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Autor(en): **Carrell, J.B. / Sommese, A.J.**

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Some topological aspects of C* actions on compact Kaehler manifolds

James B. Carrell[1] and Andrew John Sommese[2]

Ia. Introduction

T. Frankel proved in [F] the classical description of the homology of a compact Kaehler manifold X of dimension n admitting an infinitesimal isometry V with zero $(V) = F \neq \phi$.

THEOREM. *The contraction $i(V)\Omega$ of V and the Kaehler form Ω of X is exact: i.e., there exists a smooth real function f on X such that $i(V)\Omega = df$. Moreover, f is a Morse function on X whose critical manifolds are the components $F_1, \ldots, F_r$ of F. For any coefficient field, Z_p or Q,*

$$b_k(X) = \sum_j b_{k-\lambda_j}(F_j) \tag{1}$$

where λ_j is the index of f on F_j, each λ_j is an even integer, and finally, X has torsion if and only if F has torsion.

More recently, Carrell and Lieberman showed that if X admits a holomorphic vector field with variety of zeros F, then $H^p(X, \Omega^q) = 0$ if $|p-q| > \dim_c F\,[C\text{–}L_1]$. In addition, if F is finite, there exists a filtration of $H^0(F, \mathcal{O}_F)$ whose associated graded ring is $H^{\cdot}(X)$, where $H^{\cdot}(X)$ denotes cohomology of X with complex coefficients ($[C\text{–}L_2]$).

In this note, we shall enlarge upon these results in two ways. First of all, we prove in Theorem 1 a geometric version of Frankel's Theorem for a compact Kaehler manifold X having a holomorphic C^* action with fixed point set $F \neq \phi$ which is valid even for Z coefficients. It follows, for example, that if $H.(F, Z)$ admits a basis of analytic cycles, then so does $H.(X, Z)$. Moreover, Theorem 1

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leads to an interesting description of the Picard variety of X (Theorem 3) and to the Hodge numbers $h^{p,q}(X)$ in terms of certain Hodge numbers $h^{p-k,q-k}(F)$ (Theorem 2). The proofs of these results use the Bialynicki-Birula decomposition [B–B] as proved for compact Kaehler manifolds in [C-S]. An additional crucial ingredient is the fact that the Morse function f increases on the trajectories of the vector field generated by $R^+ \subset C^*$. Recently, J. Jurkiewicz [J] gave a surprising example of a compact algebraic variety with a C^* action with fixed points on which no such function can exist. It seems that because of this example, one cannot in general expect Frankel type theorems to hold for algebraic nonKaehler manifolds.

We remark in IIc that Theorem 1 is still true when a C^* action on a compact algebraic manifold has finite fixed point set provided the B–B decomposition satisfies an extra transversality condition. On the other hand, the results connecting the cohomology ring of a compact Kaehler manifold X and the zeroes of a holomorphic vector field give one a different perspective on the relation between topology and fixed points, and it is an interesting question as to how this relates to Theorem 1. In light of the differences that can arise between Kaehler and nonKaehler C^* actions, it is worthwhile to carry over to the nonKaehler case what one can say concerning vector fields.

Ib. Statement of some results

Let X denote a compact Kaehler manifold of complex dimension n with a holomorphic C^* action, $C^* \times X \to X$. The fixed point set F of the action will always be assumed nontrivial. Let $F_1, \ldots, F_r$ denote the connected components of F, and let λ_j denote $2n$-(index of f on F_j)-$\dim_R F_j$.

THEOREM 1. *There exist injective morphisms defined over Z, Q, or Z_q, for any prime q, of the form*

$$\mu_{j,k} : H_{k-\lambda_j}(F_j) \to H_k(X) \tag{2}$$

so that $\mu_k = \sum_j \mu_{j,k}$ $(1 \le j \le r)$ is an isomorphism, for any $k = 0, 1, \ldots, 2n$.

THEOREM 2. *Let $H^{\cdot}(X)$ denote deRham cohomology of X over C and let $H^k(X) = \oplus_{p+q=k} H^{p,q}(X)$ be the Hodge decomposition. Then if $0 \le k \le 2n$ and*

$$\mu_k^* : H^k(X) \to \bigoplus_j H^{k-\lambda_j}(F_j) \qquad (1 \le j \le r) \tag{3}$$

denotes the isomorphism dual to μ_k, *we have* $\mu_k^*(H^{p,q}(X)) = \oplus_j H^{p-d_j,q-d_j}(F_j)$ *where* $d_j = \lambda_j/2$. Consequently, $h^{p,q}(X) = \sum_j h^{p-d_j,q-d_j}(F_j)$.

THEOREM 5. *Let V be a special holomorphic vector field* (IIIb) *with nontrivial zero set Z on a compact complex manifold X of class $\mathcal{M}$* (IIIa). *Then for any k with $0 \le k \le 2n$,*

$$\sum_{p-q=k} h^{p,q}(X) = \sum_{p-q=k} h^{p,q}(Z) \tag{4}$$

In particular if Z has components $Z_1, \ldots, Z_m$ and $k_i = \max\{p-q : h^{p,q}(Z_i) \neq 0\}$, then if $r = \max\{k_i\}$, $h^{p,q}(X) = 0$ for $|p-q| > r$.

