Groups of finite quasi-projective dimension.

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1. Introduction

1.1. Lyndon's Identity Theorem [11] may be interpreted as a description of the structure of the relation module arising from a one-relator presentation – or, more generally, from a staggered presentation – of a group G. For such presentations, the relation module is the direct sum of the cyclic submodules generated by the images of the defining relators. Furthermore, each such submodule has the form $\mathbf{Z}G/C$, where C is the finite cyclic subgroup of G generated by the image of the root of the corresponding defining relator.

We say that a presentation has the Identity Property if its relation module has the above form. This is equivalent to the condition (I.1) of Lyndon and Schupp ([13], p. 158). We say that a group G has the Identity Property if some presentation of G has the Identity Property. In this paper, we consider a property which is weaker than the Identity Property in the following respect. Instead of considering a relation module, we look at the kernel of the nth boundary map in an RG-projective resolution of R, where R is a commutative ring with 1, and we allow this kernel to be a direct sum of cyclic modules of the form RG/S, where S is an arbitrary subgroup of G.

1.2 Let R be a commutative ring with unit and let G be a group. An exact sequence of left RG-modules

$$\mathfrak{Q}: 0 \to Q \oplus P \to P_{n-1} \to \cdots \to P_0 \to A \to 0$$

of finite length $n>0$ is called an *RG-quasi-projective resolution* of A if the modules $P, P_{n-1}, \ldots, P_0$ are RG-projective and there exists an indexed set $\{G_\alpha\}_I$ of subgroups of G such that

$$Q \cong \bigoplus_I RG/G_\alpha.$$

The set $\{G_\alpha\}_I$ can be chosen such that no RG/G_α is RG-projective. We then say that the set $\{G_\alpha\}_I$ is *associated to the resolution* $\mathfrak{Q}$. We make the convention that the sequence $0 \to A \to A \to 0$ is an RG-quasi-projective resolution of length 0 if and only if A is RG-projective.

We define the *RG-quasi-projective dimension* of A to be the shortest possible length of an RG-quasi-projective resolution $\mathfrak{Q} \twoheadrightarrow A$. If no such resolution exists, the quasi-projective dimension is said to be infinite. In particular $\operatorname{qpd}_R G$ denotes the RG-quasi-projective dimension of the trivial module R. We write $\operatorname{qpd}_{\mathbf{Z}} G$ as $\operatorname{qpd} G$. Our convention for length 0 is necessary to exclude the sequence $0 \to R \to R \to 0$, if R is not RG-projective.

Our notation, using RG-, might suggest that the above definitions depend only on the ring RG. It is, however important to recognize the explicit group-ring structure of RG. For example, even if $RG \cong SH$ as rings, an RG-quasi-projective resolution need not be an SH-quasi-projective resolution.

EXAMPLES

1. For all groups G, the inequality $\operatorname{qpd}_R G \leq \operatorname{cd}_R G$ holds.

2. Suppose G is a group with the Identity Property, then $\operatorname{qpd}_R G \leq 2$.
Particular instances of such groups are:
- one relator groups,
- groups with staggered presentations,
- certain small cancellation groups [12],
- groups with an "aspherical" presentation in the sense of Lyndon and Schupp [13].

3. If G is a finite group, and there exists a periodic RG-projective resolution of finite period k over R, then $\operatorname{qpd}_R G$ is at most k.

1.3 We now describe the structure of the article and discuss our main results.

In Section 2 we deal with some general consequences of our definition for $\operatorname{qpd}_R G$. The similarity between cd_R and qpd_R is a basic theme.

Theorem 1 states a fundamental subgroup property for groups of finite qpd over R. Suppose S is a subgroup of G and $\operatorname{qpd}_R G < \infty$. Then any RG-quasi-projective resolution may be interpreted as an RS-quasi-projective resolution and an associated set of subgroups of S may be defined in terms of a given associated set of subgroups of G. In particular, we have $\operatorname{qpd}_R S \leq \operatorname{qpd}_R G$ whenever $S \subset G$.

It follows from Theorem 1 that any set of subgroups of G associated to some quasi-projective resolution consists of finite groups. Hence, for all R-torsion-free groups G, $\operatorname{qpd}_R G = \operatorname{cd}_R G$. In particular $\operatorname{qpd}_{\mathbf{Q}} G = \operatorname{cd}_{\mathbf{Q}} G$ for all groups G. Another consequence of Theorem 1 is the following: Suppose R is torsion-free as a $\mathbf{Z}$-module and $\operatorname{qpd}_R G$ is odd, then G is R-torsion-free.

Suppose G is a fundamental group of a graph of groups whose vertex groups have bounded qpd over R and whose edge groups are R-torsion-free. Theorem 5 says that $\mathrm{qpd}_R\, G$ is finite. This is an analogue for qpd_R of a result of Chiswell for cd_R (cf. [5], p. 70).

