Comment. Math. Helvetici **54** (1979) 629–641 Birkhäuser Verlag, Basel

On Mackey's Imprimitivity Theorem*

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1. Introduction

Mackey's Imprimitivity Theorem ([1], Theorem 2; [2], Theorem 6.6) has played a fundamental role in the genesis and development of the representation theory of locally compact groups, and has found applications in other fields of mathematics as well as in quantum mechanics. In the course of the years, it has been extended to mathematical structures different from groups (cf., for instance, [3]) and new versions have appeared, some of which avoid Mackey's separability assumptions.

In the present paper, we look for a generalization of the Imprimitivity Theorem in another direction, namely, by admitting subrepresentations of induced representations. It turns out (Section 3) that the Imprimitivity Theorem is essentially still valid provided transitive systems of imprimitivity are replaced by "transitive systems of covariance" (Section 2), i.e., provided positive-operator-valued measures take the place of projection-valued measures. In particular, we show that a strongly continuous unitary representation of a second countable locally compact group G on a separable (complex) Hilbert space is unitarily equivalent to a representation induced from a closed subgroup of G if and only if there exists an associated transitive system of covariance. The same problem has been tackled by Scutaru ([4], Theorem 1) who has shown that the Imprimitivity Theorem can be generalized to subrepresentations if the positive-operator-valued measures involved satisfy a given continuity condition. Indirectly, we show here that Scutaru's condition is always satisfied (cf. Remark 5). In Section 4, we extend the result of Section 3 to projective representations.

Every group appearing in what follows will be written multiplicatively, with neutral element 1. For each topological space X, we shall denote by $\mathcal{B}_X$ the Borel structure (i.e., the σ-field) generated by the closed sets of X; whenever X is seen as a Borel space, it will always be with respect to this structure that we shall call the Borel structure of X. Every Hilbert space $\mathfrak{H}$ considered will be tacitly

* Supported by the Swiss National Science Foundation

understood to be a *complex* one, and $\mathscr{L}(\mathfrak{H})$ will stand for the (complex) vector space of all continuous linear operators in $\mathfrak{H}$. The symbol ϕ_A will denote the characteristic function of a set A.

2. Preliminaries

In preparation for formulating our generalization of Mackey's Imprimitivity Theorem, we begin by introducing some terminology which will be used extensively in the sequel.

DEFINITION 1. Let X be a topological space and let $\mathfrak{H}$ be a Hilbert space. A (weak) *Borel positive-operator-valued measure* (concisely: a *Borel* POV-*measure*) on X acting in $\mathfrak{H}$ is a mapping $M:\mathscr{B}_X \to \mathscr{L}(\mathfrak{H})$ such that

(i) M is positive, i.e., $M(\emptyset)=0$ and $M(B)\geq 0$ for all $B\in\mathscr{B}_X$;

(ii) M is (weakly) countably additive, i.e., if $(B_i)_{i\in\mathbf{N}}$ is a sequence of mutually disjoint elements of $\mathscr{B}_X$, then

$$M\left(\bigcup_{i=0}^{\infty} B_i\right)=\mathrm{w}-\sum_{i=0}^{\infty} M(B_i), \tag{2.1}$$

where "$\mathrm{w}-\sum$" means that the series $(M(B_i))$ converges in the weak operator topology on $\mathscr{L}(\mathfrak{H})$.

If $M(X)=\mathrm{Id}_{\mathfrak{H}}$, then M is said to be *normalized.*

Note that , if in addition to (i) and (ii) M satisfies

(iii) $M(B)M(B')=M(B\cap B')$

for all B, B' in $\mathscr{B}_X$, then M is a *Borel projection-valued measure* (concisely: a *Borel* PV-*measure*).

Remark 1. The increasing sequence of positive operators $(\sum_{i=0}^{n} M(B_i))$ in (2.1) is norm-bounded by $\|M(X)\|$ and therefore converges in the strong operator topology. It follows that the right-hand side of (2.1) always exists and that the weak measure M is a strong one too. The norm-boundedness of the sequence also implies that M is an ultraweak (resp. ultrastrong) measure, i.e., that $(\sum_{i=0}^{n} M(B_i))$ converges in the ultraweak (resp. ultrastrong) topology on $\mathscr{L}(\mathfrak{H})$.

Remark 2. In order that the mapping $M:\mathscr{B}_X\to\mathscr{L}(\mathfrak{H})$ be a Borel POV-measure on X acting in $\mathfrak{H}$ it is necessary and sufficient that the mapping $M_\psi:\mathscr{B}_X\to\mathbf{C}$ defined by $M_\psi(B)=(M(B)\psi\mid\psi)$ be a (bounded) positive Borel measure for all $\psi\in\mathfrak{H}$. Sufficiency follows easily by using the polarization identity.

