, q) is diffeomorphic to  $\mathbb{R}^{n-1}$ . Also, for dim M = 2, the sets M(p, q, r) are either empty or consist of a single point (Eberlein [3], Prop. 2.8, Ehrlich-Im Hof [4], Cor. 2).

In manifolds of constant sectional curvature, bisectors are totally geodesic and the half spaces determined by them are convex. These facts are widely used in the theory of fuchsian groups. However, in more general cases, the bisectors are no longer totally geodesic, nor are the half spaces they determine convex. Indeed, all bisectors M(p, q) of M are totally geodesic if and only if M has constant sectional curvature (Busemann [2], Thm. 47.4, p. 331).

A property which is weaker than the convexity of the half spaces, but which is still very useful, is the starlikeness of the half spaces H(p, q) with respect to p. To show that this starlikeness holds in our context (Prop. 2.6(2) below), we consider

the behavior of geodesic rays through p with respect to M(p, q). Let  $c_{pq}$  denote the unique geodesic with  $c_{pq}(0) = p$  and  $c_{pq}(d) = q$ , where d = (p, q).

# **PROPOSITION 2.3.**

- (1) Let  $c:[0,\infty) \to M$  be a geodesic ray with c(0) = p. If the ray c is not contained in  $c_{pq}$ , then h(c(t)) is strictly increasing for all  $t \ge 0$ . If  $c = c_{pq} | [0,\infty)$ , then h(c(t)) is strictly increasing for  $0 \le t \le d$  and h(c(t)) = d for  $t \ge d$ . If  $\dot{c}(0) = -\dot{c}_{pq}(0)$ , then h(c(t)) = -d for all  $t \ge 0$ .
- (2) Let  $c:[0,\infty) \to M$  be a geodesic ray with c(0) = p. Then c intersects M(p,q) at most once and transversely.
- (3) Let  $c : \mathbf{R} \to M$  be a geodesic with c(0) = p. Then c intersects M(p, q) at most once.

*Proof.* (1) We may assume  $\|\dot{c}\| = 1$ . Consider first a geodesic ray c with  $\dot{c}(0) \neq \pm \dot{c}_{pq}(0)$ . Suppose  $h(c(s)) \ge h(c(t))$  for  $0 \le s < t$ . This implies

$$d(q, c(t)) \ge d(q, c(s)) + d(c(s), c(t)),$$

hence

$$d(q, c(t)) = d(q, c(s)) + d(c(s), c(t))$$

by the triangle inequality. But equality occurs only when the three points q, c(s), c(t) lie on a geodesic. This contradicts our assumption on c.

The behavior of h along the geodesic  $c_{pq}$  is obvious.

(2) Uniqueness of the intersections of c with M(p, q) immediately follows from the monotonicity of  $h \circ c$  proved in (1). Now we assume that a geodesic ray c intersects M(p, q) at a point  $m = c(t) \in M(p, q)$  and we show transversality. Suppose the intersection is not transverse, i.e.,  $\langle \dot{c}(t), \operatorname{grad}_m h \rangle = 0$ . Using the same notation as in the proof of Proposition 2.2 we may write grad h =grad  $d_p - \operatorname{grad} d_p$ . Moreover,  $\operatorname{grad}_m d_p = \dot{c}(t)$ . Then our assumption implies  $\langle \operatorname{grad}_m d_p, \operatorname{grad}_m d_q \rangle = \langle \operatorname{grad}_m d_p, \operatorname{grad}_m d_p \rangle = 1$ , hence  $\operatorname{grad}_m d_p = \operatorname{grad}_m d_q$ . This is impossible, since  $m \in M(p, q)$  and  $p \neq q$ .

(3) Suppose  $c: \mathbb{R} \to M$  is a geodesic with c(0) = p intersecting M(p, q) in two points c(s) and c(t). By (2) we may assume s < 0 < t. Then we have

$$d(c(s), c(t)) = d(c(s), p) + d(p, c(t)) = d(c(s), q) + d(q, c(t)).$$

This equality is only possible if q lies on the geodesic segment between c(s) and c(t). This would imply p = q.

*Remark.* Uniqueness of the intersections of geodesic rays through p with M(p,q) (or with bd  $F_0$ , see Prop. 3.3(3) below) is well known (Busemann [1], 2.12; Ozols [10], Prop. 3.2; for n = 2, Eberlein [3], Prop. 2.6). Transversality has first been proved in Im Hof [7], Lemma 2.5.

Now we define the concept of a strictly starlike subset of M. For this purpose we set

 $(m, m') = \{c(t); t \in (0, 1)\},\$ 

where  $c:[0,1] \rightarrow M$  is the geodesic segment joining m and m'.

DEFINITION 2.4. A subset F of M is strictly starlike with respect to a point  $m \in F$  if for every point  $m' \in cl F$  the segment (m, m') is contained in int F.

Knowing that a set F is strictly starlike rather than just starlike with respect to a point has implications about the topology of F. Explicitly

LEMMA 2.5. Let F and G be two subsets of M with  $F \subseteq G$  and let m be a point of F. Suppose that for each  $m' \in G$  the segment (m, m') is contained in F. Then  $G \subseteq cl F$  and int  $G \subseteq F$ .

**Proof.** Let m' be a point of G and let U be a neighborhood of m'. The segment (m, m') lies in F and contains points of U. Hence  $m' \in cl F$ .

