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Abelian group extensions and the axiom of constructibility

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Introduction

Throughout this paper the word "group" will mean "abelian group". The results of Shelah's remarkable work on Whitehead's problem ([Sh_1], [Sh_2]) suggested the investigation of the structure of $\operatorname{Ext}(A, \mathbf{Z})$ for torsion-free A under the hypothesis of the Axiom of Constructibility, $V = L$. Applying Shelah's methods, H. Hiller, Shelah and the second-named author obtained a surprisingly simple description of the torsion-free part of $\operatorname{Ext}(A, \mathbf{Z})$ in terms of A[H–H–S].

In this paper we study, in the same spirit, the group $\operatorname{Ext}(A, G)$ in the case where A is torsion-free and G is any group satisfying suitable cardinality conditions. We are interested in characterizing pairs (A, G) such that $\operatorname{Ext}(A, G) = 0$ as well as in determining the structure of $\operatorname{Ext}(A, G)$. Herein we restrict our attention to its torsion-free part. Since in our case $\operatorname{Ext}(A, G)$ is always divisible, the structure of its torsion-free part is completely determined by its torsion-free rank. (Following [F_1] we denote the torsion-free rank of a group B by $r_0(B)$.)

Our first task is to settle the case where A is countable. For this, of course, we do not need any additional axiom of set theory. We assume G to be a group of countable torsion-free rank, thus unifying the known cases $G = \mathbf{Z}$[J, §2] and $G = T$, a torsion group ([B_1], [B_2]). In Section 1 we consider the crucial case where A is of rank 1. For such A we give a group-theoretical characterization of pairs (A, G) such that $\operatorname{Ext}(A, G) = 0$ and show that $\operatorname{Ext}(A, G) \neq 0$ implies $r_0(\operatorname{Ext}(A, G)) \geq 2^{\aleph_0}$ (Theorem 1.2). In Section 2 we study $\operatorname{Ext}(A, G)$ in case A is any coutable torsion-free group. Applying Theorem 1.2 we obtain various conditions that are necessary and (or) sufficient for the vanishing of $\operatorname{Ext}(A, G)$ (Theorems 2.1 and 2.6, Corollaries 2.4 and 2.7). In particular we have the following analogue of Pontryagin's criterion: If $\operatorname{Ext}(B, G) = 0$ for every subgroup B of A of finite rank, then $\operatorname{Ext}(A, G) = 0$ (Corollary 2.7). Using Theorem 1.2 we conclude that also in this case, $\operatorname{Ext}(A, G) \neq 0$ implies $r_0(\operatorname{Ext}(A, G)) \geq 2^{\aleph_0}$ (Theorem 2.8).

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Section 3 is devoted to the vanishing of $\operatorname{Ext}(A, G)$ for uncountable A. From now on we have to assume $V = L$ in order to be able to apply Shelah's methods. The main theorem of this section (Theorem 3.2) generalizes earlier results of the first-named author (see [E_3]). In particular it contains the following singular compactness theorem for Ext: ($V = L$). Let A be a group of singular cardinality κ and let G be a group of cardinality $< \kappa$. If $\operatorname{Ext}(B, G) = 0$ for every subgroup B of A of cardinality $< \kappa$, then $\operatorname{Ext}(A, G) = 0$. The proof of this is based on a new version [Sh_3] of the principal result of [Sh_2]. Among other consequences of Theorem 3.2 we deduce a vanishing result for $\operatorname{Ext}(A, G)$ (Theorem 3.7) which corresponds to a theorem of Hill [Hi].

The final section deals with the structure of $\operatorname{Ext}(A, G)$ for uncountable A. We show that the main result of [H–H–S] generalizes to our situation; we proceed along the lines of that proof. Theorem 4.5 may be viewed as the principal result of Section 4: ($V = L$). Let A be torsion-free and G of countable torsion-free rank such that $\operatorname{Ext}(A, G) \neq 0$. Suppose that B is a pure subgroup of A and that $\operatorname{Ext}(A/B, G) = 0$. If B is of minimal cardinality, then $r_0(\operatorname{Ext}(A, G)) \geqslant 2^{|B|}$. (As usual $|B|$ denotes the cardinality of B.) The case where A is of singular cardinality relies on a variant of Theorem 3.2 which is of interest in its own right (Theorem 4.3). Finally we deduce some corollaries concerning the torsion-free rank of $\operatorname{Ext}(A, G)$ which extend results of [H–H–S] and [Hu].

1. The rank one case

In this section we investigate the group $\operatorname{Ext}(A, G)$ in case A is torsion-free of rank 1 and G any group of (at most) countable torsion-free rank.

We first recall some definitions and known facts and state certain exact sequences which will be important tools in the proof of Theorem 1.2. Given a prime p, we denote the p-primary part of a group G by t_pG and the torsion subgroup of G by tG. The p-primary part of $\mathbf{Q}/\mathbf{Z}$ is denoted by $\mathbf{Z}(p^\infty)$ and its full preimage in $\mathbf{Q}$ by $\mathbf{Q}^{(p)}$. A group G is called *p-divisible* if $pG = G$; G is *divisible* if it is p-divisible for every prime p. A group which does not contain any nontrivial divisible subgroup is called *reduced.* It is well known that every group is the direct sum of its maximal divisible subgroup and a reduced group.