Let G be a topological group. A topological space $X \neq \emptyset$ is said to be a *topological* (*left*) *G-space* if G operates continuously on (the left of) X, i.e., if there exists a continuous mapping $(g, x) \mapsto g.x$ of the topological product space $G \times X$ into X such that, for each $x \in X$, we have $1.x = x$ and $(gg').x = g.(g'.x)$ for all g, g' in G. If H is a subgroup of G, we denote by G/H the topological homogeneous space of left cosets of H in G, which is a topological G-space in a canonical way.

DEFINITION 2. Let G be a topological group, let X be a topological G-space, let U be a strongly continuous unitary representation of G on a Hilbert space $\mathfrak{H}$, and let M be a normalized Borel POV-measure on X acting in $\mathfrak{H}$. We say that M is *G-covariant* and that the ordered pair (U, M) is a *system of G-covariance* in $\mathfrak{H}$ based on X if U, M satisfy

$$U(g)M(B)U(g)^{-1} = M(g.B) \tag{2.2}$$

for all $g \in G$ and all $B \in \mathcal{B}_X$. The system (U, M) is said to be *transitive* if so is the G-space X.

We note that if M is a Borel PV-measure, then (U, M) is a Mackey's *system of imprimitivity* for G based on X and acting in $\mathfrak{H}$.

Two systems of G-covariance, (U, M) in $\mathfrak{H}$ and (U', M') in $\mathfrak{H}'$, both based on X, are said to be *unitarily equivalent* if there exists a unitary mapping V of $\mathfrak{H}$ onto $\mathfrak{H}'$ such that

$$VU(g) = U'(g)V \quad \text{for all} \quad g \in G \tag{2.3}$$

and

$$VM(B) = M'(B)V \quad \text{for all} \quad B \in \mathcal{B}_X. \tag{2.4}$$

3. Transitive systems of covariance

Given a family $(Y_\iota)_{\iota \in I}$ of Borel spaces and, for each $\iota \in I$, a mapping f_ι of a set X into Y_ι, we shall denote by $\sigma_{\mathcal{B}}(f_\iota)$ the weakest Borel structure on X making Borel all the mappings f_ι. The symbol $\mathcal{L}_s(\mathfrak{H})$ will stand for the vector space $\mathcal{L}(\mathfrak{H})$ equipped with the strong operator topology (i.e., the topology of pointwise convergence) and $\mathcal{L}_s^w(\mathfrak{H})$ will stand for the same vector space endowed with the weak operator topology, which is the topology of pointwise convergence when $\mathfrak{H}$ is considered with the weak topology. We shall denote by $h_{\psi,\psi'}$ (ψ, ψ' in $\mathfrak{H}$) the mapping $A \mapsto (A\psi \mid \psi')$ of $\mathcal{L}(\mathfrak{H})$ into $\mathbf{C}$. To end this notational introduction, we remember that a topological space X is said to be *fully Lindelöf* if every subspace of X is Lindelöf.

LEMMA 1. *Let $(Y_\iota)_{\iota\in I}$ be a family of topological spaces and, for each $\iota \in I$, let f_ι be a mapping of a set X into Y_ι. If X, equipped with the weakest topology making continuous all the mappings f_ι, is fully Lindelöf, then $\mathscr{B}_X$ and $\sigma_{\mathscr{B}}(f_\iota)$ are identical.*

Proof. The identity mapping of the Borel space $(X, \mathscr{B}_X)$ onto $(X, \sigma_{\mathscr{B}}(f_\iota))$ is Borel. On the other hand, let U be an arbitrary open set of the topological space X. Since U is Lindelöf and since a base of the topology of X is given by the finite intersections of sets of the form $f_\iota^{-1}(O_\iota)(\iota\in I;\ O_\iota$ open set of $Y_\iota)$, there exist, for each $N\in\mathbf{N}$, a finite subfamily $(f_{\iota_{N,i}})_{1\le i\le n_N}$ of $(f_\iota)_{\iota\in I}$ and a finite sequence $(O_{\iota_{N,i}})_{1\le i\le n_N}$ of open sets with $O_{\iota_{N,i}} \subseteq Y_{\iota_{N,i}}$ such that

$$U = \bigcup_{N=0}^{\infty} \bigcap_{i=1}^{n_N} f_{\iota_{N,i}}^{-1}(O_{\iota_{N,i}}).$$

Therefore U is a Borel set of $(X, \sigma_{\mathscr{B}}(f_\iota))$ and the identity mapping of $(X, \sigma_{\mathscr{B}}(f_\iota))$ onto $(X, \mathscr{B}_X)$ is Borel. ∎

LEMMA 2. *Let $\mathfrak{H}$ be a separable Hilbert space. The Borel structure of $\mathscr{L}_s(\mathfrak{H})$ (resp. $\mathscr{L}_s^w(\mathfrak{H})$) is identical with $\sigma_{\mathscr{B}}(h_{\psi,\psi'})$.*