Now let m' be a point of int G and let U be a neighborhood of m' contained in G. The segment (m, m') can be extended to a segment (m, m'') such that  $m'' \in U$ and  $m' \in (m, m'')$ . Since  $m'' \in G$ , the whole segment (m, m'') lies in F. In particular,  $m' \in F$ .

We are now able to show

**PROPOSITION 2.6.** 

- (1)  $\operatorname{cl} H(p, q) = H(p, q) \cup M(p, q),$ int  $\operatorname{cl} H(p, q) = H(p, q),$  $\operatorname{bd} H(p, q) = M(p, q).$
- (2) H(p, q) is strictly starlike with respect to p.
- (3) The half spaces H(p, q) and H(q, p) are the connected components of M-M(p, q).

**Proof.** Obviously  $\operatorname{cl} H(p,q) \subset H(p,q) \cup M(p,q)$  and  $H(p,q) \subset \operatorname{int} (H(p,q) \cup M(p,q))$ . By Proposition 2.3(1), the hypothesis of Lemma 2.5 is satisfied

for H(p, q),  $H(p, q) \cup M(p, q)$ , and p. Thus we may conclude the proof of (1). Since  $cl H(p, q) = H(p, q) \cup M(p, q)$ , Proposition 2.3(1) also implies (2) which then implies (3).

We now consider the set of geodesic rays through p intersecting M(p,q). Denote the unit tangent sphere at p by S. For each  $v \in S$  we denote by  $c_v : [0, \infty) \to M$  the geodesic ray with  $c_v(0) = p$  and  $\dot{c}_v(0) = v$ . Since each ray  $c_v$  intersects M(p,q) at most once, we may define  $A = \{v \in S; c_v \text{ intersects } M(p,q)\}$  and two functions  $\phi : A \to M(p,q)$  and  $\varphi : A \to \mathbb{R}$  by  $\phi(v) = c_v(0,\infty) \cap M(p,q)$  and  $\varphi(v) = d(p, \phi(v))$ . The transversality of the intersection of  $c_v$  with M(p,q) implies

**PROPOSITION 2.7.** 

- (1) A is open in S.
- (2)  $\phi: A \rightarrow M(p, q)$  is a diffeomorphism.
- (3)  $\varphi: A \rightarrow \mathbf{R}$  is differentiable.

Proof. Let  $\pi$  denote the projection of  $M - \{p\}$  onto S defined by  $\pi(m) = v$ , where  $m = c_v(t)$  for suitable t. Since M(p, q) is an (n-1)-dimensional submanifold transverse to the geodesic rays  $c_v$ , i.e., transverse to the fibers of  $\pi$ , the map  $\pi | M(p,q) : M(p,q) \rightarrow S$  is a local diffeomorphism. As  $A = \pi(M(p,q))$ , property (1) holds. Now  $\phi = (\pi | M(p,q))^{-1}$  and  $\pi | M(p,q) : M(p,q) \rightarrow A$  is bijective. This implies (2) and (3).

# 3. The Dirichlet tessellation induced by a discrete set

In this section we consider an arbitrary discrete set  $D \subset M$ . By definition, D has no points of accumulation in M, or equivalently, D is locally finite, i.e., every compact set of M contains only finitely many points of D. In particular, D is at most countable and we may thus once and for all fix the notation

 $D = \{p_0, p_1, p_2, \ldots\} = \{p_i; i \in I\} \subset M,$ 

where I is a suitable set of indices.

The discrete set D may be finite or infinite, but we will always assume that it contains at least two points.

DEFINITION 3.1. Let  $D = \{p_i; i \in I\}$  be a discrete set of M.

(1) The Dirichlet region with basepoint  $p_i$  is the set

 $F_i = \{p \in M; d(p, p_i) < d(p, p_k) \text{ for all } k \neq i\}.$ 

(2) The collection  $T = \{F_i; i \in I\}$  is called the Dirichlet tessellation induced by D.

The purpose of this section is to investigate the properties of a single Dirichlet region as well as those of the whole tessellation with respect to a given discrete set D. Whenever we deal with a single region, we may restrict our attention to  $F_0$ . The following notation will be used throughout this section.

DEFINITION 3.2. For  $i \in I$ ,  $i \neq 0$  we set

- (1)  $M_i = M(p_0, p_i)$ , the bisector of  $p_0$  and  $p_i$ .
- (2)  $H_i = H(p_0, p_i)$ , the half space determined by  $M_i$  containing  $p_0$ . Using this notation we have  $F_0 = \bigcap H_i$ .

The basic properties of a single Dirichlet region are summarized in

**PROPOSITION 3.3.** Let  $F_0$  be the Dirichlet region with basepoint  $p_0 \in D$ . Then

- (1)  $F_0$  is nonempty and open in M.
- (2) cl  $F_0 = \{p \in M; d(p, p_0) \le d(p, p_i) \text{ for all } i \ne 0\}$ , and int cl  $F_0 = F_0$ .
- (3)  $F_0$  is strictly starlike with respect to  $p_0$ .

**Proof.** Obviously  $F_0 \neq \emptyset$ . Now let p be any point of  $F_0$  and denote by  $d_k$  the distance  $d(p, p_k)$ . Since  $p \in F_0$  we have  $d_0 < d_k$  for all  $k \neq 0$ . The discreteness of D implies that there exists an index  $i \in I$  with  $d_0 < d_i \leq d_k$  for all  $k \neq 0$ , *i*. Now we choose  $\varepsilon = (d_i - d_0)/2$ . By the triangle inequality the open ball  $U(p, \varepsilon) = \{q \in M; d(q, p) < \varepsilon\}$  is contained in  $F_0$ . This proves (1).