Let A be a torsion-free group of rank 1. For a nonzero element $x \in A$ and any prime p let $h_p(x)$ be the largest integer k such that p^k divides x if it exists, or $h_p(x) = \infty$ otherwise; $h_p(x)$ is called the *p-height* of x. Suppose that for every p we are given k_p which is either a nonnegative integer or ∞. Then there exists a nonzero $y \in A$ such that for every p, $h_p(y) = k_p$ if and only if the following two conditions hold:

$$k_p \neq h_p(x) \quad \text{only for finitely many } p\text{'s}; \tag{1.1a}$$

$$k_p = h_p(x) \quad \text{whenever} \quad k_p = \infty \quad \text{or} \quad h_p(x) = \infty \tag{1.1b}$$

(see e.g. (F_2, §85]). By definition of the p-heights we can associate with every nonzero $x \in A$ a short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\mu} A \to \bigoplus_p \mathbf{Z}(p^{k_p}) \to 0, \tag{1.2}$$

where μ is given by $\mu(1) = x$ and $k_p = h_p(x)$ for every p. Here $k_p = 0$ means that the p-primary part does not occur. For any group G this sequence induces a long exact sequence

$$\begin{aligned} 0 &\to \prod_p \operatorname{Hom}(\mathbf{Z}(p^{k_p}), G) \to \operatorname{Hom}(A, G) \to G \to \\ &\to \prod_p \operatorname{Ext}(\mathbf{Z}(p^{k_p}), G) \to \operatorname{Ext}(A, G) \to 0. \end{aligned} \tag{1.3}$$

In particular we shall make use of the sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}(\mathbf{Z}(p^\infty), G) \to \operatorname{Hom}(\mathbf{Q}^{(p)}, G) \to G \to \\ &\to \operatorname{Ext}(\mathbf{Z}(p^\infty), G) \to \operatorname{Ext}(\mathbf{Q}^{(p)}, G) \to 0 \end{aligned} \tag{1.4}$$

which is induced by $0 \to \mathbf{Z} \to \mathbf{Q}^{(p)} \to \mathbf{Z}(p^\infty) \to 0$.

The following facts will be applied several times; therefore we state them as a lemma.

LEMMA 1.1. (a) *For any group* G, $\operatorname{Ext}(\mathbf{Z}(p^n), G) \cong G/p^nG$. (b) $\operatorname{Ext}(\mathbf{Z}(p^\infty), G) = 0$ *if and only if* G *is* p*-divisible.*

Proof. Statement (a) is well-known (see e.g. [F_1](D), p. 222) while (b) follows from Corollary 4.3 and Theorem 4.5 of [N_1].

Finally we assign to every group G two sets of primes

$$D_1(G) = \{p \mid pG \neq G\} \quad \text{and}$$

$$D_2(G) = \{p \mid p^{k+1}G \neq p^kG \quad \text{for all} \quad k\}.$$

We are now ready to state the result of this section.

THEOREM 1.2. *Let A be a torsion-free group of rank* 1 *and let G be any group of countable torsion-free rank. Then*

(a) $\mathrm{Ext}(A, G)=0$ *if and only if for any nonzero $x \in A$ the following conditions hold:*
  (1) $\{p \in D_1(G) \mid h_p(x) \neq 0\}$ *is finite;*
  (2) *for all $p \in D_2(G)$, $h_p(x)<\infty$.*

(b) *If* $\mathrm{Ext}(A, G) \neq 0$, *then the torsion-free rank of* $\mathrm{Ext}(A, G)$ *is* $\geqslant 2^{\aleph_0}$.

Remarks

1) Combining (a) with (1.1a, b) we obtain that $\mathrm{Ext}(A, G)=0$ if and only if there is a nonzero $x \in A$ such that (1) and (2′) are satisfied, where (2′) means that $h_p(x)=0$ for all $p \in D_2(G)$.
2) Let G be a group satisfying conditions (1) and (2) for any nonzero $x \in \mathbf{Q}$. It is not hard to see that such a G is the direct sum of a divisible group and a bounded torsion group. On the other hand, there are *cotorsion* groups G, i.e. groups satisfying $\mathrm{Ext}(\mathbf{Q}, G)=0$, (of uncountable torsion-free rank) that are not of this form (cf. [F_1], §55). We conclude that the countability hypothesis on G in statement (a) cannot be dropped.
3) In (b) the hypothesis on G cannot be omitted either. By [M−V, p. 119] there are groups A and G, A torsion-free of rank 1 and G torsion-free of rank $2^{\aleph}$, such that $\mathrm{Ext}(A, G) \cong \bigoplus_{\aleph_0} \mathbf{Q}$.

Proof of Theorem 1.2. We observe that it suffices to prove the following two statements:

(a′) If there is a nonzero $x \in A$ such that (1) and (2) are satisfied, then $\mathrm{Ext}(A, G)=0$.

(b′) If there is a nonzero $x \in A$ such that (1) or (2) does not hold, then $r_0(\mathrm{Ext}(A, G)) \geqslant 2^{\aleph_0}$.

Proof of (a′): We consider the associated exact sequence (1.3). Since $\mathrm{Ext}(A, G)$ is divisible, the assertion follows if we can show that $\prod_p \mathrm{Ext}(\mathbf{Z}(p^{k_p}), G)$ is divisible, the assertion follows if we can show that $\prod_p \mathrm{Ext}(\mathbf{Z}(p^{k_p}), G)$ is a bounded torsion group. By conditin (1) and Lemma 1.1 $\mathrm{Ext}(\mathbf{Z}(p^{k_p}, G)$ nonzero only for a finite set of primes, say I. Thus it remains to show that for all $p \in I$, $\mathrm{Ext}(\mathbf{Z}(p^{k_p}), G)$ is a bounded torsion group. If $k_p<\infty$ this is obvious. In case $k_p=\infty$ condition (2) implies that $p^{k+1}G=p^kG$ for some k. Therefore by Lemma 1.1(b) the first term of the exact sequence

$$\mathrm{Ext}(\mathbf{Z}(p^\infty), p^kG) \to \mathrm{Ext}(\mathbf{Z}(p^\infty), G) \to \mathrm{Ext}(\mathbf{Z}(p^\infty), G/p^kG) \to 0$$

is trivial. Consequently, $\mathrm{Ext}(\mathbf{Z}(p^\infty), G)$ is a bounded torsion group. This completes the proof of (a′).