The results of Section 3 give information about the finite subgroups in groups of finite qpd. Our central result is the following.

THEOREM 6. *Suppose $S \neq 1$ is a finite R-torsion subgroup of G, and $\{G_\alpha\}_I$ is a set of subgroups associated to some RG-quasi-projective resolution of R of finite length. Then there exist a unique $\alpha \in I$ and a unique left coset gG_α of G_α in G, such that S is contained in $gG_\alpha g^{-1}$.*

In the case $R = \mathbf{Z}$, it follows from Theorem 6 that the associated set of subgroups $\{G_\alpha\}_I$ is a full representative set of conjugacy classes of maximal finite subgroups. Hence, up to conjugacy, the groups G_α are determined by G independently of the particular choice of a $\mathbf{Z}G$-quasi-projective resolution.

Theorem 6 is reminiscent of a theorem of Serre [9] and of results of Wall ([16], Lemma 7, Proposition 8). In fact, if $R = \mathbf{Z}$ in our situation, then the hypotheses of Serre's theorem hold, but we cannot prove this without the help of Theorems 6 and 7. Our proof of Theorem 6 is partly based on Wall's arguments.

In the special case where G is a one-relator group, Theorem 6 recovers a result of Karrass, Magnus and Solitar [10].

Further restrictions on the finite subgroups derive from Theorem 7. In the case $R = \mathbf{Z}$, it states that a finite group S satisfies qpd $S = n$ if and only if its Tate cohomology has period n. In particular, if qpd $G = 2$, then any finite subgroup of G is cyclic. Hence G has a relation module which is a direct sum of cyclic modules of the form $\mathbf{Z}G/C$, where C is a finite cyclic subgroup of G. Note the formal resemblance with the module-theoretic interpretation of the Identity Property.

As stated in 1.2, if G has the Identity Property, then qpd $G \leq 2$. We make no attempt to answer the question of whether the converse also holds. For torsion-free groups, this reduces to the question of whether cohomological and geometric dimensions always coincide (Eilenberg–Ganea problem).

The topics of Section 4 may be motivated by general properties of one-relator groups.

Let G be a one-relator group with torsion. Then either G is finite cyclic or the centre of G is trivial. We show in Corollary 8.1 that any group G of finite qpd with torsion either is finite or has trivial centre. It is known that a one-relator group has only finitely many conjugacy classes of finite subgroups. Proposition 9 gives necessary and sufficient conditions for a group of finite qpd to have only finitely many conjugacy classes of finite subgroups. This is of interest in connection with Wall's question F7 in [17].

Recall [14] that a group G is virtually torsion-free if one of its subgroups of finite index is torsion-free. The virtual cohomological dimension vcd G of a virtually torsion-free group G is defined by vcd $G = \mathrm{cd}\, S$, when S is a torsion-free subgroup of finite index. If G is not virtually torsion-free, then by definition vcd $G = \infty$.

Like qpd, the invariant vcd assumes finite values on certain classes of groups with torsion. If G is virtually torsion-free, vcd $G \leq$ qpd G holds. Our examples show that this is the most one can say in general about the relationship between vcd and qpd.

Of particular interest are groups G with vcd $G \leq$ qpd $G < \infty$. In this case, the Farrell–Tate cohomology of G is periodic and is completely determined by the cohomology of the maximal finite subgroups of G.

Two of our examples make it clear that certain general properties of one-relator groups do not follow from the Identity Property. These examples concern the inequality vcd $G \leq$ qpd G and the structure of the subgroup generated by the torsion elements.

The methods of this paper are algebraic. We have left open the question of a suitable geometric interpretation of the property of having finite qpd. By analogy with cohomological dimension, one might expect to obtain sufficient criteria for finite qpd via such an interpretation.

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2. Groups of finite qpd over a ring R

2.1 *Restriction to subgroups*

THEOREM 1. *Suppose $\mathfrak{Q} \twoheadrightarrow A$ is an RG-quasi-projective resolution of A of length n, and $\{G_\alpha\}_I$ is an associated set of subgroups. Let S be a subgroup of G. For each $\alpha \in I$, choose a set $\{t_\beta ; \beta \in J_\alpha\}$ of representatives of the double cosets SgG_α ($g \in G$). For each $\beta \in J_\alpha$, define $S_{\alpha\beta} = S \cap t_\beta G_\alpha t_\beta^{-1}$. Then $\mathfrak{Q} \twoheadrightarrow A$ is an RS-quasi-projective resolution, and there is an associated set of subgroups consisting of those $S_{\alpha\beta}$ for which $RS/S_{\alpha\beta}$ is not RS-projective.*

COROLLARY 1.1. *If S is a subgroup of G, then* $\mathrm{qpd}_R\, S \leq \mathrm{qpd}_R\, G$.