For the proof of (2) let us write

 $G_0 = \{ p \in M; d(p, p_0) \le d(p, p_i) \text{ for all } i \ne 0 \}.$ 

Clearly  $F_0 \subset G_0$ . Since  $G_0 = \bigcap \operatorname{cl} H_i$ ,  $G_0$  is closed in M and we have  $\operatorname{cl} F_0 \subset G_0$  as well as  $F_0 \subset \operatorname{int} G_0$ . Since all the half spaces  $H_i$  are strictly starlike with respect to  $p_0$ , the segment  $(p_0, p)$  is contained in  $F_0$  for all  $p \in G_0$ . Using Lemma 2.5 we may then conclude the proof of (2).

Since cl  $F_0 = G_0$ , we also obtain (3).

We now state the fundamental property of the Dirichlet tessellation.

**PROPOSITION 3.4.** Let  $T = \{F_i; i \in I\}$  be the Dirichlet tessellation induced by the discrete set D. Then

- (1)  $F_i \cap \text{cl } F_i = \emptyset$  for  $i \neq j$ .
- (2)  $M = \bigcup \operatorname{cl} F_i$ , and the covering is locally finite.

*Proof.* (1) is obvious. For the proof of (2) let p be a point of M and denote by  $d_k$  the distance  $d(p, p_k)$ . Since D is discrete, there exists an index  $i \in I$  such that  $d_i \leq d_k$  for all  $k \neq i$ . This implies  $p \in \operatorname{cl} F_i$ .

Now let K be a compact set in M, e.g., the closed ball  $K = B(p_0, R) = \{q \in M; d(q, p_0) \leq R\}$ . Assume  $p \in cl F_i \cap K$ . Then  $d(p, p_i) \leq d(p, p_0) \leq R$  and thus  $d(p_0, p_i) \leq d(p_0, p) + d(p, p_i) \leq 2R$ . This is possible only for finitely many  $i \in I$ .

Now we begin a more detailed study of the boundary of a single Dirichlet region. Again we restrict our attention to  $F_0$ . By Proposition 3.3(2) we have bd  $F_0 = \operatorname{cl} F_0 - F_0 = \bigcup B_i$ , where

$$B_i = \operatorname{cl} F_0 \cap M_i = \{ p \in M; \, d(p, p_0) = d(p, p_i) \le d(p, p_k) \text{ for all } k \ne 0, \, i \}.$$

Our investigation of bd  $F_0$  is based on a study of the geodesic rays emanating from  $p_0$ . Let S denote the unit tangent sphere at  $p_0$ . For  $v \in S$  let  $c_v : [0, \infty) \to M$ be the geodesic ray with  $c_v(0) = p_0$  and  $\dot{c}_v(0) = v$ . According to Proposition 3.3(3), each ray  $c_v$  intersects bd  $F_0$  at most once. Thus we may define  $A = \{v \in S; c_v$ intersects bd  $F_0\}$  and two functions  $\phi : A \to bd F_0$  and  $\phi : A \to \mathbb{R}$  by  $\phi(v) =$  $c_v(0, \infty) \cap bd F_0$  and  $\phi(v) = d(p_0, \phi(v))$ .

**PROPOSITION 3.5.** 

(1) A is open in S.

(2)  $\varphi: A \to \mathbf{R}$  is continuous.

(3)  $\phi: A \rightarrow bd F_0$  is a homeomorphism with respect to the induced topology on  $bd F_0 \subset M$ .

**Proof.** First we recall the analogous construction for a single bisector. Let  $A_i$  be the set  $\{v \in S; c_v \text{ intersects } M_i\}$  and define  $\phi_i : A_i \to M_i$  and  $\varphi_i : A_i \to \mathbb{R}$  by  $\phi_i(v) = c_v(0, \infty) \cap M_i$  and  $\varphi_i(v) = d(p_0, \phi_i(v))$ . According to Proposition 2.7,  $A_i$  is open in S,  $\varphi_i$  is differentiable, and  $\phi_i$  is a diffeomorphism.

Now consider  $v \in A$ . Since  $\phi(v) \in bd F_0 = \bigcup B_i$ , there is some  $i \in I$  such that  $\phi(v) \in M_i$ . Hence  $v \in A_i$ ,  $\phi(v) = \phi_i(v)$ , and  $\varphi(v) = \varphi_i(v)$ . Conversely, if  $v \in A_i$ , then  $\phi_i(v) \in M_i$ . Thus  $\phi_i(v) \notin F_0$  and so the geodesic segment from  $p_0$  to  $\phi_i(v)$  must intersect  $bd F_0$ . Therefore  $v \in A$  and  $\varphi(v) \leq \varphi_i(v)$ . These arguments show that  $A = \bigcup A_i$  and  $\varphi = \min \varphi_i$ . This implies at once that A is open in S and that  $\varphi$  is upper-semicontinuous.

In order to show that  $\varphi$  is continuous, we choose an element  $v \in A$  and a sequence  $\{v_{\alpha}\} \subset A$  converging to v. Since  $\varphi$  is upper-semicontinuous, the sequence  $\{\varphi(v_{\alpha})\}$  is bounded. Thus we may assume that  $\{\varphi(v_{\alpha})\}$  converges to a number  $d \in \mathbb{R}$ . It remains to show that  $\varphi(v) = d$ .