This result sharpens the vanishing theorem of [C–L$_1$].

Ic. The Bialynicki–Birula decomposition

The remarks made in this section are valid when X is a compact Kaehler manifold or a compact algebraic manifold. Let $C^* \times X \to X$ be a C^*-action with nontrivial fixed point set F having components $F_1, \ldots, F_r$. Each F_i is a complex submanifold of X. It is a basic fact [S_1] that for any $x \in X$, $C^* \times \{x\} \to X : (\lambda, x) \to \lambda \cdot x$ extends to a holomorphic map $P^1 \times \{x\} \to X$. Thus $\lim_{\lambda \to 0} \lambda \cdot x$ and $\lim_{\lambda \to \infty} \lambda \cdot x$ exist, and by the group action properties, both must lie in F. The immediately suggests two invariant decompositions of X, the plus and minus decompositions:

$$X = UX_j^+ = UX_j^- \quad \text{where} \quad X_j^+ = \left\{x : \lim_{\lambda \to 0} \lambda \cdot x \in F_j\right\}$$

$$\text{and} \quad X_j^- = \left\{x \in X : \lim_{\lambda \to \infty} \lambda \cdot x \in F_j\right\}.$$

These decompositions were first discussed in the algebraic case by Bialynicki-Birula [B-B] and subsequently shown to exist in the compact Kaehler case in [C-S], where proofs of the following assertions can be found. Assume X_j is one of X_j^+ or X_j^-. Each X_j is a complex submanifold of X Zariski open in its closure (i.e. $\bar{X}_j$ is an analytic subvariety: see IIb); the natural map $p_j : X_j \to F_j$ is a holomorphic C^*-equivariant maximal rank surjection; F_j is a section of X_j; and the normal bundle of F_j in X_j is a specific subbundle of the normal bundle of F_j in X which we now describe.

For each $x \in X$, $\lambda \in C^*$ acts linearly on the holomorphic tangent bundle $T(X)_x$ of X via the differential $d\lambda_x$. It is well known that the resulting complex representation of C^* is determined by the existence of a basis $v_1, \ldots, v_n$ of $T(X)_x$ and integers $a_1, \ldots, a_n$ such that $\lambda \cdot v_i = d\lambda_x(v_i) = \lambda^{a_i} v_i$. The $a_1, \ldots, a_n$ are called the weights of the action of C^* on $T(X)_x$. Therefore, for each j, one has a canonical holomorphic direct sum decomposition $T(X) | F_j = T(F_j) \oplus N(F_j)^+ \oplus N(F_j)^-$, where $N(F_j)^+$ (resp $N(F_j)^-$) is the holomorphic vector bundle on F_j whose fibre at x is generated by v_i corresponding to positive (resp. negative) weights a_i, and $T(F_j)_x$ is generated by v_i with $a_i = 0$. Then $N(F_j)^+$ (resp. $N(F_j)^-$) is the normal bundle of F_j in X_j^+ (resp. X_j^-). Note that using the definition of λ_j from Theorem 1 gives $\lambda_j = \dim_R N(F_j)_x^+$ for any $x \in F_j$.

Finally, we recall that there are exactly two distinguished components F_1 (called the source) and F_r (called the sink) such that X_1^+ and X_r^- are Zariski open in X. Note that $N(F_1)^-$ and $N(F_r)^+$ both have rank 0.

There are two important sources of examples of C^* actions. The first class is made up of the algebraic homogeneous spaces G/P, where G is a complex semi-simple Lie group and P a parabolic subgroup. Every regular one parameter subgroup H of P defines a C^* action on G/P with finite F, and, by a result of E. Akyildiz [A], the B-B decomposition coincides with a Bruhat decomposition. Thus for most actions, the $\bar{X}_j^+$ may be regarded as generalized Schubert cycles. The second class of examples, in which one finds some nonprojective actions, consists of the torus embeddings (i.e. equivariant completions of $((C^*)^n)$. These are studied in [K–K–M–St. D] and [M–O].

IIa. The Lyupanov function

Associated to a C^* action on X are the circle and radial actions arising from the circle $S^1 \subset C^*$ and the radial subgroup R^+ of C^* consisting of all positive real numbers. In this section we will show that the Morse function f of §Ia is a Lyupanov function, i.e. is strictly increasing along the radial orbits in X. We will also prove some geometric consequences of the existence of this function.

Let X have Kaehler form Ω which we may suppose is invariant under the circle action. The two actions give rise to a pair of vector fields V and W on X such that $JV = W$, where J is the complex structure tensor of X. For any smooth function g on X, set

$$V_x g = \frac{d}{d\theta} g(e^{i\theta} \cdot x) \big|_{\theta=0} \qquad (\theta \in R)$$

$$W_x g = \frac{d}{dr} g(r \cdot x) \big|_{r=1} \qquad (r \in R^+)$$

Note that zero $(V)=$ zero $(W)=F$.

The trajectories of W, that is, the rays $r\cdot x$, have limits in F as $r\to 0$ or ∞. The next lemma shows that these lie in different components of F.