Proof of Theorem 1. Suppose $\mathfrak{Q} \to A$ has the form $0 \to Q \oplus P \to P_{n-1} \to \cdots \to P_0 \to A \to 0$, where P is an RG-projective, and $Q \cong \oplus_I RG/G_\alpha$. Now Q is the free

R-module on a left G-set T, whose decomposition into G-orbits has the form $T = \bigsqcup_I G/G_\alpha$.

The S-orbit decomposition of T has the form $T = \bigsqcup_I (\bigsqcup_{J_\alpha} S/S_{\alpha\beta})$, so as left RS-module

$$Q \cong \oplus_I (\oplus_{J_\alpha} RS/S_{\alpha\beta}).$$

Define $J = \{(\alpha, \beta); \alpha \in I, \beta \in J_\alpha, RS/S_{\alpha\beta}$ is not RS-projective$\}$. Then Q has an RS-direct sum decomposition $Q \cong P' \oplus Q'$, where P' is RS-projective and $Q' = \oplus_J RS/S_{\alpha\beta}$.

COROLLARY 1.2. *In the situation of Theorem* 1, *the subgroups* G_α *are all finite.*

Proof. Fix $\alpha \in I$ and let $S = G_\alpha$ in the proof of Theorem 1. Then some $S_{\alpha\beta}$ coincides with S, so Q contains the trivial module R as an RS-direct summand. Hence the RS-projective P_{n-1} contains R as a trivial RS-submodule. This is possible only if S is finite.

COROLLARY 1.3. *If G is R-torsion-free, then* $\mathrm{qpd}_R\, G = \mathrm{cd}_R\, G$.

Proof. It is sufficient to show that $\mathrm{cd}_R\, G \leq \mathrm{qpd}_R\, G$ and we may assume that $\mathrm{qpd}_R\, G = n < \infty$. For every finite subgroup S of G, the order $|S| = |S| \cdot 1 \in R$ is a unit of R. Thus the canonical epimorphism $RG \twoheadrightarrow RG/S$ splits via an RG-homomorphism $\sigma: RG/S \to RG$, where

$$\sigma(gS) = \frac{1}{|S|} \sum_{h \in S} gh,$$

and so RG/S is RG-projective.

It follows that any RG-quasi-projective resolution is an RG-projective resolution.

COROLLARY 1.4. *If S is an R-torsion-free subgroup of G, then* $\mathrm{cd}_R\, S \leq \mathrm{qpd}_R\, G$.

COROLLARY 1.5. *Suppose G is virtually R-torsion-free, then* $\mathrm{vcd}_R\, G \leq \mathrm{qpd}_R\, G$.

2.2 *Quasi-projective and projective resolutions*

Let $0 \to Q \oplus P \to P_{n-1} \to \cdots \to P_0 \to A \to 0$ be an RG-quasi-projective resolution of A and let $0 \to K \to M_{n-1} \to \cdots \to M_0 \to A \to 0$ be an exact sequence of RG-modules with all the M_i RG-projective. By Schanuel's lemma ([1], [15]) there exist RG-projectives M' and P' such that $(Q\oplus P)\oplus P'\cong K\oplus M'$. Hence the exact sequence

$$\begin{array}{l} 0\to K\to M_{n-1}\to M_{n-2}\to\cdots\to M_0\to A\to 0\\ \qquad\oplus\ \cdot\ \oplus\\ \qquad M\underset{1}{\to} M' \end{array}$$

is an RG-quasi-projective resolution of A.

If $0\to(\oplus_I RG/G_\alpha)\oplus P\to P_{n-1}\to\cdots\to P_0\to A\to 0$ is a quasi-projective resolution of A and $\{G_\alpha\}_I$ is an associated set of subgroups, then an RG-projective resolution can be obtained by the following construction. For each $\alpha\in I$, choose an RG_α-projective resolution of R, $\mathcal{M}^\alpha\to RG_\alpha\to R\to 0$. The functor $RG\otimes G_\alpha-$ applied to this resolution gives an RG-projective resolution of RG/G_α. We thus obtain an RG-projective resolution of A of the form

$$\cdots\to\oplus_I(RG\otimes_{RG_\alpha}M_1^\alpha)\xrightarrow{\partial_{n+1}}(\oplus_I RG)\oplus P\xrightarrow{\partial_n}P_{n-1}\to\cdots\to P_0\to A\to 0. \quad (*)$$

We use the resolution (*) for R to calculate the homology and cohomology of G in high dimensions.