Proof. Our first remark is that $\mathscr{L}(\mathfrak{H})$ equipped with the compact-open topology is a Lusin space ([5], Part I, Ch. II, Theorem 7); hence the Borel structure generated by its closed sets is identical with the Borel structure of the Lusin space $\mathscr{L}_s(\mathfrak{H})$ (resp. $\mathscr{L}_s^w(\mathfrak{H})$) by virtue of a well-known Souslin's theorem ([6], TG.IX, §6, Prop. 14). It follows that the lemma is also proven for $\mathscr{L}_s(\mathfrak{H})$ once it is proven for $\mathscr{L}_s^w(\mathfrak{H})$. On the other hand, the desired result for $\mathscr{L}_s^w(\mathfrak{H})$ follows from Lemma 1 because a Lusin space is fully Lindelöf. In fact, every subspace of a Lusin space is image by a continuous mapping of a subspace of a Polish space, thus it is Lindelöf ([6], TG.IX, Appendice I, Prop. 1). ∎

LEMMA 3. *Let G be a second countable locally compact group and let X be a transitive Hausdorff G-space. If $\mathfrak{H}$ is a separable Hilbert space and M is a Borel* POV*-measure on X acting in $\mathfrak{H}$, then the mapping $g \mapsto M(g.B)$ of G into $\mathscr{L}_s^w(\mathfrak{H})$ (resp. $\mathscr{L}_s(\mathfrak{H})$) is Borel for all $B\in\mathscr{B}_X$.*

Proof. We first show that, for each $\psi\in\mathfrak{H}$ and each $B\in\mathscr{B}_X$, the real-valued function

$$g \mapsto M_\psi(g.B) = (M(g.B)\psi \mid \psi)$$

defined in G is Borel. To prove this, we remark that the function

$$f_B : (g, x) \mapsto \phi_B(g^{-1}.x)$$

of the Borel product space $G \times X$ into $\mathbf{R}$ is Borel. In fact, X is a Souslin space since, for a fixed $x \in X$, the mapping $g \mapsto g.x$ of the Polish group G onto X is continuous; therefore we have $\mathscr{B}_{G\times X} = \mathscr{B}_G \times \mathscr{B}_X$ ([7], Ch. III, §2, Theorem 2), namely, the Borel structure of the topological product space $G \times X$ is identical with the product of the Borel structures of G and X. This implies that the mapping $(g, x) \mapsto g^{-1}.\, x$ of the product Borel space $G \times X$ onto X, and hence f_B, are Borel. Following Mackey ([8], Lemma 7.1), we can now apply a part of Fubini's theorem (cf. [9], §36, Theorem B) and conclude that the function

$$g \mapsto M_\psi(g.B) = \int \phi_{g.B}(x)\, dM_\psi(x) = \int f_B(g, x)\, dM_\psi(x)$$

is Borel.

By using the polarization identity, we see that the complex-valued function

$$g \mapsto (M(g.B)\psi \mid \psi') = h_{\psi,\psi'}(M(g.B))$$

defined in G is Borel for all ψ, ψ' in $\mathfrak{H}$. The desired result follows from Lemma 2, since a mapping l of a Borel space into $\mathscr{L}(\mathfrak{H})$ equipped with $\sigma_{\mathscr{B}}(h_{\psi,\psi'})$ is Borel if and only if $h_{\psi,\psi'} O l$ is Borel for all ψ, ψ' in $\mathfrak{H}$. ■

Remark 3. Lemma 3 is also valid if X is a Borel G-space (not necessarily transitive), i.e., if X is a G-space, is endowed with a Borel structure, and the mapping $(g, x) \mapsto g.x$ of the product Borel space $G \times X$ onto X is Borel. The definitions of a Borel POV-measure (Definition 1) and of a system of G-covariance (Definition 2) are obviously also meaningful if X is a Borel G-space and $\mathscr{B}_X$ the Borel structure of X. The same is true for the notions of a system of imprimitivity and of unitary equivalence of two systems of G-covariance.

PROPOSITION 1. *Let G be a second countable locally compact group, let X be a countably generated Borel G-space, and let $\mathfrak{H}$, $\mathfrak{H}'$ be separable Hilbert spaces. If (U, M) is a system of G-covariance in $\mathfrak{H}$ based on X, there exist a separable Hilbert space $\mathfrak{H}_e$, an isometric mapping W of $\mathfrak{H}$ into $\mathfrak{H}_e$, and a system of imprimitivity*

(U_e, P) for G based on X and acting in $\mathfrak{H}_e$ satisfying

$$WU(g) = U_e(g)W \quad \textit{for all} \quad g \in G, \tag{3.1}$$

$$WM(B) = P(B)W \quad \textit{for all} \quad B \in \mathcal{B}_X, \tag{3.2}$$

and such that the set

$$\mathfrak{M} = \{P(B)W\psi \mid B \in \mathcal{B}_X \quad \textit{and} \quad \psi \in \mathfrak{H}\}$$

is total in $\mathfrak{H}_e$.

The mapping W is surjective if and only if (U, M) is a system of imprimitivity.