Proof of (b′): Suppose first that (1) does not hold. Again we consider the associated exact sequence (1.3). By hypothesis and Lemma 1.1 the product $\prod_p \mathrm{Ext}(\mathbf{Z}(p^{k_p}), G)$ has infinitely many nontrivial factors. Therefore it has a quotient isomorphic to $\prod_{p\in J} \mathbf{Z}(p)$ for some infinite set of primes J. It follows that $r_0(\prod_p \mathrm{Ext}(\mathbf{Z}(p^{k_p}), G)) \geqslant 2^{\aleph_0}$. As $r_0(G)$ is countable, we see from (1.3) that $r_0(\mathrm{Ext}(A, G)) \geqslant 2^{\aleph_0}$ as well.

Now suppose that x does not satisfy (2). So there is a prime p such that A contains a copy of $\mathbf{Q}^{(p)}$ and the chain

$$G \supseteq pG \supseteq \cdots \supseteq p^k G \supseteq \cdots$$

is properly descending. Since $\mathrm{Ext}(\mathbf{Q}^{(p)}, G)$ is then an epimorphic image of $\mathrm{Ext}(A, G)$, it suffices to show that $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, G)) \geqslant 2^{\aleph_0}$. For this purpose we distinguish two cases. First assume that G/tG is not p-divisible. Then its p-adic completion $(G/tG)\hat{}_p$ is torsion-free of cardinality $2^{\aleph_0}$. But by [N_1, p. 233], $(G/tG)\hat{}_p$ is an epimorphic image of $\mathrm{Ext}(\mathbf{Z}(p^\infty), G/tG)$; hence $r_0(\mathrm{Ext}(\mathbf{Z}(p^\infty), G/tG)) \geqslant 2^{\aleph_0}$. Using the sequence (1.4) for G/tG, we conclude that $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, G/tG)) \geqslant 2^{\aleph_0}$, and hence $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, G)) \geqslant 2^{\aleph_0}$.

In the second case assume that G/tG is p-divisible. Then the reduced part of t_pG must be unbounded. Therefore, by the main result of [Sz], there is an epimorphism

$$tG \twoheadrightarrow \bigoplus_{k<\omega} \mathbf{Z}(p^k) = H.$$

We claim that $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, H)) = 2^{\aleph_0}$. By Lemma 1 of [R] we have $|\mathrm{Ext}(\mathbf{Z}(p^\infty), H)| = 2^{\aleph_0}$. Thus, using exactness of (1.4) for $G = H$, we conclude that $|\mathrm{Ext}(\mathbf{Q}^{(p)}, H)| = 2^{\aleph_0}$. But $\mathrm{Ext}(\mathbf{Q}^{(p)}, H)$ is torsion-free, hence the claim is proved. Now $\mathrm{Ext}(\mathbf{Q}^{(p)}, H)$ is an epimorphic image of $\mathrm{Ext}(\mathbf{Q}^{(p)}, tG)$, and the latter fits into an exact sequence

$$\mathrm{Hom}(\mathbf{Q}^{(p)}, G/tG) \to \mathrm{Ext}(\mathbf{Q}^{(p)}, tG) \to \mathrm{Ext}(\mathbf{Q}^{(p)}, G).$$

As $\mathrm{Hom}(\mathbf{Q}^{(p)}, G/tG)$ is countable, it follows that $r_0(\mathrm{Ext}(\mathbf{Q}^{(p)}, G)) \geqslant 2^{\aleph_0}$ also in this case. This completes the proof of Theorem 1.2.

2. The countable case

Throughout this section G denotes a given group of countable torsion-free rank. We now study the group Ext (A, G) in case A is any countable torsion-free group. We start with

THEOREM 2.1. *For any countable torsion-free group A the following statements are equivalent:*

(a) $\operatorname{Ext}(A, G)=0$;

(b) *A is the union of an ascending chain of pure subgroups $\{A_n \mid n<\omega\}$ such that $A_0=0$ and for all n, $r_0(A_{n+1}/A_n)\leqslant 1$ and* $\operatorname{Ext}(A_{n+1}/A_n, G)=0$.

For the proof of this theorem we need the following auxiliary result.

LEMMA 2.2. *Suppose that G is reduced. If A is torsion-free of finite rank such that* $\operatorname{Ext}(A, G)=0$, *then the torsion-free rank of* $\operatorname{Hom}(A, G)$ *is countable.*

Proof. We proceed by induction on the rank of A. Suppose first that A is of rank 1 and let $x\in A$, $x\neq 0$. Then we consider the associated exact sequences (1.2) and (1.3). Since G is reduced, we have $\operatorname{Hom}(\mathbf{Z}(p^{k_p}), G)=0$ if $k_p=\infty$ or if $pG=G$. It remains to consider those primes for which $pG\neq G$ and $k_p<\infty$. But we know from Theorem 1.2(a) that $k_p\neq 0$ only for a finite number of them. Therefore $\prod_p \operatorname{Hom}(\mathbf{Z}(p^{k_p}), G)$ is a bounded torsion group, and hence exactness of (1.3) implies that $r_0(\operatorname{Hom} A, G))$ is countable.