Since  $\phi(v_{\alpha}) = c_{v_{\alpha}}(\varphi(v_{\alpha})) \in bd F_0$  and  $bd F_0$  is closed, the sequence  $\{\phi(v_{\alpha})\}$  converges to  $c_v(d) \in bd F_0$ . This implies that  $\phi(v) = c_v(d)$ , hence  $\varphi(v) = d$ .

Using geodesic polar coordinates at  $p_0$ , we may regard  $\phi: A \rightarrow bd F_0$  as the graph of the continuous function  $\varphi$ . Therefore  $\phi$  is a homeomorphism.

As an immediate consequence of Proposition 3.5 we have

THEOREM 3.6. bd  $F_0$  is an (n-1)-dimensional topological submanifold of M. Moreover, the homeomorphism  $\phi : A \rightarrow bd F_0$  induces a differential structure on bd  $F_0$ . Hence bd  $F_0$  is triangulable.

Recall the presentation bd  $F_0 = \bigcup B_i$ . Our aim is to simplify this presentation by omitting those members of the collection  $\{B_i\}$  which are redundant. Whenever  $B_i$  contains a point p which is not contained in any  $B_k$  for  $k \neq i$ , then certainly  $B_i$ may not be omitted. This motivates the following

**DEFINITION 3.7.** 

- (1)  $S_i = \{p \in M; d(p, p_0) = d(p, p_i) < d(p, p_k) \text{ for all } k \neq 0, i\}.$
- (2) If  $S_i \neq \emptyset$ , we call  $S_i$  a side of  $F_0$ .

Remark. This definition of sides is identical with Ozols' definition of faces (cf. Ozols [10], p. 225) and essentially equivalent to Eberlein's definition of bounding sides (cf. Eberlein [3], Def. 2.2). If dim M = 2, arguments of Eberlein (loc. cit., Lemma 2.11, p. 38) show that the sides are connected. An essential ingredient in his proof is the fact noted above that for dim M = 2, the sets M(p, q, r) are either empty or consist of a single point. However, examples show that the sides may fail to be connected for dim  $M \ge 3$ .

The next lemma is crucial for the fitting together of neighboring Dirichlet regions.

### **LEMMA 3.8.**

- (1) If  $S_i$  is a side of  $F_0$ , then it is also a side of  $F_i$ .
- (2)  $F_0 \cup S_i \cup F_i$  is open in M.

**Proof.** (1) is an obvious consequence of the definition of sides. For the proof of (2) it suffices to show that each point of  $S_i$  has a neighborhood contained in  $F_0 \cup S_i \cup F_i$ . Let  $E_i$  be the Dirichlet region based at  $p_0$  with respect to  $D - \{p_i\}$ . Then  $E_i$  is open, and it is easily verified that  $S_i \subset E_i \subset F_0 \cup S_i \cup F_i$ .

The following result is a topological characterization of the sides of bd  $F_0$ .

**PROPOSITION 3.9.**  $S_i = \operatorname{int}_{bd F_0} B_i = \operatorname{int}_{M_i} B_i$ .

Proof. Recall  $S_i \subset B_i = \operatorname{bd} F_0 \cap M_i$ . We first show that  $\operatorname{int}_{\operatorname{bd} F_0} B_i = \operatorname{int}_{M_i} B_i$ . It suffices to prove that a set  $U \subset B_i$  which is open in  $\operatorname{bd} F_0$  is also open in  $M_i$ , and vice-versa. Let  $\phi: A \to \operatorname{bd} F_0$  and  $\phi_i: A_i \to M_i$  be the maps studied in Proposition 3.5. Given  $U \subset B_i$  we set  $V = \phi^{-1}(U) \subset A$  and  $V_i = \phi_i^{-1}(U) \subset A_i$ . Since  $U \subset B_i = \operatorname{bd} F_0 \cap M_i$ , the sets V and  $V_i$  coincide, and so do the maps  $\phi | V$  and  $\phi_i | V_i$ . Now it is clear that U is open in  $\operatorname{bd} F_0$  if and only if it is open in  $M_i$ . From now on we will write  $\operatorname{int} B_i$  instead of  $\operatorname{int}_{\operatorname{bd} F_0} B_i$  (or  $\operatorname{int}_{M_i} B_i$ ).

Next we show that  $S_i = \operatorname{int} B_i$ . Let p be a point of  $S_i$ . According to Lemma 3.8(2), there is a neighborhood U of p in M which is contained in  $F_0 \cup S_i \cup F_i$ . Then  $U \cap M_i = U \cap S_i$  and  $V = U \cap M_i$  is an  $M_i$ -neighborhood of p contained in  $S_i$ . Hence p is an interior point of  $B_i$ .

Conversely, let p be a point of  $B_i - S_i$ . We claim that p cannot be an interior point of  $B_i$  (with respect to  $M_i$ ). Let U be any  $M_i$ -neighborhood of p. We will construct a point  $q \in U$  which does not belong to  $B_i$ . Since  $p \in B_i - S_i$ , there is an index  $k \neq i$  such that  $p \in B_i \cap B_k \subset M_i \cap M_k$ . Let  $h_k$  denote the function defined by  $h_k(p) = d(p, p_0) - d(p, p_k)$ . Since  $M_i$  and  $M_k$  intersect transversely, grad  $(h_k | M_i)$ does not vanish on  $M_i \cap M_k$ . Let  $\mu$  be an integral curve of grad  $(h_k | M_i)$  through p. Then for sufficiently small  $\varepsilon > 0$  the point  $q = \mu(\varepsilon)$  lies in U, but  $h_k(q) > 0$ , so q lies outside cl  $F_0$ .