Let (U', M') be a system of G-covariance in $\mathfrak{H}'$ based on X and unitarily equivalent to (U, M). If there exist $\mathfrak{H}'_e$, W', P', U'_e, $\mathfrak{M}'$ mutually satisfying the same relations as, respectively, $\mathfrak{H}_e$, W, P, U_e, $\mathfrak{M}$ when $\mathfrak{H}'$, U', M', replace $\mathfrak{H}$, U, M, then the systems of imprimitivity (U_e, P) and (U'_e, P') are unitarily equivalent.

Proof. By a theorem of Neumark [10], there exist a Hilbert space $\mathfrak{H}_e$, an isometric mapping W of $\mathfrak{H}$ into $\mathfrak{H}_e$, and a normalized Borel PV-measure P on X acting in $\mathfrak{H}_e$ such that

$$WM(B) = P(B)W$$

for all $B \in \mathcal{B}_X$. We realize $\mathfrak{H}_e$, W, and P as follows. Let $\mathscr{C}_{\mathfrak{H}}(\mathcal{B}_X)$ be the complex vector space of all step functions based on $\mathcal{B}_X$ taking values in $\mathfrak{H}$. Define a positive Hermitian sesquilinear form $\langle\cdot|\cdot\rangle$ on $\mathscr{C}_{\mathfrak{H}}(\mathcal{B}_X)$ by

$$\left\langle \sum_i \psi_i \phi_{B_i} \,\middle|\, \sum_j \psi_j \phi_{B_j} \right\rangle = \sum_{i,j} \left(M\left(B_i \cap B_j\right)\psi_i \mid \psi_j \right), \tag{3.3}$$

where the sums are finite and $(\cdot \mid \cdot)$ is the inner multiplication on $\mathfrak{H}$. The right-hand side of (3.3) is independent of the particular form the elements of $\mathscr{C}_{\mathfrak{H}}(\mathcal{B}_X)$ can have as finite sums of terms $\psi\phi_B$ $(\psi \in \mathfrak{H};\, B \in \mathcal{B}_X)$. This follows from the additivity of M and from the fact that, given a finite family of elements of $\mathcal{B}_X$, it is always possible to choose a finite family of mutually disjoint elements of $\mathcal{B}_X$ such that every element of the first family is the union of elements of the second. The positivity of $\langle\cdot|\cdot\rangle$ is a consequence of the positivity of M. Let $\mathcal{J}$ be the subspace of all $f \in \mathscr{C}_{\mathfrak{H}}(\mathcal{B}_X)$ such that $\langle f \mid f\rangle = 0$; then $\mathfrak{H}_e$ is the completion of the quotient space $\mathscr{C}_{\mathfrak{H}}(\mathcal{B}_X)/\mathcal{J}$ equipped with the extended quotient form which we shall denote by $(\cdot|\cdot)_e$. The mapping W is defined by $W\psi = [f_\psi]$, where $f_\psi \in \mathscr{C}_{\mathfrak{H}}(\mathcal{B}_X)$

is the constant mapping with the value ψ and $[f_\psi]$ denotes the equivalence class of f_ψ modulo $\mathscr{J}$; the PV-measure P is given by

$$P(B)\Big[\sum_i \psi_i\phi_{B_i}\Big]=\Big[\sum_i \psi_i\phi_{B\cap B_i}\Big]\quad(\psi_i\in\mathfrak{H};\,B_i\in\mathscr{B}_X)$$

and extension by continuity. We remark that, for each $B\in\mathscr{B}_X$, we have $M(B)=W^*P(B)W$ and $W^*W=\mathrm{Id}_{\mathfrak{H}}$, where W^* is the adjoint of W. The set

$$\mathfrak{M}=\{P(B)W\psi\mid B\in\mathscr{B}_X \text{ and } \psi\in\mathfrak{H}\}$$

is total in $\mathfrak{H}_e$; it follows that P is weakly countably additive because the subset $\{P(B)\mid B\in\mathscr{B}_X\}$ of $\mathscr{L}(\mathfrak{H}_e)$ is norm-bounded and, for each sequence (B_i) of mutually disjoint elements of $\mathscr{B}_X$ and each pair $P(B)W\psi$, $P(B')W\psi'$ of elements of $\mathfrak{M}$, we have

$$\begin{aligned}\Big(P\Big(\bigcup_{i=0}^{\infty}B_i\Big)P(B)W\psi\mid P(B')W\psi'\Big)_e&=\Big(M\Big(\bigcup_{i=0}^{\infty}(B\cap B'\cap B_i)\Big)\psi\mid\psi'\Big)\\&=\sum_{i=0}^{\infty}(M(B\cap B'\cap B_i)\psi\mid\psi')\\&=\sum_{i=0}^{\infty}(P(B_i)P(B)W\psi\mid P(B')W\psi')_e.\end{aligned}$$