Now assume that the lemma holds for all torsion-free groups of rank $\leqslant n$. Let A be torsion-free of rank $n+1$ such that $\operatorname{Ext}(A, G)=0$, and let B be a pure subgroup of A of rank n. Then there is an exact sequence

$$0\to \operatorname{Hom}(A/B, G)\to \operatorname{Hom}(A, G)\to \operatorname{Hom}(B, G)\to \operatorname{Ext}(A/B, G)\to 0.$$

As by induction hypothesis $r_0(\operatorname{Hom}(B, G))$ is countable, we conclude that $r_0(\operatorname{Ext}(A/B, G))$ is countable too. But then we have $\operatorname{Ext}(A/B, G)=0$ by Theorem 1.2(b). Therefore by the first part, $r_0(\operatorname{Hom}(A/B, G))$ is countable; hence r $(\operatorname{Hom}(A, G))$ is countable as well.

Proof of Theorem 2.1. The implication (b)$\Rightarrow$(a) is a special case of [E_2, Theorem 1.2]. Conversely, suppose that A is a countable torsion-free group such that $\operatorname{Ext}(A, G)=0$. Let A be represented as the union of an ascending chain of pure subgroups $\{A_n \mid n<\omega\}$ such that $A_0=0$ and $r_0(A_{n+1}/A_n)\leqslant 1$. Clearly such a chain of subgroups exists. Then we have $\operatorname{Ext}(A_n, G)=0$ for all n, and therefore there are exact sequences

$$\mathrm{Hom}\,(A_n, G) \to \mathrm{Ext}\,(A_{n+1}/A_n, G) \to 0.$$

Note that we may assume G to be reduced. Thus $r_0(\mathrm{Hom}\,(A_n, G))$ is countable by Lemma 2.2 and hence $r_0(\mathrm{Ext}\,(A_{n+1}/A_n, G))$ is countable too. But then by Theorem 1.2(b) $\mathrm{Ext}\,(A_{n+1}/A_n, G)=0$ for all n. This completes our proof.

Theorems 1.2 and 2.1 provide a number of interesting consequences. The first generalizes Stein's theorem (see e.g. [E_1, Theorem 4.1]).

COROLLARY 2.3. *Suppose that G is countable torsion-free and for all primes p, G is not p-divisible. If A is any group of countable torsion-free rank, then* $\mathrm{Ext}\,(A, G)=0$ *implies A free.*

Proof. First it follows from [N_1, Theorem 4.5] that for any A, $\mathrm{Ext}\,(A, G)=0$ implies A torsion-free. Therefore, if A is of countable rank, we may apply Theorem 2.1. Hence A is the union of an ascending chain $\{A_n \mid n<\omega\}$ of pure subgroups such that $A_0=0$ and for all n, $r_0(A_{n+1}/A_n)\leq 1$ and $\mathrm{Ext}(A_{n+1}/A_n, G)=0$. Now by the hypothesis on G we have $D_1(G)=D_2(G)=P$ (the set of all primes). Thus, by remark 1) from Theorem 1.2, there is for every n a nonzero $x\in A_{n+1}/A_n$(except that $A_{n+1}=A_n$) such that $h_p(x)=0$ for all p. But this means that for all n, A_{n+1}/A_n is free; hence by [E_1, Theorem 2.6] A is free.

COROLLARY 2.4. *Let G' be a pure subgroup of G. If A is any countable torsion-free group, then* $\mathrm{Ext}\,(A, G)=0$ *if and only if* $\mathrm{Ext}\,(A, G')=0$ *and* $\mathrm{Ext}\,(A, G/G')=0$.

Proof. The "if" part holds trivially. Conversely, suppose that $\mathrm{Ext}\,(A, G)=0$. Then clearly $\mathrm{Ext}\,(A, G/G')=0$, and by Theorem 2.1, A is the union of an ascending chain of pure subgroups $\{A_n \mid n<\omega\}$ such that $A_0=0$ and for all n, $r_0(A_{n+1}/A_n)\leq 1$ and $\mathrm{Ext}\,(A_{n+1}/A_n, G)=0$. Now $D_i(G')$ is contained in $D_i(G)$ for $i=1, 2$, since G' is pure in G. Therefore by Theorem 1.2(a) we have $\mathrm{Ext}\,(A_{n+1}/A_n, G')=0$ for all n, and hence $\mathrm{Ext}\,(A, G')=0$ by Theorem 2.1.

COROLLARY 2.5. *There is a countable quotient H of G such that for any countable torsion-free group A,* $\mathrm{Ext}\,(A, G)=0$ *if and only if* $\mathrm{Ext}\,(A, H)=0$.