PROPOSITION 2. *Suppose $\{G_\alpha\}_I$ is a set of subgroups associated to some RG-quasi-projective resolution of R of Length n. Then for each $q>n$, there are natural isomorphisms*

$$H^q(G;-)\cong\prod_I H^{q-n}(G_\alpha;-);\qquad H_q(G;-)\cong\oplus_I H_{q-n}(G_\alpha;-)$$

of functors from RG-modules to R-modules.

PROPOSITION 3. *If R admits an RG-quasi-projective resolution of length n, then $H_n(G;R)$ embeds into a free R-module.*

COROLLARY 3.1. *If R is a* PID *and R admits an RG-quasi-projective resolution of length n, then $H_n(G;R)$ is R-free.*

Proof of Proposition 3. We use the special resolution (*) to calculate $H_n(G;R)$. For each $\alpha\in I$, the αth direct summand $RG\otimes_{RG_\alpha}M_1^\alpha$ is mapped under ∂_{n+1} into the augmentation ideal of the αth direct summand $RG\cong RG\otimes_{RG_\alpha}RG_\alpha$. It follows that $\mathbf{1}_R\otimes_{RG}\partial_{n+1}=0$, and so $H_n(G;R)\cong\ker(\mathbf{1}_R\otimes_{RG}\partial_n)$ is isomorphic to a submodule of the R-projective $(\oplus_I R)\oplus(R\otimes_{RG}P)$ and hence also of some free R-module.

COROLLARY 3.2. *Suppose R is torsion-free as an abelian group. If $qpd_R G = 2k+1$ is odd, then G is R-torsion-free, and so* $\mathrm{cd}_R G = 2k+1$.

Proof. Suppose C is a finite cyclic subgroup of G of order $m > 1$. By Theorem 1, R admits an RC-quasi-projective resolution of length $2k+1$. By Proposition 3, $H_{2k+1}(C, R)$ embeds into a free R-module and thus is $\mathbf{Z}$-torsion-free. On the other hand, $R/mR = H_{2k+1}(C, R)$ has obvious $\mathbf{Z}$-torsion unless $m \cdot 1$ is a unit in R and $R/mR = 0$.

If G is R-torsion-free, then Proposition 3 is relevant only for $n = \mathrm{cd}_R G$. However if G has non-trivial information about $H_i(G, R)$ for infinitely many $i \geq \mathrm{qpd}_R G$ as shown by the next result.

PROPOSITION 4. *Suppose there exists an RG-quasi-projective resolution of R of length i, let $\{G_\alpha\}_I$ be its associated set of subgroups, and suppose, for each $\alpha \in I$, there exists an RG_α-quasi-projective resolution of R of length k. Then there exists an RG-quasi-projective resolution of R of length $k+i$.*

Proof. In case G is R-torsion-free, there exist RG-projective resolutions of R of arbitrary length $1 \geq \mathrm{cd}_R G$. Otherwise, the set $\{G_\alpha\}_I$ is non-empty. Using the idea of the construction of the resolution (*), but replacing each RG_α-projective resolution $\mathcal{M}^\alpha \to RG_\alpha \to R \to 0$ by an RG_α-quasi-projective resolution of length k, we obtain an RG-quasi-projective resolution of length $k+i$ by an analogous procedure. Here the following fact is used: Let $U \subset V \subset G$ be subgroups, then $RG \otimes_V RV/U \cong RG/U$.

Remark. Inductive arguments based on Proposition 4 lead to the following:

(i) In the circumstances of Proposition 4, there exist RG-quasi-projective resolutions of R of length $m \cdot k + i$ for arbitrary integers $m \geq 0$.

(ii) If $\mathrm{qpd}_R G = n$, then there are RG-quasi-projective resolutions of R of length $m \cdot n$ for all integers $m > 0$.

2.3. *Graphs of groups of finite* qpd

THEOREM 5. *Let Γ be a graph of groups, $\{G_v\}_V$ its set of vertex groups, $\{G_e\}_E$ its set of edge groups and G its fundamental group. Suppose there is an integer j such that $\mathrm{qpd}_R G_v < j$ for all $v \in V$ and that all the groups G_e are R-torsion-free, then* $\mathrm{qpd}_R G < \infty$.