Since $\mathscr{B}_X$ is countably generated and $\mathfrak{H}$ is separable, $\mathfrak{H}_e$ is separable. For let $\{B_i\}_{i\in\mathbf{N}}$ be a clan (or ring) of elements of $\mathscr{B}_X$ generating $\mathscr{B}_X$ and let $\{\psi_i\}_{i\in\mathbf{N}}$ be a dense subset of elements of $\mathfrak{H}$; the set

$$\mathfrak{M}_0=\{P(B_k)\psi_l\mid B_k\in\{B_i\}\quad\text{and}\quad\psi_l\in\{\psi_i\}\}$$

is dense in $\mathfrak{M}$. In fact, for each $P(B)W\psi\in\mathfrak{M}$ and an arbitrary positive real number ε, we can choose $B_k\in\{B_i\}$ such that $(P(B\Delta B_k)W\psi\mid W\psi)_e^{1/2}<\varepsilon/2$ ([9], §13, Theorem D) and $\psi_l\in\{\psi_i\}$ such that $\|\psi-\psi_l\|<\varepsilon/2$; then we have

$$\begin{aligned}\|P(B)W\psi-P(B_k)W\psi_l\|_e&\le\|(P(B)-P(B_k))W\psi\|_e\\&\quad+\|P(B_k)W(\psi-\psi_l)\|_e\\&\le(P(B\Delta B_k)W\psi\mid W\psi)_e^{1/2}\\&\quad+\|\psi-\psi_l\|<\varepsilon.\end{aligned}$$

Now, for each $g \in G$, let $U_e(g)$ be the unitary operator in $\mathfrak{H}_e$ defined in $\mathfrak{M}$ by

$$U_e(g)P(B)W\psi = P(g.B)WU(g)\psi \tag{3.4}$$

and extended to $\mathfrak{H}_e$ by linearity and continuity. This definition makes sense since

$$\begin{aligned}(U_e(g)P(B)W\psi \mid U_e(g)P(B')W\psi')_e &= (M(g.(B \cap B'))U(g)\psi \mid U(g)\psi') \\ &= (M(B \cap B')\psi \mid \psi') \\ &= (P(B)W\psi \mid P(B')W\psi')_e\end{aligned}$$

for all $P(B)W\psi$, $P(B')W\psi'$ in $\mathfrak{M}$. Let $\mathscr{L}_s(\mathfrak{H}_e)_1$ be the closed unit ball of $\mathscr{L}(\mathfrak{H}_e)$ equipped with the strong operator topology. The mapping $g \mapsto (P(g.B), WU(g)\psi)$ of G into the topological product space $\mathscr{L}_s(\mathfrak{H}_e)_1 \times \mathfrak{H}_e$ is Borel for all $B \in \mathscr{B}_X$ and all $\psi \in \mathfrak{H}$ by Lemma 3 (Remark 3) and because the Borel structure of $\mathscr{L}_s(\mathfrak{H}_e)_1 \times \mathfrak{H}_e$ coincides with the product Borel structure. In addition, the mapping $(A, \psi) \mapsto A\psi$ of $\mathscr{L}_s(\mathfrak{H}_e)_1 \times \mathfrak{H}_e$ into $\mathfrak{H}_e$ is continuous. From this, from Lemma 1, and from the uniform equicontinuity of the unitary group $\mathbf{U}(\mathfrak{H}_e)$, we can conclude ([11], Chap. III, §3, Prop. 5) that the homomorphism $g \mapsto U_e(g)$ of G into $\mathbf{U}(\mathfrak{H}_e)$ equipped with the strong operator topology is Borel, hence continuous. (Note that the strong operator topology is identical on $\mathbf{U}(\mathfrak{H}_e)$ with the weak operator one and makes $\mathbf{U}(\mathfrak{H}_e)$ into a Polish group ([12], Lemme 4)). Finally, we get that (U_e, P) is a system of imprimitivity for G based on X and acting in $\mathfrak{H}_e$, as can be easily checked in $\mathfrak{M}$.

Given the system of G-covariance (U', M'), suppose that we have a Hilbert space $\mathfrak{H}'_e$ an isometric mapping W' of $\mathfrak{H}'$ into $\mathfrak{H}'_e$, a system of imprimitivity (U'_e, P') for G based on X and acting in $\mathfrak{H}'_e$ satisfying $W'^*P'(B)W' = M'(B)$ for all $B \in \mathscr{B}_X$, $W'^*U'_e(g)W' = U'(g)$ for all $g \in G$, and suppose that the set

$$\mathfrak{M}' = \{P'(B)W'\psi \mid B \in \mathscr{B}_X \text{ and } \psi \in \mathfrak{H}'\}$$

is total in $\mathfrak{H}'_e$. If Z is a unitary mapping of $\mathfrak{H}$ onto $\mathfrak{H}'$ establishing the equivalence of (U, M) to (U', M'), then the mapping $P(B)W\psi \mapsto P'(B)W'Z\psi$ of $\mathfrak{M}$ onto $\mathfrak{M}'$ extends by linearity and continuity to a unitary mapping of $\mathfrak{H}_e$ onto $\mathfrak{H}'_e$ making (U_e, P) and (U'_e, P') unitarily equivalent. The assertion about the surjectivity of W follows at once. ∎