Proof. By the main result of [Sz] and the proof of [E_3, Theorem 2.2] there is a countable torsion group T and an epimorphism $\varepsilon: tG \twoheadrightarrow T$ such that for any countable torsion-free A, $\mathrm{Ext}\,(A, tG)=0$ if and only if $\mathrm{Ext}\,(A, T)=0$. We denote the induced homomorphism $\operatorname{Ext}(G/tG, tG) \to \operatorname{Ext}(G/tG, T)$ by ε_* and let H be a representative of the class $\varepsilon_*[G]$. Thus there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & tG \arrow[r] \arrow[d, "\varepsilon"] & G \arrow[r] \arrow[d, "\eta"] & G/tG \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & T \arrow[r] & H \arrow[r] & G/tG \arrow[r] & 0 \end{tikzcd}$$

with exact rows. We claim that H has the required properties. First it is clear that H is countable. Furthermore we see from the diagram that η is an epimorphism. Hence for any group A, $\operatorname{Ext}(A, G)=0$ implies $\operatorname{Ext}(A, H)=0$. Conversely, suppose that A is countable torsion-free such that $\operatorname{Ext}(A, H)=0$. Then we have $\operatorname{Ext}(A, G/tG)=0$ and $\operatorname{Ext}(A, T)=0$ by Corollary 2.4. Thus $\operatorname{Ext}(A, tG)=0$ as well and hence $\operatorname{Ext}(A, G)=0$. This completes our proof.

THEOREM 2.6. *If A is a countable torsion-free group such that* $\operatorname{Ext}(A, G)=0$, *then* $\operatorname{Ext}(A/B, G)=0$ *for every pure subgroup B of A of finite rank.*

Proof. Let B be any pure subgroup of A of finite rank. We can choose an ascending chain of pure subgroups $\{B_n \mid n<\omega\}$ of A of finite rank with union A such that $B_0=B$ and for all n, $r_0(B_{n+1}/B_n)\leq 1$. Then the same argument as in the proof of Theorem 2.1 shows that $\operatorname{Ext}(B_{n+1}/B_n, G)=0$ for all n. Now let $A_n = B_n/B$, so we have $A/B=\bigcup_{n<\omega} A_n$ where $A_0=0$ and for all n, A_n is a pure subgroup of A/B of finite rank and $A_{n+1}/A_n \cong B_{n+1}/B_n$. Therefore we have $\operatorname{Ext}(A/B, G)=0$ by Theorem 2.1.

Remark. Combining Theorems 1.2, 2.1 and 2.6 we obtain another proof of Baer's criterion which characterizes pairs of groups (A, T), A countable torsion-free and T a torsion group, such that every extension of T by A splits [B_1].

The following result is an analogue of Pontryagin's criterion (see e.g. [F_1], Theorem 19.1).

COROLLARY 2.7. *If A is a countable torsion-free group such that* $\operatorname{Ext}(B, G)=0$ *for every subgroup B of A of finite rank, then* $\operatorname{Ext}(A, G)=0$.

Proof. Let A be represented as the union of an ascending chain of pure subgroups $\{A_n \mid n<\omega\}$ such that $A_0=0$ and for all n, $r_0(A_{n+1}/A_n)\leq 1$. Then for all n, $\operatorname{Ext}(A_n, G)=0$ by hypothesis. Using Theorem 2.6, we conclude that for all n, $\operatorname{Ext}(A_{n+1}/A_n, G)=0$. Hence we have $\operatorname{Ext}(A, G)=0$ by Theorem 2.1.

The next result contains Proposition 5 of [Hu] and, in particular, the well-known fact that for every countable torsion-free nonfree group A, $r_0(\operatorname{Ext}(A, \mathbf{Z}))$ is $2^{\aleph_0}$ (see e.g. [J], Théorème 2.7).

THEOREM 2.8. *Let* A *be a countable torsion-free group such that* $\operatorname{Ext}(A, G) \neq 0$. *Then the torsion-free rank of* $\operatorname{Ext}(A, G)$ *is* $\geqslant 2^{\aleph_0}$.

Proof. Clearly we may assume that G is reduced. If $\operatorname{Ext}(A, G) \neq 0$, then by Corollary 2.7 there exists a subgroup B of A of finite rank such that $\operatorname{Ext}(B, G) \neq 0$. Suppose that B is of minimal rank, and let B' be a pure subgroup of B such that $r_0(B/B') = 1$. Then we consider the exact sequence

$$\operatorname{Hom}(B', G) \to \operatorname{Ext}(B/B', G) \to \operatorname{Ext}(B, G) \to \operatorname{Ext}(B', G).$$

The minimality of $r_0(B)$ implies that $\operatorname{Ext}(B', G) = 0$. Thus the exact sequence yields that $\operatorname{Ext}(B/B', G) \neq 0$, and hence by Theorem 1.2(b), $r_0(\operatorname{Ext}(B/B', G)) \geqslant 2^{\aleph_0}$. On the other hand, by Lemma 2.2, $r_0(\operatorname{Hom}(B', G))$ is countable. It follows that $r_0(\operatorname{Ext}(B, G)) \geqslant 2^{\aleph_0}$, hence $r_0(\operatorname{Ext}(A, G)) \geqslant 2^{\aleph_0}$.

Remark. For A a countable torsion-free group and T an arbitrary torsion group, Baer [B_2] has shown that $\operatorname{Ext}(A, T)$ is torsion-free. If in addition T is countable, we conclude that

$$\operatorname{Ext}(A, T) \cong \prod_{\aleph_0} \mathbf{Q} \quad (\text{cf. } [B_2], \text{ pp. } 229\text{–}230).$$

It is well-known that this group admits a *compact topology*. Furthermore we know from [J, Corollaire 2.8] that the same holds for groups of the form $\operatorname{Ext}(A, \mathbf{Z})$, A being countable torsion-free. These facts led us to ask the following

Question. Does $\operatorname{Ext}(A, G)$ admit a compact topology whenever A is countable torsion-free and G countable of finite torsion-free rank?