As a consequence of Propositions 3.9 and 2.7(2) we have

COROLLARY 3.10.  $S_i$  is an open submanifold of bd  $F_0$  and of  $M_i$ . The differential structures of bd  $F_0$  (as given by Theorem 3.6) and of  $M_i$  (as a submanifold of M) coincide on  $S_i$ .

**DEFINITION 3.11.** 

- (1) A point  $p \in bd F_0$  is called a *regular boundary point* of  $F_0$  if it belongs to a side of  $F_0$ . The set of regular boundary points of  $F_0$  is denoted by reg bd  $F_0$ .
- (2) The complement bd  $F_0$ -reg bd  $F_0$  is called the set of singular boundary points of  $F_0$  and denoted by sing bd  $F_0$ .

**PROPOSITION 3.12.** 

- (1) sing bd  $F_0$  is closed in bd  $F_0$ .
- (2) The topological dimension of sing bd  $F_0$  does not exceed n-2.

**Proof.** We have the presentation sing bd  $F_0 = \bigcup (\operatorname{bd} F_0 \cap M_i \cap M_j)$ , where the union is taken over all indices i, j with  $0 \neq i \neq j \neq 0$ . Each of the sets bd  $F_0 \cap M_i \cap M_j$  is closed in bd  $F_0$ , and their union is locally finite. Thus sing bd  $F_0$  is closed in bd  $F_0$ .

By Proposition 2.2(3),  $M_i \cap M_j$  is either empty or is an (n-2)-dimensional submanifold of M. Hence bd  $F_0 \cap M_i \cap M_j$  has topological dimension at most n-2 (*cf.* Hurewicz-Wallman [6], Thm. III.1, p. 26). As a countable union of such spaces, bd  $F_0$  itself has topological dimension at most n-2 (*cf.* Hurewicz-Wallman [6], Thm. III.2, p. 30).

COROLLARY 3.13. reg bd  $F_0$  is open and dense in bd  $F_0$ .

*Proof.* It suffices to observe that sing bd  $F_0$  cannot contain an open subset of bd  $F_0$ , since dim (bd  $F_0$ ) = n-1, whereas dim (sing bd  $F_0$ )  $\leq n-2$ .

*Remark.* For Dirichlet regions induced by a discrete group acting freely and isometrically on *M*, Corollary 3.13 is a consequence of Sugahara [11], Theorem A. A proof for general discrete groups using the Baire Category Theorem was communicated to us by Eberlein (personal communication).

The closures of  $S_i$  with respect to bd  $F_0$ ,  $M_i$ , or M, all coincide. In the following theorem we may therefore use the notation cl  $S_i$  without ambiguity.

THEOREM 3.14. bd  $F_0 = \bigcup \operatorname{cl} S_i$ .

**Proof.** Let p be a point of bd  $F_0$ . If p is a regular boundary point, then  $p \in S_i$  for some *i*. In general, choose a basis  $\{U_\alpha\}$  of bd  $F_0$ -neighborhoods with  $p \in U_{\alpha+1} \subset U_\alpha$ . According to Corollary 3.13, each  $U_\alpha$  contains a regular boundary point  $q_\alpha$  belonging to some side  $S_{i_\alpha}$ . The sequence  $\{q_\alpha\}$  converges to p.

We may assume that all  $U_{\alpha}$  lie in some compact ball  $B(p_0, R)$ . Then  $q_{\alpha} \in B(p_0, R)$  implies  $p_{i_{\alpha}} \in B(p_0, 2R)$ . Therefore only finitely many different indices can occur in the sequence  $\{i_{\alpha}\}$ . Hence there exists an index *i* that occurs infinitely many times. This determines a subsequence of  $\{q_{\alpha}\}$  contained in  $S_i$  and converging to *p*. Thus  $p \in \text{cl } S_i$ .

*Remark.* Obviously cl  $S_i \subset B_i$ , but examples show that  $S_i \neq \emptyset$  and cl  $S_i \neq B_i$  is possible. Thus the presentation bd  $F_0 = \bigcup \operatorname{cl} S_i$  may be finer than the presentation bd  $F_0 = \bigcup \{B_i; S_i \text{ is a side of } F_0\}$ . The latter was obtained by Eberlein (personal communication) for Dirichlet regions induced by a discrete group (see Section 4) using the density of reg bd  $F_0$  in bd  $F_0$ . For dim M = 2, see also Eberlein [3], Proposition 2.9.

So far we have only considered the boundary of a single Dirichlet region, but it is clear that the classification into regular and singular boundary points applies to all regions of the Dirichlet tessellation. More precisely, we have

LEMMA 3.15. Let N(p) denote the cardinality of the set  $\{i \in I; p \in cl F_i\}$ . Then  $1 \leq N(p) < \infty$  and the following hold:

- (1) N(p) = 1 if and only if p belongs to  $F_i$  for some  $i \in I$ .
- (2) N(p) = 2 if and only if p belongs to a side of some  $F_i$ .
- (3)  $N(p) \ge 3$  if and only if p is a singular boundary point of some  $F_i$ .