Proof. Associated to the graph Γ there is an exact sequence of RG-modules $A \rightarrowtail B \twoheadrightarrow R$ where $A \cong \bigoplus_E RG/G_e$ and $B \cong \bigoplus_V RG/G_v$ (cf. [5]). Let $\mathcal{P}^e \twoheadrightarrow R$ be an RG_e-projective resolution, then $RG \otimes_{RG_e} \mathcal{P}^e \to RG/G_e$ is an RG-projective resolution. Similarly $RG \otimes_{RG_v} \mathfrak{Q}^v \twoheadrightarrow RG/G_v$ is an RG-quasi-projective resolution, provided $\mathfrak{Q}^v \twoheadrightarrow R$ is RG_v-quasi-projective. We thus get an

RG-projective resolution $\mathcal{P} \twoheadrightarrow A$ of length $p \leq j$, and, using the remark after Proposition 4, an RG-quasi-projective resolution $\mathcal{Q} \twoheadrightarrow B$ of length $q > p$. The monomorphism $A \rightarrowtail B$ lifts to an RG-morphism of complexes $F: \mathcal{P} \to \mathcal{Q}$. The mapping cone construction described by Bass ([1] p. 30) yields an RG-quasi-projective resolution $MC(F) \twoheadrightarrow R$ of length q.

3. Finite subgroups in groups of finite qpd

3.1. *Conjugacy classes of finite subgroups*

THEOREM 6. *Suppose $S \neq 1$ is a finite R-torsion subgroup of G, and $\{G_\alpha\}_I$ is a set of subgroups associated to some RG-quasi-projective resolution of R of finite length. Then there exist a unique $\alpha \in I$ and a unique left coset gG_α of G_α in G, such that S is contained in $gG_\alpha g^{-1}$.*

The proof is split into three parts.

(a) The conclusion of the theorem holds if $p = |S|$ is prime.

Let J denote the set of all pairs (α, β) with $\alpha \in I$ and $\beta = gG_\alpha \in G/G_\alpha$ such that $S_{\alpha\beta} = S \cap gG_\alpha g^{-1} \neq 1$. Since p is not invertible in R, it follows from Theorem 1 that $\{S_{\alpha\beta}\}_J$ is a set of subgroups associated to some RS-quasi-projective resolution of finite length n, say. Now $S_{\alpha\beta} = S$ for all $(\alpha, \beta) \in J$ and so applying Proposition 2 twice, we get R-isomorphisms

$$R/pR \cong H_{2n+1}(S, R) \cong \oplus_J H_{n+1}(S, R) \cong \oplus_J \oplus_J H_1(S, R) \cong \oplus_J \oplus_J R/pR.$$

Comparing the ranks of the free R/pR-modules R/pR and $\oplus_J \oplus_J R/pR$, we find that J is a singleton, as required.

(b) If $\alpha, \alpha' \in I$, $g \in G$ are such that $G_{\alpha'} \cap gG_\alpha g^{-1}$ is not R-torsion-free, then $\alpha = \alpha'$ and $g \in G_\alpha$.

Otherwise, choose an R-torsion subgroup S of $G_{\alpha'} \cap gG_\alpha g^{-1}$ of prime order, and apply (a).

(c) The general case follows by induction on the order of S. For the inductive step, we apply an argument of Wall ([16], Proposition 8). Here (a) is the initial case of the induction and (b) plays the rôle of Wall's Lemma 7. Note that if S is an R-torsion group, so is any subgroup of S.

COROLLARY 6.1. *Suppose $R = \mathbf{Z}$ and G, $\{G_\alpha\}_I$ are as in Theorem 6. Then*

(i) *The set $\{G_\alpha\}_I$ is a complete set of representatives of conjugacy classes of maximal finite subgroups of G.*

(ii) *If $G \neq 1$ is finite, then I is a singleton, say $I = \{0\}$, and $G_0 = G$.*

Thus, in the particular case $R = \mathbf{Z}$, the subgroups G_α are determined up to conjugacy by G itself independently of the choice of a $\mathbf{Z}G$-quasi-projective resolution. In this sense we may speak about "a set of subgroups $\{G_\alpha\}_I$ associated to G."

In the case $R=\mathbf{Z}_{(p)}$, the localisation of $\mathbf{Z}$ at a prime p, a weaker form of Corollary 6.1 holds. We state it as a second corollary, since we refer to it in the proof of the next theorem.

COROLLARY 6.2. *Suppose $R=\mathbf{Z}_{(p)}$ and G, $\{G_\alpha\}_I$ are as in Theorem 6. For each $\alpha \in I$, choose a p-Sylow subgroup S_α in G_α. Then*

(i) *The set $\{S_\alpha\}$ is a complete set of representatives of conjugacy classes of maximal finite p-subgroups of G.*

(ii) *If G is finite and not p-torsion-free, then I is a singleton, say $I=\{0\}$, and S_0 is a p-Sylow subgroup of G.*

3.2. *Periodicity*

Recall ([4] Ch. XII) that a finite group G has p-period $k>0$ if the p-component of its Tate cohomology satisfies $\hat{H}^i(G,-)_{(p)} \cong \hat{H}^{i+k}(G,-)_{(p)}$ and k is minimal with this property. This is equivalent to $\hat{H}^*(G,-)$ having period k on the category of $\mathbf{Z}_{(p)}G$-modules.