3. The uncountable case: vanishing of Ext (A, G)

In order to extend the results of the previous sections to groups of uncountable cardinality, we shall need to assume the Axiom of Constructibility, $V = L$. Before stating the main result of this section, let us recall a definition from [E_3]. For any set U and infinite cardinal λ, let $K_{\lambda^+}(U)$ denote the filter on $\mathcal{P}(U)$, the power set of U, generated by all $X \in \mathcal{P}(\mathcal{P}(U))$ satisfying

(i) X is closed under unions of chains; and
(ii) for all $S \subseteq U$, there exists $H \in X$ such that $S \subseteq H$ and $|H| \leq |S| + \lambda$.

(Such an X will be called a *generating element* of $K_{\lambda^+}(U)$.) We shall say that a property P of subsets of U *holds for almost all subsets* (w.r.t. $K_{\lambda^+}(U)$) if $\{S \subseteq U \mid S$ satisfies $P\}$ belongs to $K_{\lambda^+}(U)$.

LEMMA 3.1. (1) *If $\lambda \leq \mu$, then $K_{\lambda^+}(U) \subseteq K_{\mu^+}(U)$.*
(2) *$K_{\lambda^+}(U)$ is λ^+-complete, i.e. if X_ν, $\gamma < \lambda$ are elements of $K_{\lambda^+}(U)$, then $\bigcap \{X_\gamma \mid \nu < \lambda\}$ belongs to $K_{\lambda^+}(U)$.*
(3) *If $V \subseteq U$ and $X \in K_{\lambda^+}(V)$, then $\{H \subseteq U \mid H \cap V \in X\}$ belongs to $K_{\lambda^+}(U)$.*
(4) *If A is a group, B a subgroup of A and $X \in K_{\lambda^+}(A/B)$, then $\{H \subseteq A \mid (H+B)/B \in X\}$ belongs to $K_{\lambda^+}(A)$.*
(5) *If A is a group, then almost all subsets of A (w.r.t. $K_{\omega_1}(A)$) are pure subgroups of A.*
(6) *If B is a pure subgroup of A, then for almost all subsets H of A (w.r.t. $K_{\omega_1}(A)$), $B+H$ is a pure subgroup of A.*

Proof. (1)–(4) are easy consequences of the definition. Part (5) follows from the facts that (i) the union of a chain of pure subgroups is a pure subgroup, and (ii) every infinite subset of A is contained in a pure subgroup of A of the same cardinality (cf. [F_1], Proposition 26.2). By (5) applied to the group A/B we obtain an element X of $K_{\omega_1}(A/B)$ consisting of pure subgroups of A/B. Then by part (4), $Y = \{H \subseteq A \mid (H+B)/B \in X\}$ belongs to $K_{\omega_1}(A)$; it is readily verified that for all $H \in Y$, $B+H$ is a pure subgroup of A. This proves (6).

For any infinite cardinal κ, let cf (κ) denote the confinality of κ. By definition, κ is *singular* if cf $(\kappa) < \kappa$ and otherwise κ is *regular*. Recall that a group is said to be *κ-generated* if it has a set of generators of cardinality $< \kappa$.

THEOREM 3.2. ($V = L$). *Let A be a group of uncountable cardinality κ and let G be a group of cardinality $\lambda < \kappa$. Then*
(1) *If κ is singular and* $\text{Ext}(B, G) = 0$ *for every κ-generated subgroup B of A, then* $\text{Ext}(A, G) = 0$.
(2) $\text{Ext}(A, G) = 0$ *if and only if A is the union of a continuous ascending chain $\{A_\nu \mid \nu < \text{cf}(\kappa)\}$ of κ-generated pure subgroups of A such that* $\text{Ext}(A_0, G) = 0$ *and for all $\nu < \text{cf}(\kappa)$,* $\text{Ext}(A_{\nu+1}/A_\nu, G) = 0$.
(3) *If* $\text{Ext}(A, G) = 0$, *then for almost all subgroups H of A (w.r.t. $K_{\lambda^+}(A)$),* $\text{Ext}(A/H, G) = 0$.

The above result is proved in [E_3] for the case of a torsion-free group A and G a *torsion* group, but the same proof works for arbitrary A and G any *countable* group. That proof was based on [Sh_2]; here we shall give a proof based on a new simplified version [Sh_3] of the principal result of [Sh_2]. For the convenience of the reader we shall state a version (for abelian groups) of the main theorem of [Sh_3].

THEOREM 3.3 (Shelah). *Let κ be a singular cardinal and let $\{\kappa_i \mid i<\mathrm{cf}(\kappa)\}$ be an increasing and continuous sequence of cardinals satisfying: $\kappa_0=0$, $\mathrm{cf}(\kappa)\leqslant\kappa_1$ and $\sup\{\kappa_i \mid i<\mathrm{cf}(\kappa)\}=\kappa$. Let A be a group of cardinality κ; let $S_i=$ the set of all subgroups of A of cardinality κ_i and let $S_i'=\{0\}\cup S_i$. Suppose $\mathcal{F}$ is a class of pairs (C_2, C_1) of subgroups of A such that $C_1\subseteq C_2$. Suppose $\mathcal{F}$ satisfies the following two properties:*

(HI) *For every $i<\mathrm{cf}(\kappa)$, there is a function $g_i: S_i'\times S_i\to S_i$ such that whenever $A_1\subsetneqq A_2$ are in S_i' and $A_1\in\{0\}\cup(\text{range } g_i)$, then $A_2\subseteq g_i(A_1, A_2)$ and $(g_i(A_1, A_2), A_1)\in\mathcal{F}$;*