Now we define

**DEFINITION 3.16.** 

- (1) reg  $T = \{p \in M; N(p) \le 2\},\$
- (2) sing  $T = \{p \in M; N(p) \ge 3\}.$

## **PROPOSITION 3.17.**

- (1) sing T is closed in M.
- (2) The topological dimension of sing T does not exceed n-2.

**Proof.** We have the presentation sing  $T = \bigcup (\operatorname{cl} F_i \cap \operatorname{cl} F_j \cap \operatorname{cl} F_k)$ , where the union is taken over all triples of pairwise distinct indices. Since this union is locally finite and countable, the rest of the proof is similar to that of Proposition 3.12.

COROLLARY 3.18.

(1) reg T is open in M.

(2) reg T is connected.

**Proof.** It suffices to observe that a subspace of topological dimension at most n-2 cannot disconnect M (cf. Hurewicz-Wallman [6], Thm. IV.4, p. 48).

*Remark.* The properties of the sets  $\{p \in M; N(p) = 1\}$  and  $\{p \in M; N(p) = 2\}$ , which by Lemma 3.15 follow from our study of  $F_0$  and its sides, might suggest that a stratification of M could be obtained using  $\{p \in M; N(p) = k\}$  as strata. However, these sets are not necessarily manifolds if  $k \ge 3$ , because intersections of the form  $M(p, q, r) \cap M(p, s)$  need not be transverse.

# 4. The Dirichlet tessellation induced by a discrete group

In this section we come to the most important application of the Dirichlet tessellation. Here we no longer begin with an arbitrary discrete set  $D \subseteq M$ , but with a particular discrete set obtained from the action of a discrete group of isometries of M.

Let I(M) be the full group of isometries of M. A subgroup  $\Gamma$  of I(M) is called *discrete* if it is a discrete subset of I(M) with respect to the compact-open topology of I(M).

Throughout this section we consider a fixed discrete subgroup  $\Gamma$  of I(M). The fact that we are dealing with isometries of a riemannian manifold has some important consequences.

LEMMA 4.1. Let  $\Gamma$  be a discrete subgroup of I(M).

- (1) For all  $p \in M$  the isotropy group  $\Gamma_p = \{\psi \in \Gamma; \psi(p) = p\}$  is finite. Denote its order by i(p).
- (2) For all  $p \in M$  the orbit  $\Gamma(p) = \{\psi(p) \in M; \psi \in \Gamma\}$  is a discrete subset of M.
- (3) The canonical map from  $\Gamma$  to  $\Gamma(p)$  given by  $\psi \mapsto \psi(p)$  is i(p)-to-one.
- (4)  $\Gamma$  is countable.

*Proof.* (1) The full isotropy group  $I_p(M) = \{\psi \in I(M); \psi(p) = p\}$  is compact (Kobayashi-Nomizu [9], vol. I, p. 239), and hence  $\Gamma_p = \Gamma \cap I_p(M)$  is finite.

(2) Suppose that  $\Gamma(p)$  is not discrete. Then there exists a sequence  $\{\psi_{\alpha}(p)\} \subset \Gamma(p)$  of pairwise distinct points converging to some point of M. This implies the existence of a subsequence of  $\{\psi_{\alpha}\} \subset \Gamma$  which converges to an element of I(M) (Kobayashi-Nomizu [9], vol. I, p. 47-48). Since the elements of  $\{\psi_{\alpha}\}$  are pairwise distinct, this contradicts the discreteness of  $\Gamma$ .

(3) By definition the canonical map  $\Gamma \to \Gamma(p)$  is i(p)-to-one at the point  $p \in \Gamma(p)$ . For any other point  $\psi(p) \in \Gamma(p)$  it suffices to observe that  $\Gamma_{\psi(p)} = \psi \circ \Gamma_p \circ \psi^{-1}$ , hence  $i(\psi(p)) = i(p)$ .

(4) Since  $\Gamma(p)$  is discrete in *M*, it is countable. Together with (3) this implies that  $\Gamma$  is countable.

*Remark.* By the same type of arguments as above one shows that the following properties are equivalent.

- (1)  $\Gamma$  is a discrete subgroup of I(M).
- (2)  $\Gamma$  acts discontinuously at a point of M.
- (3)  $\Gamma$  acts discontinuously on M.
- (4)  $\Gamma$  acts properly discontinuously on M.

Hence we will not use the concepts of discontinuity.

Since  $\Gamma$  is countable we may once and for all fix the notation

$$\Gamma = \{\psi_0, \psi_1, \psi_2, \ldots\} = \{\psi_i; i \in I\} \subset I(M),$$

where I is a suitable set of nonnegative indices. We will also use the notation  $\psi_0 = id \in \Gamma$  and  $\psi_{-i} = \psi_i^{-1}$  for  $i \neq 0$ .

LEMMA 4.2. Let Fix ( $\Gamma$ ) denote the set { $p \in M$ ;  $\psi_i(p) = p$  for some  $i \neq 0$ }. Then  $M - \text{Fix}(\Gamma)$  is dense in M.

**Proof.** Recall that Fix  $(\psi_i) = \{p \in M; \psi_i(p) = p\}$  is a submanifold of dimension at most n-1 (Kobayashi [8], p. 59). As Fix  $(\Gamma) = \bigcup$  Fix  $(\psi_i)$ , the topological dimension of Fix  $(\Gamma)$  is at most n-1. Therefore M -Fix  $(\Gamma)$  is dense in M.