LEMMA 1. *The Morse function f is increasing along the trajectories of W.*

Proof. One need only show that if $x\notin F$, then $df(W)>0$ at x, and this is obvious since

$$df(W)=(i(V)\Omega)W=\Omega(V,W)=\Omega(V,JV)>0$$

unless $V_x=0$.

COROLLARY 1. *The index of f at $x\in F_i$ is $\dim_{\mathbf{R}} N(F_i)_x^-$. Consequently the index is even.*

COROLLARY 2. *The source F_1 of X is $\{x\in X: f$ assumes its absolute minimum at $x\}$. Similarly, the sink F_r* of *X is $\{x\in X: f$ assumes its absolute maximum at $x\}$.*

Proof. This follows from the fact that $T(X)\mid F_1=T(F_1)\oplus N(F_1)^+$ and consequently f is increasing in every direction normal to F_1. The other assertion is similar.

Therefore the fixed point components can be indexed $F_1, F_2, \ldots, F_r$ so that

$$f(F_1)<f(F_2)\leq\cdots\leq f(F_{r-1})<f(F_r) \tag{5}$$

J. Jurkiewicz has given an example in [J] of a 3 dimensional torus imbedding containing nonfixed points $x_1,\ldots,x_6$ for which

$$\lim_{r\to\infty} r\cdot x_i=\lim_{r\to 0} r\cdot x_{i+1} \qquad (1\leq i\leq 5)$$

and

$$\lim_{r\to\infty} r\cdot x_6=\lim_{r\to 0} r\cdot x_1$$

Such X cannot admit a Lyupanov function.

It is useful to have the following example since by Blanchard's Theorem, any projective manifold X admitting a C^* action with $F\neq\phi$ admits an equivariant projective imbedding.

EXAMPLE. Suppose $X = P^n$ and let $a_0, \ldots, a_n$ be integers such that $a_0 \leq \cdots \leq a_n$. One defines a C^* action by on X by $\lambda \cdot [x_0, \ldots, x_n] = [\lambda^{a_0} x_0, \ldots, \lambda^{a_n} x_n]$. Then it is not hard to see that with respect to the Fubini–Study metric Ω on P^n, $i(V)\Omega = df$, where

$$f[x_0, \ldots, x_n] = \sum a_i \, |x_i|^2 / \sum |x_i|^2$$

This function separates the components of F.

One obtains some interesting information from this example for a projective X embedded equivariantly in P^n. In fact, if Ω^*(resp. f^*) denoted the pull back of Ω (resp. f) to X, then $i(V)\Omega^* = df^*$, so f^* is a Morse function of X. It follows that C^* has fixed points on X (hence the Borel Fixed Point Theorem). Indeed, the B-B decomposition for X is induced on X from the B–B decomposition of P^n. The following fact is slightly more surprising.

PROPOSITION 1. *Suppose X is as above and, in addition, X is contained in no C^* invariant hyperplane of P^n. If F_1 (resp. F_1^*) denotes the source of P^n (resp. X), then $F_1^* = X \cap F_1$. An analogous result holds for the sinks.*

Proof. Since the hyperplane $x_0 = 0$ is invariant, there exists a point x in X of the form $[1, x_1, \ldots, x_n]$. Since $\lim_{r \to 0} r \cdot x$ lies in F_1, it must also lie in F_1^* by Lemma 1. It follows that

$$F_1^* = f^{*-1}(a_0), \quad \text{so} \quad F_1^* = f = {}^{-1}(a_0) = f^{-1}(a_0) \cap X = F_1 \cap X.$$

This completes the proof.

Therefore $f(X) = f(P^n)$, however it is not true that X meets every component of $(P^n)^{C^*}$. We conjecture the following: if X^{C^*} is finite and X is equivariantly imbedded in P^n, then $\chi(X) = \# X^{C^*} \leq \chi(P^n) = n+1$, where χ denotes Euler characteristic.

IIb. Construction of the maps $\mu_{j,k}$

Let X be compact Kaehler, let C^* act on X, and let F_j be any component of the fixed point set F. In this section, we shall construct morphisms $\mu_{j,k} : H_{k-\lambda_j}(F_j) \to H_k(X)$ which may be viewed as realizations of the following construction: for any cycle z in F_j, let $z^+ = p_j^{+-1}(|z|)$ and let $\mu_{j,k}(z)$ be the closure of $p_j^{+-1}(z)$. (Actually, we should denote $\mu_{j,k}$ by $\mu_{j,k}^+$ since there is dually a natural

$\mu_{j,k}^-$ for the minus decomposition.) Then $\mu_{j,k}(z)$ is a cycle in X. Note that the degree of $\mu_{j,k}(z)$ is (degree of z)$+\lambda_j$, since $\lambda_j = \dim_{\mathbf{R}} N(F_j)^+$.

Recall that it was stated, but not explicitly proved, in [C–S] that $\bar{X}_j^+$ is a subvariety of X in which X_j^+ is Zariski open. We shall now explicitly prove this by constructing a certain compactification B_j of X_j^+ so that the holomorphic equivariant map $i: X_j^+ \to X$ extends meromorphically to $\phi_j: B_j \to X$.