(HII) *For every $i<\mathrm{cf}(\kappa)$ and every $A_1\subsetneqq A_2$ in S_{i+1}', if $(A_2, A_1)\in\mathcal{F}$ then player II has a winning strategy in the following game: in the n^{th} move ($n<\omega$), player I chooses $B_n\in S_i'$ such that $C_{n-1}\subseteq B_n$ (where $C_{-1}=0$) and then player II chooses $C_n\in S_i'$ such that $B_n\subseteq C_n$. Player II wins if*

$$(A_2+\bigcup_{n<\omega} C_n, A_1+\bigcup_{n<\omega} C_n)\in\mathcal{F}.$$

Then A is the union of a continuous ascending chain $\{A_\nu \mid \nu<\omega\,\mathrm{cf}(\kappa)\}$ of κ-generated subgroups of A such that $A_0=0$ and $(A_{\nu+1}, A_\nu)\in\mathcal{F}$ for all $\nu<\omega\,\mathrm{cf}(\kappa)$.

Proof of Theorem 3.2. Let A and G be as in he hypotheses of 3.2. We shall prove (1), (2) and (3) simultaneously by induction on κ. Part (3) is proved as in the proof of Theorem 3.4 of [E_3]. (In place of Lemma 3.5(3) we require the straightforward generalization in which K_{ω_1} is replaced by K_{λ^+}). The sufficiency of the condition in part (2) is just Theorem 1.2 of [E_2], and when κ is regular, necessity is Theorem 1.5 of [E_2]. (Note that we can assume the chain $\{A_\nu \mid \nu<\kappa\}$ consists of *pure* subgroups by Lemma 3.1(5).) Thus it remains only to prove (1) and necessity in (2) when κ is singular.

Let $\mathcal{F}$ be the class of pairs (C_2, C_1) of subgroups of A such that C_1 is a pure subgroup of C_2 and $\mathrm{Ext}(C_2/C_1, G)=0$. Choose an increasing sequence $\{\kappa_i \mid i<\mathrm{cf}(\kappa)\}$, whose limit is κ such that $\kappa_0=0$ and for all $i\geqslant 0$, $\kappa_{i+1}\geqslant\max\{\mathrm{cf}(\kappa), \lambda\}$. We shall show that $\mathcal{F}$ satisfies (HI) and (HII) of Theorem 3.3. First we prove a lemma.

LEMMA 3.4. ($V=L$). *Let κ be a limit cardinal, let A be a group of cardinality κ and let G be a group of cardinality $\lambda<\kappa$. Suppose that* $\mathrm{Ext}(B, G)=0$ *for every*

κ-generated subgroup B of A. Let C be a subgroup of A of infinite cardinality μ, where $\lambda \leqslant \mu^+ < \kappa$. Then there is a pure subgroup C^* of A of cardinality μ such that C^* contains C and

(*) Ext $(C'/C^*, G) = 0$ *for all subgroups C' of A of cardinality μ* that contain C^*.

Proof. If no such group C^* exists, then we can construct by induction a continuous ascending chain $\{C_\nu \mid \nu < \mu^+\}$ of subgroups of A of cardinality μ such that $C_0 = C$ and for all ν, $1 \leqslant \nu < \mu^+$, C_ν is pure in A and Ext $(C_{\nu+1}/C_\nu, G) \neq 0$. Let $\tilde{C} = \bigcup_{\nu<\mu^+} C_\nu$. Then $|\tilde{C}| = \mu^+ < \kappa$, but Ext $(\tilde{C}, G) \neq 0$ by Lemma 1.4 of [E$_2$], which contradicts the hypothesis. This completes the proof of the lemma.

Now we can verify (HI) by defining g_i so that for all $(A_1, A_2) \in S_i' \times S_i$, $A_2 \subseteq g_i(A_1, A_2)$ and $C^* = g_i(A_1, A_2)$ satisfies (*) of Lemma 3.4. (Note that if $A_1 = 0$, then $(g_i(A_1, A_2), 0) \in \mathcal{F}$ by the hypothesis of Theorem 3.2(1).)

It remains to verify (HII). Given $A_1 \subsetneqq A_2$ in $S_{i+1}'(i \geqslant 1)$ such that $(A_2, A_1) \in \mathcal{F}$, there exists by Lemma 3.1(6) a generating element X of $K_{\lambda^+}(A_2)$ such that for all $H \in X$, $A_1 + H$ is a pure subgroup of A_2. Moreover since by inductive hypothesis A_2/A_1 satisfies Theorem 3.2(3) we may assume – using Lemma 3.1(4) – that every element H of X satisfies Ext $(A_2/(A_1 + H), G) = 0$. Hence for all $H \in X$, $(A_2, A_1 + H) \in \mathcal{F}$. Now the winning strategy of player II is as follows. Suppose $C_{n-1} \in S_i'$ has been chosen so that $C_{n-1} \cap A_2 \in X$. If player I chooses $B_n \in S_i'$ with $C_{n-1} \subseteq B_n$, then player II chooses $C_n \in S_i'$ such that $B_n \subseteq C_n$ and $C_n \cap A_2 \in X$; this is possible because $\kappa_i \geqslant \lambda$. Now

$$(A_2 + \bigcup_{n<\omega} C_n)/(A_1 + \bigcup_{n<\omega} C_n) \cong A_2/(A_1 + \bigcup_{n<\omega} (C_n \cap A_2)),$$

and $(A_2, A_1 + \bigcup_{n<\omega} (C_n \cap A_2)) \in \mathcal{F}$ since X is closed under unions of chains. It follows easily that $(A_2 + \bigcup_{n<\omega} C_n, A_1 + \bigcup_{n<\omega} C_n) \in \mathcal{F}$.