Now we choose a point  $p_0 \notin \text{Fix}(\Gamma)$ . According to Lemma 4.1, the orbit  $\Gamma(p_0)$  is a discrete set of M, and since  $p_0 \notin \text{Fix}(\Gamma)$ , the canonical map from  $\Gamma$  to  $\Gamma(p_0)$  is one-to-one. In particular, the points  $p_i = \psi_i(p_0)$  are pairwise distinct and  $\Gamma(p_0)$  is given as  $\{p_i; i \in I\}$ . We will fix this notation for the rest of this section.

In the preceeding section the Dirichlet tessellation has been defined with respect to an arbitrary discrete set. Now we apply this construction to the orbit  $\Gamma(p_0)$ . Let us denote by  $F_i$  the Dirichlet region based at  $p_i = \psi_i(p_0)$  with respect to  $\Gamma(p_0)$  and by  $T = \{F_i; i \in I\}$  the Dirichlet tessellation so obtained. Clearly all the results of Section 3 apply to the present situation. Here we consider the additional properties of T which result from taking the discrete set to be the orbit of a nonfixed point of a discrete group of isometries. The most important new property is the congruence of all of the regions  $F_i$ . More precisely,

LEMMA 4.3. Consider the Dirichlet tessellation  $T = \{F_i; i \in I\}$  with respect to  $\Gamma(p_0)$ . Then  $F_i = \psi_i(F_0)$ , and a similar statement holds for cl  $F_i$ , bd  $F_i$ , reg bd  $F_i$ , sing bd  $F_i$ , respectively. In particular, if S is a side of  $F_0$ , then  $\psi_i(S)$  is a side of  $F_i$ .

**Proof.** We only prove  $F_i = \psi_i(F_0)$ . Assume  $p \in F_0$ . Then  $d(\psi_i(p), p_i) = d(p, p_0) < d(p, p_k) = d(\psi_i(p), \psi_i(p_k))$  for all  $k \neq 0$ . Now observe that if k runs over all indices different from 0, then  $\psi_i \circ \psi_k$  runs over all elements of  $\Gamma$  different from  $\psi_i$ . Thus  $\psi_i(p_k) = \psi_i \circ \psi_k(p_0)$  runs over all points of  $\Gamma(p_0)$  different from  $p_i$ . Therefore the inequalities for  $\psi_i(p)$  imply that  $\psi_i(p) \in F_i$ .

Conversely, if  $q \in F_i$ , then the same argument shows that  $\psi_{-i}(q) \in F_0$ , hence  $q \in \psi_i(F_0)$ .

Together with Lemma 4.3, the basic property of the Dirichlet tessellation (Proposition 3.4) translates into the well known fact that any single Dirichlet region provides a fundamental region for the action of  $\Gamma$  on M.

In Lemma 3.15 we have classified the points of M with respect to a discrete set by the number

 $N(p) = \#\{i \in I; p \in \operatorname{cl} F_i\}.$ 

Here we may equivalently define

$$N(p) = \#\{i \in I; \psi_i(p) \in cl F_0\}.$$

Let us define in addition  $n(p) = #(\Gamma(p) \cap \operatorname{cl} F_0)$ . Then Lemma 4.1(3) implies  $N(p) = i(p) \cdot n(p)$ , where i(p) is the order of the isotropy group  $\Gamma_p$ .

**PROPOSITION 4.4.** Let p be a point of M.

- (1) If N(p) = 1, then  $p \in F_i$  for some *i*, *p* is not a point of Fix ( $\Gamma$ ), and *p* has no point equivalent under  $\Gamma$  in cl  $F_i$ .
- (2) If N(p) = 2, then  $p \in \text{reg bd } F_i$  for some *i*, and one of the following holds.
  - (a)  $p \notin Fix(\Gamma)$ , and p has exactly one point equivalent under  $\Gamma$  in cl  $F_i$  (actually in reg bd  $F_i$ ).
  - (b) p is a fixed point of an isometry  $\psi \in \Gamma$  with  $\psi^2 = id$ , and p has no point equivalent under  $\Gamma$  in cl  $F_i$ .
- (3)  $N(p) \ge 3$ , then  $p \in \text{sing bd } F_i$  for some *i*.

**Proof.** Recall Lemma 3.15. For the additional assertions, if suffices to observe that N(p) = 1 implies i(p) = n(p) = 1, whereas N(p) = 2 implies either i(p) = 1 and n(p) = 2, or i(p) = 2 and n(p) = 1.

Now we turn our attention to the sides of a given Dirichlet region.

DEFINITION 4.5. Two sides  $S_i$  and  $S_j$  of  $F_0$  are called *conjugate sides* if there is an element  $\psi \in \Gamma$ ,  $\psi \neq id$ , with the property  $\psi(S_i) = S_j$ . Such an isometry is called a *conjugating isometry*.

**PROPOSITION** 4.6. Let  $S_i$  be a side of  $F_0$ . Then there exists a unique conjugate side, namely  $S_{-i} = \psi_{-i}(S_i)$ , and the conjugating isometry  $\psi_{-i}$  is uniquely determined. Moreover, if  $S_i$  is self-conjugate, then  $\psi_i^2 = id$ .

**Proof.** Let  $S_i$  be a side of  $F_0$ . Then  $S_i$  is also a side of  $F_i$  by Lemma 3.8 (1). Thus  $S_{-i} = \psi_{-i}(S_i)$  is a side of  $\psi_{-i}(F_i) = F_0$  (cf. Lemma 4.3). Therefore  $S_{-i}$  is a conjugate side of  $S_i$  with conjugating isometry  $\psi_{-i}$ .