Let τ denote the trivial line bundle on F_j and let $\tilde{B}_j = (X_j^+ \oplus \tau) - F_j$. On $\tilde{B}_j$ take the product C^* action and denote $C^* \backslash \tilde{B}_j$ by B_j. Clearly p_j makes B_j a holomorphic fibre bundle over X_j^+. The points of B_j will be denoted $[v, t]$. Note that X_j^+ sits naturally inside B_j as the set of all points of the form $[v, 1]$. The space $P_j = B_j - X_j^+$ is a holomorphic fibre bundle over F_j whose fibre over x is isomorphic to the V-manifold $C^* \backslash (N(F_j)_x^+ - \text{zero})$.

LEMMA 2. *$i: B_j - P_j \to X$ extends meromorphically to $\phi_j: B_j \to X$.*

Proof. Begin by choosing a nonsingular point $[v_0, 0]$ of P_j, where, say, $v_0 \in X_j^+$. Near $[v_0, 0]$, B_j may be parameterized by $\Delta^m \times \Delta$, where Δ is a sufficiently small disc about 0 in C and $m = \dim X_j^+ - 1$. Define $\phi_j: \Delta^m \times \Delta^* \to X$ by $\phi_j(v, \delta) = i([\delta^{-1} \cdot v, 1]) = i(\delta^{-1} \cdot v)$. By Lemma IIA of [S$_1$], $\lim_{\delta \to 0} (\delta^{-1} \cdot v)$ exists and, in fact, for each v, $\delta \to (\delta^{-1} \cdot v)$ is holomorphic at the origin. Therefore ϕ_j extends holomorphically to $\{v\} \times \Delta$ for each $v \in \Delta^m$. It follows from Lemma IA and Siu's Extension Theorem [S$_1$] that ϕ_j extends meromorphically to B_j.

The assertion that $\bar{X}_j^+$ is a subvariety of X in which X_j^+ is Zariski open follows immediately from Remmert's Proper Mapping Theorem and the fact that the closure Γ_j of the graph of ϕ_j in $B_j \times X$ is an irreducible subvariety of $B_j \times X$. By Hironaka's resolution of singularities [H], there exists smooth compact complex manifold $\tilde{\Gamma}_j$ and a holomorphic map $\theta_j: \tilde{\Gamma}_j \to \Gamma_j$. One obtains holomorphic maps $h_j: \tilde{\Gamma}_j \to X$ and $g_j: \tilde{\Gamma}_j \to F_j$ from the following diagram

$$\begin{tikzcd}
& & \tilde{\Gamma}_j \arrow[d] & B_j \times X \\
& & \Gamma_j \arrow[ur, hook] \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\
F_j & B_j \arrow[l] & & Z
\end{tikzcd} \tag{7}$$

Let $H_c^{\cdot}$ denote cohomology with compact supports, and let $b_j = \dim_c F_j$. The map $\mu_{j,k}: H_{k-\lambda_j}(F_j) \to H_k(X)$ is defined (for Z or field coefficients) by the composition

$$H_{k-\lambda_j}(F_j) \to H_c^{2b_j-k+\lambda_j}(F_j) \xrightarrow{g_j^*}$$

$$\to H_c^{2b_j-k+\lambda_j}(\tilde{\Gamma}_j) \to H_k(\tilde{\Gamma}_j) \xrightarrow{h_{j*}} H_k(X) \tag{8}$$

where the first and third maps are Poincaré duality maps. For $\alpha \in H_{p-\lambda_j}(F_j)$, we denote $\mu_{j,p}(\alpha)$ simply by α^+. As mentioned there is a corresponding class α^- obtained by replacing X_j^+ by X_j^- in the above discussion.

The construction we have just described works of course without the Kaehler assumption. The next lemma, however, requires the existence of the Lyupanov function.

LEMMA 3. *Suppose the Morse function f has value a_j on F_j, and let z be a cycle representing $\alpha \in H.(F_j)$. Then there exists a cycle z^+ in X representing α^+ with $|z^+| \subset \bar{X}_j^+$. Consequently $f(|z^+|) \subset [a_j, \infty[$. Similarly, there exists a representing cycle z^- for α^- such that $f(|z^-|) \subset]-\infty, a_j]$*

Proof. Since $\tilde{\Gamma}_j \to X$ factors through $\bar{X}_j^+$, and since f is increasing along the radial trajectories, $f(X_j^+) \subset [a_j, \infty[$. This proves the first assertion. The second assertion is similar.

Remark. It is clear from the construction of α^+ that if α is an analytic cycle on F_j, then α^+ is an analytic cycle on X. Consequently, if $H.(F_j)$ admits a basis of analytic cycles, then Theorem 1 implies that the same is true for $H.(X)$.

IIc. Proof of Theorem 1

The importance of the construction of IIb is that for any $\alpha, \beta \in H.(F_j)$, the intersection product $\alpha^+ \cdot \beta^-$ can be computed on F_j, as we shall show. Note that if $F_j = F_1$, the source (resp. F_r, the sink), then α^- (resp α^+) is $i_*(\alpha)$ where $i: F_j \to X$ is the inclusion.