If π is a non-empty set of primes, then G is π-periodic if and only if it is p-periodic for each $p \in \pi$. The π-period of G is the least common multiple of the the p-periods for all $p \in \pi$.

THEOREM 7. *Suppose $R=\mathbf{Z}_{(\pi)}$, where π is a set of primes. Let G be a finite group which is not R-torsion-free. Then G has π-period k if and only if* $\mathrm{qpd}_R\, G = k$.

Proof. Suppose that G has finite π-period k. By a theorem of Swan [15], there exists an RG-quasi-projective resolution of the form

$$0 \to R \to P_{k-1} \to \cdots \to P_0 \to R \to 0.$$

Therefore, $\mathrm{qpd}_R\, G \leq k$.

Conversely, suppose $\mathrm{qpd}_R\, G = k < \infty$ and let $p \in \pi$. If G is p-torsion-free, then G has p-period $1<k$, so we may assume that G has p-torsion.

Choose an RG-quasi-projective resolution $\mathfrak{Q} \twoheadrightarrow R$ of length k and apply the exact functor $\mathbf{Z}_{(p)} \otimes_R -$ to $\mathfrak{Q}$ to obtain a $\mathbf{Z}_{(p)}G$-quasi-projective resolution $\mathfrak{Q}_{(p)} \twoheadrightarrow \mathbf{Z}_{(p)}$ of length k.

Since G has p-torsion, it follows from Corollary 6.2. that any set of subgroups associated to $\mathfrak{Q}_{(p)}$ consists of a single subgroup G_0 whose index in G is prime to p. By Proposition 2, we have natural isomorphisms

$$(1)\ H^{2k+1}(G;-) \cong H^{k+1}(G_0;-) \qquad (2)\ H^{k+1}(G;-) \cong H^1(G_0;-).$$

By Theorem 1 and Corollary 6.2, the set $\{G_0\}$ is also associated to $\mathfrak{Q}_{(p)}$ regarded as a $\mathbf{Z}_{(p)}G_0$-quasi-projective resolution. By another application of Proposition 2, we have a natural isomorphism

(3) $H^{k+1}(G_0;-)\cong H^1(G_0;-)$

Combining (1), (2) and (3) and interpreting the result as Tate cohomology, we have a natural isomorphism

$$\hat{H}^{2k+1}(G,-)\cong\hat{H}^{k+1}(G,-).$$

Now dimension shifts may be used to establish, for all $i\in\mathbf{Z}$, the natural isomorphisms

$$\hat{H}^{i+k}(G,-)\cong\hat{H}^{i}(G,-)$$

of functors on $\mathbf{Z}_{(p)}G$-modules. Hence the p-period of G divides k for all $p\in\pi$.

COROLLARY 7.1. *If* G$\leqslant$2, *then every finite subgroup of* G *is cyclic.*

Proof. Let F be a finite subgroup of G. Then F has period 1 or 2 and hence $F_{ab}\cong\hat{H}^{-2}(F,\mathbf{Z})\cong\hat{H}^{0}(F,\mathbf{Z})\cong\mathbf{Z}/|F|\mathbf{Z}$

COROLLARY 7.2. *Suppose* qpd $G<\infty$ *and* $\{G_\alpha\}_I$ *is a full set of representatives of conjugacy classes of maximal finite subgroups of G. Then for all* $q>$qpd G, *there are natural isomorphisms*

$$H_q(G,-)\cong\oplus_I H_q(G_\alpha,-)\qquad H^q(G,-)\cong\prod_I H^q(G_\alpha,-)$$

of functors from $\mathbf{Z}G$*-modules to* $\mathbf{Z}$*-modules.*

COROLLARY 7.3. *Suppose* G, $\{G_\alpha\}$ *are as in Corollary 7.2. Then for each* $\alpha\in I$, qpd G_α *divides* qpd G.

4. Applications, Examples and Comments

4.1. *Groups of finite* qpd *with torsion*

Our theory provides some insight into the structure of groups with non-trivial torsion and finite qpd. The following results include some known facts about one-relator groups with torsion. No new results about torsion-free groups are to be expected, since in this case qpd $G=$ cd G.

PROPOSITION 8. *Suppose* qpd $G<\infty$ *and* N *is the normaliser in* G *of a non-trivial finite subgroup. Then* N *is finite.*

Proof. By hypothesis, N contains a non-trivial finite normal subgroup S. By Corollary 1.1, qpd $N<\infty$. Let $\{N_\alpha\}_I$ be set of subgroups of N associated to some $\mathbf{Z}N$-quasi-projective resolution of $\mathbf{Z}$. By Theorem 6, there exist a unique $\alpha\in I$ and a unique left coset nN_α of N_α in N such that $S\subset nN_\alpha n^{-1}$. Since S is normal in N, we have $S\subset xN_\alpha x^{-1}$ for all x in N. It follows that $N=N_\alpha$, so by Corollary 1.2, N is finite.