Set  $\Sigma_i = \{ \psi \in \Gamma; \psi(S_i) \subset \operatorname{cl} F_0 \}$ . Since

 $\{\psi_0, \psi_{-i}\} \subset \Sigma_i \subset \{\psi \in \Gamma; \psi(p) \in \operatorname{cl} F_0\}$ 

for any  $p \in S_i$  and N(p) = 2 for all  $p \in S_i$ , we have  $\Sigma_i = \{\psi_0, \psi_{-i}\}$ . This proves both uniqueness assertions.

If  $\psi_{-i}(S_i) = S_i$ , then  $\psi_{-i}^2(S_i) = S_i$ . Hence  $\psi_{-i}^2 \in \Sigma_i$ . Since  $\psi_{-i} \neq id$ , this implies  $\psi_{-i}^2 = id$ . Hence  $\psi_i^2 = id$ .

We now come to the main result of Section 4.

THEOREM 4.7. Let  $\Gamma$  be a discrete subgroup of I(M) and let  $F_0$  be the Dirichlet region based at  $p_0 \notin Fix(\Gamma)$  with respect to  $\Gamma(p_0)$ . Denote by  $\Sigma$  the set of isometries

 $\Sigma = \{\psi_i \in \Gamma; S_i \text{ is a side of } F_0\}.$ 

Then  $\Sigma$  generates  $\Gamma$ . In particular, if  $F_0$  has only finitely many sides, then  $\Gamma$  is finitely generated.

**Proof.** Let  $\psi_j$  be an arbitrary element of  $\Gamma$ . By Corollary 3.18 (2), we may join  $p_0$  and  $p_j$  by a path  $c:[0, 1] \rightarrow \text{reg } T$ . The compact set c([0, 1]) meets only finitely many members of the tessellation  $T = \{F_i; i \in I\}$ .

Set  $t_0 = \sup \{t \in [0, 1]; c(t) \in cl F_0\}$ . Since  $cl F_0$  is closed, we actually have  $c(t_0) \in cl F_0$ . If  $t_0 = 1$ , then  $\psi_j = id$  and there is nothing to prove. On the other hand,  $t_0 > 0$ , because  $c(0) = p_0$  lies in the open set  $F_0$ . By our choice of the path c and the definition of  $t_0$ , the point  $c(t_0)$  is a regular boundary point of  $F_0$ . Hence there is a well defined index  $i_1 \in I$  such that  $c(t_0) \in S_{i_1} \subset cl F_0 \cap \psi_{i_1}(cl F_0)$ . By definition of  $\Sigma$  the isometry  $\psi_{i_1}$  is contained in  $\Sigma$ .

Now we set  $t_1 = \sup \{t \in [0, 1]; c(t) \in cl F_{i_1}\}$ . If  $t_1 = 1$  the process stops and  $\psi_i = \psi_{i_1}$ . Otherwise  $t_1 < 1$ . We claim that  $t_1 > t_0$ . It is clear that  $t_1 \ge t_0$ . According to Lemma 3.8(2), there is a neighborhood of  $c(t_0)$  which is contained in  $F_0 \cup S_{i_1} \cup F_{i_1}$ , and since c must leave cl  $F_0$  for  $t > t_0$ , it has to stay in  $F_{i_1}$  for some  $t > t_0$ . This implies that  $t_1 > t_0$ .

Again  $c(t_1)$  is a regular boundary point and there is a well defined  $i_2 \in I$  such that  $c(t_1) \in \operatorname{cl} F_{i_1} \cap \operatorname{cl} F_{i_2}$ . Moreover,  $i_2 \neq 0$ ,  $i_1$ . Since  $F_{i_1}$  and  $F_{i_2}$  have a common side, the isometry  $\psi_{i_2} \circ \psi_{-i_1}$  belongs to  $\Sigma$ . Therefore  $\psi_{i_2}$  can be written as a product of elements of  $\Sigma$ .

After finitely many steps this process ends with  $\psi_{i_k} = \psi_j$ . Thus  $\psi_j$  can be written as

$$\psi_{j} = (\psi_{i_{k}} \circ \psi_{-i_{k-1}}) \circ \cdots \circ (\psi_{i_{2}} \circ \psi_{-i_{1}}) \circ \psi_{i_{1}}.$$

*Remarks.* (1) The proof of Theorem 4.7 follows the classical scheme (cf. Busemann [1], Thm. 2.10 for general spaces). The new ingredient is Corollary 3.18, which enables us to choose a path in reg T. Thus we get  $\Sigma = \{\psi_i \in \Gamma; S_i \text{ is a side of } F_0\}$  as a set of generators, rather than the larger set  $\{\psi_i \in \Gamma; cl F_0 \cap \psi_i(cl F_0) \neq \emptyset\}$ .

(2) As has been noted by L. Danzer, the group generated by the translations  $z \mapsto z+1$  and  $z \mapsto z+e^{i\pi/3}$ ,  $z \in \mathbb{C}$ , shows that the set of generators given by Theorem 4.7 is not necessarily minimal.

(3) For n=2 and  $\Gamma$  acting freely, Eberlein has proved that  $\Gamma$  is finitely

generated if and only if one (and hence all) Dirichlet regions have finitely many sides (Eberlein [3], Thm. A). For  $n \ge 3$  this is no longer true (cf. Greenberg [5], Thm. 2).

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