LEMMA 4. *Assume $\alpha \in H_r(F_j)$ and $\beta \in H_s(F_j)$. Then $\alpha^+ \cdot \beta^- = i_*(\alpha \cdot \beta)$.*

Proof. It is an easy exercise to show that $\alpha^+ \cdot \beta^- \in H_{r+s-2b}(X)$ where $b = \dim_c F_j$. The proof therefore follows from Lemma 3.

In light of Frankel's Theorem, to prove Theorem 1, one need only show that for each q, $0 \le q \le 2n = 2 \dim_c X$, and, for coefficient field Z_p,

$$\mu_q = \sum_j \mu_{j,q} : \bigoplus_j H_{q-\lambda_j}(F_j) \to H_q(X)$$

is a monomorphism. Let $\alpha_j \in H_{q-\lambda_j}(F_j)$ for each j and suppose $\Psi = \sum_j \mu_{j,q}(\alpha_j) = 0$. We first claim that α_1, the class at the lowest level of f, vanishes. For if $\alpha_1 \ne 0$, then one can find $\beta_1 \in H_{2b-q+\lambda_1}(F_1)$ so that $\alpha_1 \cdot \beta_1 \neq 0$ in $H_0(F_1)$. Now

$$\beta_1^- \cdot \Psi = \beta_1^- \cdot \alpha_1^+ = i_*(\beta_1 \cdot \alpha_1)$$

Hence $i_*(\beta_1 \cdot \alpha_1) = 0$, a contradiction

Similarly, having renumbered the F_j so that $F_2, \ldots, F_m$ are at the next critical level, we can show by the same technique that $\alpha_2, \ldots, \alpha_m$ all vanish. By continuing in this manner, we finally deduce that all $\alpha_j = 0$.

Remark. The main construction in the proof of Theorem 1 will go over unchanged to any compact complex manifold for which the Bialynicki–Birula decomposition exists. The injectivity argument only breaks down when one has cycles ala IIa, and thus the Morse function is useful because it guarantees that no cycles exist. If X is compact complex, F is finite, the B–B decomposition exists, and all X_j^+ and X_k^- are transverse, then a theorem of Smale [Sm] guarantees the existence of a Morse function f on X so that: (a) the critical point set of f coincides with F, (b) the index of f at $x_j \in F$ is $\dim_{\mathbf{R}} N(\{x_j\})^-$, (c) $f(x_j) = \dim_{\mathbf{R}} N(\{x_j\})^-$, and (d) f is increasing on radial orbits. Thus the injectivity of the maps (2), when F is finite, is true as long as the B–B decomposition exists. For X in a large class of compact complex manifolds, namely those with a C^* action with finite $F \neq \phi$ in the class $\mathcal{M}$ defined in IIIa, the surjectivity is also true, and may be seen by applying Theorem 4 of IIIa, Corollary 4 of [C–L$_1$], Corollary 1.7 of [Fu], and the fact that the B–B decomposition exists if X is in $\mathcal{M}$.

COROLLARY 3. *If X is of class $\mathcal{M}$, F is finite, and all X_j^+ and X_k^- are transverse, then Theorem 1 is valid.*

IId. Proof of Theorem 2

In this section we will be concerned with studying the duals of the isomorphisms $\mu_k : \oplus_j H_{k-\lambda_j}(F_j) \to H_k(X)$ defined by (8). We shall consider complex coefficients only, and the complex dual of H_k will be understood to be the complex deRham cohomology group H^k. Let $H^k(X) = \oplus_{p+q=k} H^{p,q}(X)$ be the Hodge decomposition. This decomposition arises from the fact that on a compact Kaehler manifold, the Laplacian Δ preserves type. Consequently if $\tau = \sum_{p+q=k} \tau_{p,q}$ is a k-form where $\tau_{p,q}$ has type (p, q) and if $\Delta\tau = 0$, then $\Delta\tau_{p,q} = 0$ for all $\tau_{p,q}$. We shall show that if μ_k^* denotes the complex dual of μ_k, then

$$\mu_k^*(H^{p,q}(X)) \subset \bigoplus_j H^{p-d_j, q-d_j}(F_j)$$

where $2d_j = \lambda_j$. The bijectivity of μ_k^* will then imply Theorem 2.

Reversing arrows in (8) gives

$$H^k(\tilde{\Gamma}_j) \to (H_c^{2b_j-k+\lambda_j}(\tilde{\Gamma}_j))^* \to (H_c^{2b_j-k+\lambda_j}(F_j))^* \to H^{k-\lambda_j}(F_j) \tag{9}$$

The composition of the maps in (9) is the classical Gysin map $g_{j*} = g_*$ associated to $g_j : \tilde{\Gamma}_j \to F_j$. In DeRham cohomology, $g_*\tau$, for $\tau \in H^k(\tilde{\Gamma}_j)$, is determined by the condition

$$\int_{F_j} g_*\tau \wedge \sigma = \int_{\tilde{\Gamma}_j} \tau \wedge g^*\sigma \tag{10}$$

for an arbitrary class $\sigma \in H^{2b_j-k+\lambda_j}(F_j)$. Finally, by definition, $\mu^* = g_*h^*$ where h^* is induced by the holomorphic map $h : \tilde{\Gamma}_j \to X$.