Let G be a locally compact group and let H be a closed subgroup of G. We denote by $\mathrm{Ind}_H^G U$ the (strongly continuous unitary) representation of G induced from H by a strongly continuous unitary representation U of H on a Hilbert space, say $\mathfrak{H}$. In what follows, whenever G is second countable and $\mathfrak{H}$ separable, we shall always assume that $\mathrm{Ind}_H^G U$ is realized on $l^2(G/H, \mu)$, the Hilbert space of all equivalence classes of μ-square-integrable mappings of G/H into $\mathfrak{H}$, where μ is a G-quasi-invariant measure on G/H. Moreover, we shall denote by $P_{\mathfrak{H}}$ the standard Borel PV-measure on G/H acting in $L^2_{\mathfrak{H}}(G/H, \mu)$ defined by

$$P_{\mathfrak{H}}(B)f = \phi_B f \quad (f \in L^2_{\mathfrak{H}}(G/H, \mu))$$

(with the familiar abuse of notation of using the same symbol for a mapping and for its equivalence class).

PROPOSITION 2. *Let G be a second countable locally compact group, let H be a closed subgroup of G, let μ be a G-quasi-invariant measure on G/H, and let $\mathfrak{H}, \mathfrak{H}'$ be separable Hilbert spaces. If (U, M) is a system of G-covariance in $\mathfrak{H}$ based on G/H, there exist a strongly continuous unitary representation $\gamma(U)$ of H on a separable Hilbert space $\mathfrak{K}$ and an isometric mapping V of $\mathfrak{H}$ into $L^2_{\mathfrak{K}}(G/H, \mu)$ satisfying*

$$VU(g) = (\mathrm{Ind}_H^G \gamma(U))(g)V \quad \textit{for all} \quad g \in G, \tag{3.5}$$

$$VM(B) = P_{\mathfrak{K}}(B)V \quad \textit{for all} \quad B \in \mathscr{B}_{G/H}, \tag{3.6}$$

and such that the set

$$\{P_{\mathfrak{K}}(B)V\psi \mid B \in \mathscr{B}_{G/H} \text{ and } \psi \in \mathfrak{H}\}$$

is total in $L^2_{\mathfrak{K}}(G/H, \mu)$.

The mapping V is surjective if and only if (U, M) is a system of imprimitivity.

If (U', M') is a system of G-covariance in $\mathfrak{H}'$ based on G/H and unitarily equivalent to (U, M), and if $\mathfrak{K}'$ is the carrier space of $\gamma(U')$, then the systems of imprimitivity $(\mathrm{Ind}_H^G \gamma(U), P_{\mathfrak{K}})$ and $(\mathrm{Ind}_H^G \gamma(U'), P_{\mathfrak{K}'})$ are unitarily equivalent.

Proof. We apply Mackey's Imprimitivity Theorem to the system of imprimitivity (U_e, P) constructed in the proof of Proposition 1 with $X = G/H$; so we get $\gamma(U)$ and a unitary mapping W_e of $\mathfrak{H}_e$ onto $L^2_{\mathfrak{K}}(G/H, \mu)$ making $(\mathrm{Ind}_H^G \gamma(U), P_{\mathfrak{K}})$ unitarily equivalent to (U_e, P) and such that (3.5), (3.6) are satisfied with $V = W_e W$. Obviously, V is onto $L^2_{\mathfrak{K}}(G/H, \mu)$ if and only if W is onto $\mathfrak{H}_e$, i.e., if and only if (U, M) is a system of imprimitivity. The other assertions follow immediately. ■

Remark 4. Equation (3.5) expresses the unitary equivalence of U to a subrepresentation of $\mathrm{Ind}_H^G\gamma(U)$; conversely, if an isometric mapping V of $\mathfrak{H}$ into $L^2_{\mathfrak{K}}(G/H, \mu)$ establishes such an equivalence, then (3.5) is satisfied and we have $V^*V=\mathrm{Id}_{\mathfrak{H}}$. Moreover, if M is defined by (3.6), i.e., by $M(B)=V^*P_{\mathfrak{K}}(B)V$, then (U, M) is a system of G-covariance in $\mathfrak{H}$ based on G/H. Note that (3.4) implies $U(g)=W^*U_e(g)W$ for all $g\in G$; the corresponding equation with $\mathrm{Ind}_H^G\gamma(U)$ instead of U_e and V instead of W is obviously valid.

Remark 5. A posteriori, we see that, for each $B\in\mathscr{B}_{G/H}$, the mapping $g\mapsto P(g.B)$ of G into $\mathscr{L}_s^w(\mathfrak{H}_e)$ considered above is actually continuous since

$$(P(g.B)\psi \mid \psi')_e=(P(B)U_e(g)^{-1}\psi \mid U_e(g)^{-1}\psi')_e$$

for all ψ, ψ' in $\mathfrak{H}_e$.