COROLLARY 8.1. *Suppose* qpd $G<\infty$ *and* G *has non-trivial torsion. Then either* G *is finite or* G *has trivial centre.*

COROLLARY 8.2. *Let* G *be an abelian group satisfying the hypotheses of Corollary* 8.1. *Then* G *is finite cyclic.*

Proof. Since $H_{2n}(\mathbf{Z}/p\mathbf{Z}\times\mathbf{Z}/p\mathbf{Z};\mathbf{Z})$ has non-trivial torsion, it follows from Corollary 3.1 that qpd $(\mathbf{Z}/p\mathbf{Z}\times\mathbf{Z}/p\mathbf{Z})=\infty$. Hence any finite abelian group of finite qpd is cyclic. (cf. [4], p. 262).

COROLLARY 8.3. *Suppose* $G\cong H\times K$ *with* $H\neq 1\neq K$ *and* G *satisfies the hypotheses of Corollary* 8.1. *Then* H *and* K *are finite of coprime order.*

4.2. *Conjugacy classes of finite subgroups*

PROPOSITION 9. *Suppose* qpd $G=n<\infty$. *Then the following are equivalent:*
(i) *There are only finitely many conjugacy classes of finite subgroups in* G.
(ii) *There exists an integer* j *such that for all* $i>j$ *the groups* $H^i(G,\mathbf{Z})$ *are finite.*
(iii) $H^{2n}(G,\mathbf{Z})$ *is finite.*

Proof. Use Theorems 6, 7 and Corollary 7.1 together with the isomorphisms

$$H^{2n}(G,\mathbf{Z})\cong \Pi_I H^{2n}(G_\alpha,\mathbf{Z})\cong \Pi_I\hat{H}^0(G_\alpha,\mathbf{Z})\cong \Pi_I\mathbf{Z}/|G_\alpha|\mathbf{Z}.$$

Remark. Proposition 9 is a partial answer to Wall's question F7 in [17]. It implies, for example, that a group of type FP_∞ and of finite qpd admits only finitely many conjugacy classes of finite subgroups.

4.3. *Virtually torsion-free groups of finite* qpd

(a) *Virtual cohomological dimension*

PROPOSITION 10. *Let* G *be a one-relator group and* R *a ring. Then*
(i) $\mathrm{vcd}_R G\leq 2$.
(ii) *For any* R*-torsion-free subgroup* H *of* G, $\mathrm{cd}_R H\leq 2$ *holds.*

Proof. Corollary 1.5 applies since G is virtually torsion-free [7]. Hence $\mathrm{vcd}_R\, G \leqslant \mathrm{qpd}_R\, G \leqslant 2$. Statement (ii) follows from Corollary 1.4. A proof based on combinatorial arguments is given in ([2], Theorem 7.7).

The inequality $\mathrm{vcd}_R\, G \leqslant \mathrm{qpd}_R\, G$ holds whenever $\mathrm{vcd}_R\, G$ is finite. In the case $R = \mathbf{Z}$, this is the most one can say about the relationship between $\mathrm{vcd}\, G$ and $\mathrm{qpd}\, G$. We refer to the examples 1, 2 below and also Example 3 in 4.4.

EXAMPLE 1. Let G and H be finite groups. Then $\mathrm{vcd}\,(G * H) = 1$ and $\mathrm{qpd}\,(G * H)$ equals the least common multiple of $\mathrm{qpd}\, G$ and $\mathrm{qpd}\, H$.

EXAMPLE 2. The matrix group $SL(2, \mathbf{Z})$ is infinite, has torsion and non-trivial centre. Hence $\mathrm{qpd}\, SL(2, \mathbf{Z}) = \infty$ by Corollary 8.1. However $\mathrm{vcd}\,(SL(2, \mathbf{Z})) = 1$.

There is a decomposition for $SL(2, \mathbf{Z})$:

$$SL(2, \mathbf{Z}) \cong \mathbf{Z}/4\mathbf{Z} \underset{\mathbf{Z}/2\mathbf{Z}}{*} \mathbf{Z}/6\mathbf{Z}$$

This shows that some care is needed if one wishes to weaken the hypothesis of Theorem 5. However, the condition set there on the edge-groups is probably stronger than is necessary for the conclusions to hold.