Let $\tau \in H^{p,q}(X)$ with $p+q=k$, and let $\tilde{\tau}$ be a smooth closed (p,q) form on X representing τ. We shall show that for any $\sigma \in H^{r,s}(F_j)$,

$$\int_{F_j} g_*h^*\tau \wedge \sigma = 0 \tag{11}$$

unless $r = b_j + d_j - p$ and $s = b_j + d_j - q$. Let $\tilde{\sigma}$ be a closed (r,s)-form representing σ. Then by (10)

$$\int_{F_j} g_*h^*\tau \wedge \sigma = \int_{\Gamma_j} h^*\tau \wedge g^*\sigma = \int_{\Gamma_j} h^*\tilde{\tau} \wedge g^*\tilde{\sigma}$$

But, since $h^*\tilde{\tau}$ has type (p,q) and $g^*\tilde{\sigma}$ has type (r,s), (11) holds unless $p+r = q+s = b_j + d_j = \dim_c \tilde{\Gamma}_j$. It follows immediately that $g_*h^*\tau \in H^{p-d_j,q-d_j}(F_j)$, and Theorem 2 is proved.

The following result was suggested to us by F. Connelly.

COROLLARY 4. *Let C^* act on X with fixed point components $F_1, \ldots, F_n$. Then index $(X) = \sum(-1)^{d_i}$ index (F_i) where $d_i = \lambda_i/2$.*

Proof. By the Hodge Index Theorem,

$$\begin{aligned}
\text{index } (X) &= \sum_{p,q} (-1)^p h^{p,q}(X) \\
&= \sum_{p,q} \sum_i (-1)^p h^{p-d_i,q-d_i}(F_i) \\
&= \sum_i (-1)^{d_i} \sum_{j,k} (-1)^j h^{j,k}(F_i) \\
&= \sum (-1)^{d_i} \text{ index } (F_i).
\end{aligned}$$

IIe. Some applications

Let X be compact Kaehler and admit a C^* action with source F_1 (resp. sink F_r). We give two applications of the B–B decomposition and Theorem 1 to compute two basic groups on X in terms of the source (or sink).

THEOREM 3. *Let $i:F_1\to X$ be the inclusion. Then* (a) $i_*:\pi_1(F_1, x_0)\to (X, x_0)$ *is an isomorphism, and* (b) *There exists an exact sequence for Picard varieties*

$$0\to K\to \operatorname{Pic}(X)\to \operatorname{Pic}(F_1)\to 0,$$

where K is isomorphic to the Z-module of divisors generated by the $\bar{X}_i^+$ where $\chi(F_i)=\operatorname{rank} N(F_i)^-=1$.

Proof. For (a), recall that by the construction of IIb, X is bimeromorphic to a V-manifold B via a meromorphic map $\phi:B\to X$ such that there is a commutative diagram

$$\begin{tikzcd} B \arrow[r, "\phi"] \arrow[dr, "\pi"'] & X \\ & F_1 \arrow[u, "i"'] \end{tikzcd}$$

where π is a locally trivial holomorphic map. Now it is well known that ϕ induces an isomorphism on fundamental groups. Since each fibre of π is a compactification of an affine space, π also induces an isomorphism. Thus i_* is an isomorphism.

To prove part (b), consider the commutative diagram

$$\begin{tikzcd} H^1(X,\mathcal{O}_X) \arrow[r] \arrow[d, "\alpha"'] & H^1(X,\mathcal{O}_X^*) \arrow[r] \arrow[d, "\beta"'] & H^2(X,Z) \arrow[r] \arrow[d, "i^*"'] & H^2(X,\mathcal{O}_X) \arrow[d, "\gamma"'] \\ H^1(F_1,\mathcal{O}_{F_1}) \arrow[r] & H^1(F_1,\mathcal{O}_{F_1}^*) \arrow[r] & H^2(F_1,Z) \arrow[r] & H^2(F_1,\mathcal{O}_{F_1}) \end{tikzcd}$$

where the vertical maps are all induced by $i:F_1\to X$, and the rows are exact, being induced by the exponential sequences $0\to Z\to \mathcal{O}\to \mathcal{O}^*\to 0$ on X and F_1. By the argument of part (a), α and γ are isomorphisms. We will first show exactness of $\operatorname{Pic}(X)\to \operatorname{Pic}(F_1)\to 0$ by showing β is onto. To do this, it will suffice, by a standard argument, to show that i^* is onto. Suppose $\nu\in H^2(F_1)$, and let $\mu\in H_{2b_1-2}(F_1)$ be the Poincaré dual of ν. Let $\mu^+\in H_{2n-2}(X)$ be the image of μ

and, finally, let $\tau \epsilon H^2(X)$ be the Poincaré dual of μ^+. Then it is not hard to show, by a diagram chase, that $i^*\tau = \nu$. To complete the proof of (b), note that, by definition,

$$0 \to \operatorname{Ker} i^* \to \operatorname{Pic}(X) \to \operatorname{Pic}(F_1)$$

is exact, so it remains to identify $\operatorname{Ker} i^* = \operatorname{Ker}(H^2(X) \to H^2(F_1))$. But if $\alpha \epsilon \operatorname{Ker} i^*$, then the Poincaré dual of α in $H_{2n-2}(X)$ must come from $\bigoplus_{\chi(F_i)=1} H_{2b_i}(F_i)$. In other words, the Poincaré dual of a class in $\operatorname{Ker} i^*$ is a linear combination over Z of the divisors $\bar{X}_i^+$, for which $\chi(F_i) = 1$, as was to be shown.