COROLLARY. *Let $G, H, \mathfrak{H}$ be as in Proposition 2 and let U be a strongly continuous unitary representation of G on $\mathfrak{H}$. The following conditions are equivalent*:

(i) *There exists a normalized Borel* POV-*measure M on G/H acting in $\mathfrak{H}$ such that (U, M) is a system of G-covariance in $\mathfrak{H}$ based on G/H;*

(ii) *U is unitarily equivalent to a subrepresentation of a strongly continuous unitary representation of G induced from H.*

Every (weak) Borel POV-measure M on a locally compact space X acting in $\mathfrak{H}$ determines a *Radon vector measure* M_{R} on X taking values in $\mathscr{L}_s^w(\mathfrak{H})$ ([13], §2), defined by

$$M_{\mathrm{R}}(f)=\int f(x)\,dM(x)\qquad (f\in\mathscr{C}^0_{\mathrm{C}}(X)),$$

which is positive, i.e., such that $M_{\mathrm{R}}(f)\geqslant 0$ if $f\geqslant 0$, and satisfies

$$\|M_{\mathrm{R}}(f)\|\leqslant\sup_{x\in X}|f(x)|.$$

This follows at once by noting that, for each $f\in\mathscr{C}^0_{\mathrm{C}}(X)$, the mapping

$$(\psi, \psi')\mapsto\left(\left(\int f(x)\,dM(x)\right)\psi \mid \psi'\right)=\int f(x)\,dM_{\psi,\psi'}(x)$$

is a bounded sesquilinear form, where the bounded complex Borel measure $M_{\psi,\psi'}$ is defined by $M_{\psi,\psi'}(B)=(M(B)\psi \mid \psi')$. The converse is also true with $M(B)=M_{\mathrm{R}}(\phi_B)$ $(B\in\mathscr{B}_X)$ after extension of M_{R} to the M_{R}-integrable functions ([13], §2, Prop. 3). One checks that M is a PV-measure if and only if $M_{\mathbf{R}}$ is a $*$-representation of the C^*-algebra $\mathscr{C}^0_{\mathbf{C}}(X)$.

If X is locally compact, we can then define a system of G-covariance also as an ordered pair $(U, M_{\mathbf{R}})$, where the Borel POV-measure M of Definition 2 is replaced by a normalized norm-bounded positive Radon vector measure on X taking values in $\mathscr{L}^w_s(\mathfrak{H})$ and (2.2) is replaced by

$$U(g)M_{\mathbf{R}}(f)U(g)^{-1}=M_{\mathbf{R}}(g.f) \qquad (f\in\mathscr{C}^0_{\mathbf{C}}(X)),$$

with $(g.f)(x)=f(g^{-1}.x)$ for all $x\in X$. Proposition 2 and its corollary can be formulated accordingly and the theorem of Neumark used in the proof follows in this context from ([14], Theorems 1 and 4), where another realization of the Hilbert space $\mathfrak{H}_e$ is given; moreover, the separability assumptions on $G, \mathfrak{H}, \mathfrak{H}'$, and $\mathfrak{K}$ can be dropped by choosing Blattner's realization of $\mathrm{Ind}_H^G U$ [15]. In fact, the subalgebra of $\mathscr{C}^b_{\mathbf{C}}(X)$ (with the sup-norm) generated by $\mathscr{C}^0_{\mathbf{C}}(X)$ and the constant functions can be chosen as the C^*-algebra $\mathscr{A}$ of Stinespring's Theorem 1. The unitary representation U_e of G on $\mathfrak{H}_e$ defined by

$$U_e(g)[f\otimes\psi]=[g.f\otimes U(g)\psi] \qquad (f\in\mathscr{A};\ \psi\in\mathfrak{H}),$$

where [ ] denotes an equivalence class modulo the elements with vanishing norms, is strongly continuous because the linear representation $g\mapsto g.f$ of G on $\mathscr{A}$, with $g.f=f$ whenever f is a constant function, is continuous ([16], Chap. 8, §2, Props. 5 and 2). Finally, use is made of Blattner's version of the Imprimitivity Theorem ([17], Corollary to Theorem 2).

4. Transitive projective systems of covariance

Let $\mathbf{P}(\mathfrak{H})$ be the projective space deduced from a separable Hilbert space $\mathfrak{H}$ and let $\omega_{\mathfrak{H}}$ be the canonical surjection of $\mathfrak{H}-\{0\}$ onto $\mathbf{P}(\mathfrak{H})$. We shall denote by $\Omega_{\mathfrak{H}}$ the canonical surjection of $\mathbf{U}(\mathfrak{H})$ onto the projective unitary group $\mathbf{PU}(\mathfrak{H})$ and assume that $\mathbf{U}(\mathfrak{H})$ is equipped with the strong operator topology and $\mathbf{PU}(\mathfrak{H})$ with the final topology for $\Omega_{\mathfrak{H}}$. A *strongly continuous unitary projective representation* of a topological group G on $\mathbf{P}(\mathfrak{H})$ is then a continuous homomorphism $\tilde{U}$ of G into $\mathbf{PU}(\mathfrak{H})$. Since $\mathbf{U}(\mathfrak{H})$ (and therefore $\mathbf{PU}(\mathfrak{H})$) is Polish, we can lift $\tilde{U}$ to a *Borel unitary multiplier representation* of G([12], Lemme 3), i.e., to a Borel mapping U of G into $\mathbf{U}(\mathfrak{H})$ satisfying $U(1)=\mathrm{Id}_{\mathfrak{H}}$ and