(b) *Farrell–Tate cohomolgy*

Recall that the Farrell–Tate cohomology $\hat{H}^*$ is defined in [6] for any group of finite vcd. Now suppose that G is virtually torsion-free and $\mathrm{qpd}\, G$ is finite, then by Corollary 1.5 we have $\mathrm{vcd}\, G \leqslant \mathrm{qpd}\, G$. Since any finite subgroup of G has periodic Tate cohomology, the Farrell–Tate functors $\hat{H}^*(G, -)$ are periodic ([3], § 14). For all $q > \mathrm{vcd}\, G$, there are natural isomorphisms $\hat{H}^q(G, -) = H^q(G, -)$. Hence the next proposition is a consequence of our results in § 3.

PROPOSITION 11. *Suppose G is virtually torsion-free,* $\mathrm{qpd}\, G < \infty$ *and* $\{G_\alpha\}_I$ *is a set of associated subgroups of G. Let k be the least common multiple of* $\{\mathrm{qpd}\, G_\alpha\}_I$ *and m the least common multiple of* $\{|G_\alpha|\}_I$. *Then*

(i) *The Farrell–Tate cohomology $\hat{H}^*(G, -)$ has period k dividing* $\mathrm{qpd}\, G$.

(ii) *For any $\mathbf{Z}G$-module M, the groups $\hat{H}^*(G, M)$ are annihilated by m.*

Remarks. (1) Part (ii) of Proposition 11 answers a question of ([3], § 11) in the special case of groups of finite qpd.

(2) The group $SL(2, \mathbf{Z})$ has periodic Farrell–Tate cohomology with period 2, but $\mathrm{qpd}\, SL(2, \mathbf{Z}) = \infty$.

4.4 *Groups with the Identity Property*

Subgroups of one-relator groups have certain properties not shared by groups of finite qpd in general. We give two examples of groups of finite qpd, each of which violates a general property of subgroups of one-relator groups. In each case, the group in question has the Identity Property.

EXAMPLE 3. Let H denote Higman's group

$$\langle a,b,c,d \mid a^2=a^b,\ b^2=b^c,\ c^2=c^d,\ d^2=d^a\rangle,$$

and let S denote the one-relator group

$$\langle x,y \mid (x^{-1}y)^r=1\rangle, \qquad (r>1).$$

Choose non-trivial elements $h,h' \in H$ and define

$$G = H \underset{h=x}{*} S \underset{y=h'}{*} H$$

The two canonical embeddings of H into G extend to an epimorphism of $H*H$ onto G. Since H has no proper subgroups of finite index [8], neither has G. But the element $x^{-1}y$ of G has finite order $r>1$, so G is not virtually torsion-free and $\operatorname{vcd} G=\infty$.

However $\operatorname{cd} H=2$ ([2], p. 167) and $\operatorname{qpd} S=2$ since S is a one-relator group. Now the proof of Theorem 5 gives a $\mathbf{Z}G$-quasi-projective resolution of $\mathbf{Z}$ of length 2, so $\operatorname{qpd} G=2$.

EXAMPLE 4. Let D denote the infinite dihedral group $\mathbf{Z}/2\mathbf{Z} * \mathbf{Z}/2\mathbf{Z}$ and let C denote the infinite cyclic subgroup of index 2 in D. Define $G=D *_C D$. Then $\operatorname{qpd} G$ is finite, by Theorem 5. In fact, the mapping cone construction in the proof of Theorem 5 yields a $\mathbf{Z}G$-quasi-projective resolution of $\mathbf{Z}$ of length 2. It has four associated subgroups, each of order 2, and together they generate G. By Corollary 6.1, these four subgroups of order 2 form a complete set of representatives of conjugacy classes of finite subgroups.

Now suppose G is isomorphic to a free product $*_I G_\alpha$ of finite groups $G_\alpha \neq 1$ $(\alpha \in I)$. From the above discussion, we must have $\operatorname{card}(I)=4$ and, for each $\alpha \in I$, $|G_\alpha|=2$. In other words, $G \cong *_4 \mathbf{Z}/2\mathbf{Z}$, and so $G_{ab} \cong \oplus_4 \mathbf{Z}/2\mathbf{Z}$. However, it follows from the decomposition $G \cong D *_C D$ that $G_{ab} \cong \oplus_3 \mathbf{Z}/2\mathbf{Z}$. Thus we have obtained a contradiction, and so G cannot be expressed as a free product of finite groups.

This example contrasts with the following general fact about one-relator groups. Let H be a one-relator group and let N be the subgroup of H generated by all the torsion elements of H. Then N is a free product of finite cyclic subgroups of H. ([7], Theorem 1). More generally, if S is any subgroup of H, generated by torsion elements, then $S \subset N$. Applying the Kurosh subgroup theorem, we deduce that S is a free product of finite cyclic groups.

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Note added in proof: Recently, K. W. Gruenberg has informed us of a version of Serre's Theorem stronger than the form presented in [9]. For groups of finite *qpd* over Z it contains our Theorem 6.

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