Remark. Assertion (a) of Theorem 3 is true in much more generality than we have stated. For example, it holds for X in the class $\mathcal{M}$ of IIIa, so in particular for compact algebraic varieties. The isomorphism $H_1(X) \cong H_1(F_1) \cong H_1(F_r)$ follows directly from (a).

Note that Theorem 3(a) implies that the source and sink have isomorphic fundamental groups.

IIIa. Holomorphic vector felds and the class $\mathcal{M}$

In this section we will develop ideas needed for the proof of Theorem 5. In so doing, we generalize the main theorems of [C–L$_1$, CL$_2$] to the class $\mathcal{M}$ consisting of all compact complex manifolds M such that

M is bimeromorphic to a compact Kaehler manifold Y via a holomorphic map $f: Y \to M$ (12)

Suppose $M \in \mathcal{M}$. Then it is well known that if V is a holomorphic vector field on M and Y satisfies (12), then there exists a holomorphic vector field W on Y so that $f_* W = V$. This follows, essentially, from Hartog's Lemma, since the set of points where a meromorphic map is not defined has codimension at least two.

Lemma 5. *W has zeros if and only if V has zeros.*

Proof. Clearly V has zeros if W does. Suppose that V has a zero. By [C–L$_1$] or [S$_2$], a holomorphic vector field W on Y has zeros if and only if the contraction operator $i(W)$ annihilates $H^0(Y, \Omega_Y^1)$. In our case, this follows from the fact that $H^0(Y, \Omega_Y^1)$ is a birational invariant. That is, given $\omega \in H^0(Y, \Omega_Y^1)$, there exists an

$\omega' \in H^0(M, \Omega^1_M)$ such that $f^*\omega' = \omega$. Thus

$$i(W)\omega = i(W)f^*\omega' = f^*i(V)\omega' = 0$$

since $i(V)\omega'$ is a holomorphic function on M with a zero (since V has zeros) and consequently is identically zero.

Associated to the contraction operator $i(V): \Omega^p_M \to \Omega^{p-1}_M$ is a complex of sheaves

$$0 \to \Omega^n_M \to \Omega^{n-1}_M \to \cdots \to \Omega^1_M \to \mathcal{O}_M \to 0$$

and spectral sequences

$$\begin{aligned} 'E_1^{-p,q} &= H^q(M, \Omega^p_M) \\ ''E_2^{p,q} &= H^p(M, \mathcal{H}^q_M) \end{aligned} \tag{13}$$

where $\mathcal{H}^q_M$ denotes the cohomology sheaf

$$\operatorname{Ker} i(V) \mid \Omega^{-q}_M / i(V)\Omega^{-q+1}_M \quad [\text{C–L}_1].$$

The key fact is that if M is compact Kaehler and V has zeros, then all the differentials d_r in the $'E$ spectral sequence are 0. We shall show that this is also true if $M \in \mathcal{M}$. Let $f: Y \to M$ satisfy (12), suppose $f_*W = V$, and assume V has zeros. Then f induces a mapping of spectral sequences

$$f^*: {}'E_r^{-p,q} \to {}'E_r^{-p,q}, \quad \text{i.e.} \quad f^*d_r = d_r f^*,$$

which is injective on the $'E_1$ level by [W]. Since $d_1 = 0$ on Y (lemma 5), $d_1 = 0$ on M also. The degeneracy of the $'E$ spectral sequence for $M \in \mathcal{M}$ follows immediately by induction

Let Z denote the variety of zeros of V having sheaf of rings $\mathcal{O}_Z = \mathcal{O}_M / i(V)\Omega^1_M$ as structure sheaf.