$$U(g)U(g')=\alpha(g,g')U(gg')$$

for all g, g' in G. The *multiplier* α is a 2-cocycle of G with values in the circle group (equipped with the trivial operation of G). A *projective system of G-covariance* in $\mathfrak{H}$ based on a topological G-space X is an ordered pair $(\tilde{U}, M)$, where $\tilde{U}$ is a strongly continuous unitary projective representation of G on $\mathbf{P}(\mathfrak{H})$ and M is a normalized Borel POV-measure on X acting in $\mathfrak{H}$ such that (2.2) is satisfied by a Borel lifting U of $\tilde{U}$ (and hence by every lifting of $\tilde{U}$). Two such systems, $(\tilde{U}, M)$ in $\mathfrak{H}$ and $(\tilde{U}', M')$ in $\mathfrak{H}'$, both based on X, are said to be *unitarily equivalent* if there exists a unitary mapping V of $\mathfrak{H}$ onto $\mathfrak{H}'$ such that (2.4) is satisfied together with

$$\tilde{V} \circ \tilde{U}(g) = \tilde{U}'(g) \circ \tilde{V} \quad \text{for all} \quad g \in G,$$

where $\tilde{V}$ is defined by $\tilde{V} \circ \omega_{\mathfrak{H}} = \omega_{\mathfrak{H}'} \circ (V \mid \mathfrak{H} - \{0\})$.

Let now G be a second countable locally compact group and let $\tilde{U}$ be a strongly continuous unitary projective representation of a closed subgroup H of G on $\mathbf{P}(\mathfrak{H})$. We shall denote by $\mathrm{Ind}_H^G \tilde{U}$ the projective representation of G induced from H by $\tilde{U}$, namely, the (strongly continuous unitary) projective representation

$$\Omega_{L_{\mathfrak{H}}^2(G/H,\mu)} \circ \mathrm{Ind}_H^G U,$$

where μ is a G-quasi-invariant measure on G/H and $\mathrm{Ind}_H^G U$ is the Borel unitary multiplier representation of G induced from H by a Borel lifting U of $\tilde{U}$([2], §4). We remark that $\mathrm{Ind}_H^G \tilde{U}$ is independent of the particular Borel lifting chosen.

PROPOSITION 3. *Let $G, H, \mu, \mathfrak{H}$ and $\mathfrak{H}'$ be as in Proposition 2 and let $(\tilde{U}, M)$ be a projective system of G-covariance in $\mathfrak{H}$ based on G/H. There exist a strongly continuous unitary projective representation $\gamma(\tilde{U})$ of H on $\mathbf{P}(\mathfrak{K})$, with $\mathfrak{K}$ a separable Hilbert space, and an isometric mapping V of $\mathfrak{H}$ into $L_{\mathfrak{K}}^2(G/H, \mu)$ such that*

$$\tilde{V} \circ \tilde{U}(g) = (\mathrm{Ind}_H^G \gamma(\tilde{U}))(g) \circ \tilde{V} \quad \text{for all} \quad g \in G,$$

$$VM(B) = P_{\mathfrak{K}}(B) V \quad \text{for all} \quad B \in \mathcal{B}_{G/H},$$

where $\tilde{V}$ is defined by $\tilde{V} \circ \omega_{\mathfrak{H}} = \omega_{L_{\mathfrak{K}}^2(G/H,\mu)} \circ (V \mid \mathfrak{H} - \{0\})$.

The set

$$\{P_{\mathfrak{K}}(B) V \psi \mid B \in \mathcal{B}_{G/H} \text{ and } \psi \in \mathfrak{H}\}$$

is total in $L_{\mathfrak{K}}^2(G/H, \mu)$; the mapping V is surjective if and only if $(\tilde{U}, M)$ is a projective system of imprimitivity.

If $(\tilde{U}', M')$ is a projective system of G-covariance in $\mathfrak{H}'$ based on G/H and unitarily equivalent to $(\tilde{U}, M)$, and if $\mathbf{P}(\mathfrak{K}')$ is the carrier space of $\gamma(\tilde{U}')$, then the projective systems of imprimitivity $(\mathrm{Ind}_H^G \gamma(\tilde{U}), P_{\mathfrak{K}})$ *and* $(\mathrm{Ind}_H^G \gamma(\tilde{U}'), P_{\mathfrak{K}'})$ *are unitarily equivalent.*

The proof of Proposition 3 parallels that of Proposition 2, but uses the Projective Imprimitivity Theorem ([2], Theorem 6.6). The corollary to Proposition 2 may also be restated for projective representations accordingly, and Proposition 3, as well as the corollary, can be formulated in terms of multiplier representations.

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Received March 8, 1979