THEOREM 4. *Let V be a holomorphic vector field on $M \in \mathcal{M}$ with $Z \neq \phi$. Then*

(a) $H^p(M, \Omega^q_M) = 0$ if $|p-q| > \dim_c Z$

(b) if $\dim_c Z = 0$, then there exists a filtration

$$H^0(Z, \mathcal{O}_Z) = F_{-n} \supset F_{-n+1} \supset \cdots \supset F_0 = C, \qquad n = \dim_c M, \tag{15}$$

such that $F_iF_j \subset F_{i+j}$ *with the property that*

$$F_{-i}/F_{-i+1} \cong H^i(M, \Omega^i_M)$$

By the Chow Lemma, compact algebraic varieties lie in the class $\mathcal{M}$. *Consequently,*

COROLLARY 4. *The conclusions of Theorem* 4 *hold if* M *is a compact algebraic manifold.*

IIIb. Proof of Theorem 5

We now consider a class of holomorphic vector fields which includes the vector fields generated by C^* actions. Suppose V is a holomorphic vector field on a complex manifold M with zero set $Z \neq \phi$. Then say V is *special* if near any $x \in Z$, there exists a coordinate neighbourhood U_x with coodinates $(y_1, \ldots, y_r, z_1, \ldots, z_s)$ such that

(a) $Z \cap U_x = \{(y, z)\colon y = \cdots = y_r = 0\}$
(b) $X = \sum_{i=1}^r a_i \partial/\partial y_i$, where $a_1, \ldots, a_r$ are holomorphic functions on U_x.
(c) $\|\partial a_i/\partial y_j\|$ has rank r on $Z \cap U_x$.

The special vector fields are always nondegenerate in the sense of Bott and the variety Z is always nonsingular. A key fact about special vector fields is

LEMMA 6. (D. Lieberman). *For every* p, q *the injection* $i\colon Z \to M$ *induces isomorphisms* $H^p(M, \mathcal{H}^{-q}_M) \cong H^p(Z, \Omega^q_Z)$.

Proof. For each $z = (z_1, \ldots, z_s)$, let $U(z) = \{(y, z)\colon y \text{ arbitrary}\}$ be the slice through z. Then

$$0 \to \Omega^r_{U(z)} \to \Omega^{r-1}_{U(z)} \to \cdots \to \Omega^1_{U(z)} \to \mathcal{O}_{U(z)} \to 0 \tag{16}$$

is a Koszul complex with differential $i(V \mid U(z))$.

Note that $\mathcal{H}^k_{U(z)} = 0$ if $k \neq 0$ and, because $V \mid U(z)$ vanishes to first order at $(0, z)$, $\mathcal{H}^0_{U(z)} = C_{(0,z)}$, the sheaf on $U(z)$ supported at $(0, z)$ with stalk C at $(0, z)$. Now the proof of (16), given say in [S], works with parameters. Consequently, if $\Omega^{i,0}_U$ denotes the subsheaf of Ω^i_U generated by differentials containing no dz

terms, then

$$\operatorname{Ker} i(V) \mid \Omega_U^{j,0}/i(V)\Omega_U^{j+1,0} = \begin{cases} 0, & \text{if } j \neq 0 \\ \mathcal{O}_{Z\cap U}, & \text{if } j = 0 \end{cases}$$

The lemma now results from the fact that tensoring (16) by $\Omega^q_{Z\cup U}$ over $\mathcal{O}_{Z\cap U}$ yields the isomorphism $\mathcal{H}_U^{-q} = \Omega^q_{Z\cap U}$.

We shall now prove Theorem 5. Consider the diagram of complexes

$$\begin{array}{ccccccccccc} 0 \to & \Omega_M^n & \to & \Omega_M^{n-1} & \to & \cdots & \to & \Omega_M^1 & \to & \mathcal{O}_M & \to 0 \\ & \uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\ 0 \to & \Omega_Z^n & \to & \Omega_Z^{n-1} & \to & \cdots & \to & \Omega_Z^1 & \to & \mathcal{O}_Z & \to 0 \end{array}$$

with differentials $i(V)$ and $i(V \mid Z) = 0$ respectively, and with vertical mappings given by i^*. By the above lemma, these complexes have spectral sequences with isomorphic $''E_2^{p,q}$ terms. Let $\mathcal{U}$ be a Leray covering of M, and let $\mathcal{U}'$ denote the restrictions of the open sets of $\mathcal{U}$ to Z. $\mathcal{U}'$ may be assumed to be a Leray cover of Z. Form the double complexes $\{C^p(\mathcal{U}, \Omega_M^q), i(V), \delta\}$ $\{C^p(\mathcal{U}', \Omega_Z^q), 0, \delta\}$, where δ denotes Cech differential. The total complex K_M, where

$$K_M^r = \bigoplus_{p-q=r} C^p(\mathcal{U}, \Omega_M^q)$$

with total differential D given on $C^p(\mathcal{U}, \Omega_M^q)$ by $i(V) + (-1)^q\delta$ gives rise to the spectral sequences (13) and (14) of V. A similar remark is true for the analogous total complex $K_Z^{\cdot}$. From the isomorphism of $''E_2$ terms, one concludes that $H^r(K_Z^{\cdot}) \cong H^r(K_M^{\cdot})$ for any r. But $H^r(K_Z^{\cdot}) = \oplus_{p-q=r} H^p(Z, \Omega_Z^q)$ while, because of the degeneration of $E_1^{-p,q} = H^q(M, \Omega_M^p)$, if M is in the class $\mathcal{M}$, one has $\dim H^r(K_M^{\cdot}) = \sum_{p-q=r} h^{p,q}(M)$. Theorem 5 follows from these considerations.

Remark. Hopefully Theorem 5 will give some information about the mysterious filtration (15) of Theorem 4. For some specific calculations of this filtration, see [C], [C–L$_2$].

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