Therefore by Theorem 3.3 we have a continuous ascending chain $\{A_\nu \mid \nu < \omega\,\mathrm{cf}(\kappa)\}$ of κ-generated subgroups with union A such that $(A_{\nu+1}, A_\nu) \in \mathcal{F}$ for all $\nu < \omega\,\mathrm{cf}(\kappa)$. Note that the continuity of the chain implies that for all ν, A_ν is pure in A. By choosing a continuous subchain of length cf (κ) we obtain the chain required for Theorem 3.2(2). This completes the proof of Theorem 3.2.

COROLLARY 3.5 ($V = L$). *Let G be a group of countable torsion-free rank. There is a countable quotient H of G such that for any torsion-free group A,* Ext $(A, G) = 0$ *if and only if* Ext $(A, H) = 0$.

Proof. Let H be the countable quotient of G given by Corollary 2.5. For any group A, Ext $(A, G) = 0$ implies Ext $(A, H) = 0$. We shall prove by induction on $|A|$ that the converse is also true if A is torsion-free. For countable A this is

Corollary 2.5. If A is uncountable and $\operatorname{Ext}(A, H)=0$, then by Theorem 3.2(2), A is the union of a continuous chain $\{A_\nu \mid \nu<\operatorname{cf}(\kappa)\}$ of κ-generated subgroups such that $\operatorname{Ext}(A_0, H)=0$ and for all $\nu<\operatorname{cf}(\kappa)$, $A_{\nu+1}/A_\nu$ is torsion-free and $\operatorname{Ext}(A_{\nu+1}/A_\nu, H)=0$. By induction, $\operatorname{Ext}(A_0, G)=0$ and for all $\nu<\operatorname{cf}(\kappa)$, $\operatorname{Ext}(A_{\nu+1}/A_\nu, G)=0$; hence (by Theorem 1.2 of [E_2]) $\operatorname{Ext}(A, G)=0$.

Remark. As an immediate consequence of Corollary 3.5 we obtain Theorem 3.2 for A torsion-free and G any group of countable torsion-free rank (and arbitrary cardinality) with $\lambda=\omega$ in 3.2(3). This generalizes Theorem 3.4 of [E_3].

We can also generalize Shelah's solution of Whitehead's problem in L using Corollary 2.3.

COROLLARY 3.6 ($V=L$). *Suppose that G is a countable torsion-free group such that for all primes p, G is not p-divisible. For any group A, if* $\operatorname{Ext}(A, G)=0$ *then A is free.*

Proof. We proceed by induction on the cardinality of A, using Corollary 2.3 and Theorem 3.2(2).

Remark. By [Sh_1], Corollary 3.6 is independent of ZFC. Moreover the same holds for Corollary 3.5 and Theorem 3.2(2) and (3) (see [E_3]). We do not know, however, if Theorem 3.2(1) is independent of ZFC.

The following result is related to Corollary 2.5 as Hill's theorem ([Hi]) is related to Pontryagin's criterion.

THEOREM 3.7 ($V=L$). *Let A be a torsion-free group and G a group of countable torsion-free rank. Suppose $A=\bigcup_{n<\omega} A_n$, where $\{A_n \mid n<\omega\}$ is a chain of pure subgroups of A such that* $\operatorname{Ext}(A_n, G)=0$ *for all $n<\omega$. Then* $\operatorname{Ext}(A, G)=0$.

Proof. By Corollary 3.5 we may assume that G is countable. the proof of the theorem will be by induction on $|A|$. If A is countable the result follows easily from Corollary 2.7. Suppose now that $|A|=\kappa>\aleph_0$. Theorem 3.2(3) and Lemma 3.1(2) and (3) imply that there is a generating element X of $K_{\omega_1}(A)$ consisting of subgroups H such that for all $n<\omega$, $\operatorname{Ext}(A_n/(H\cap A_n), G)=0$. Moreover by Lemma 3.1(6) we may assume that for all $H\in X$ and all $n<\omega$, A_n+H is pure in A. Now using the properties of a generating element we can define by transfinite induction a continuous ascending chain $\{H_\nu \mid \nu<\kappa\}$ of elements of X such that $H_0=0$, $A=\bigcup_{\nu<\kappa} H_\nu$, and for all $\nu<\kappa$, $|H_\nu|<\kappa$. For all $\nu<\kappa$

$$H_{\nu+1}/H_\nu=\bigcup_{n<\omega}((H_{\nu+1}\cap A_n)+H_\nu)/H_\nu \quad \text{and} \quad ((H_{\nu+1}\cap A_n)+H_\nu)/H_\nu \cong (H_{\nu+1}\cap A_n)/(H_\nu\cap A_n).$$

Hence Ext $(((H_{\nu+1}\cap A_n)+H_\nu)/H_\nu, G)=0$ since by choice of X, Ext $(A_n/(H_\nu\cap A_n), G)=0$. Moreover $((H_{\nu+1}\cap A_n)+H_\nu)/H_\nu$ is pure in $H_{\nu+1}/H_\nu$ since by choice of X, A_n+H_ν is pure in A. Therefore by inductive hypothesis, Ext $(H_{\nu+1}/H_\nu, G)=0$. Since this is true for all $\nu<\kappa$, it follows that Ext $(A, G)=0$. This completes the proof of the theorem.

4. The uncountable case: structure of Ext (A, G)

The aim of this final section is to determine the torsion-free rank of Ext (A, G) in the case where A is uncountable torsion-free and G satisfies suitable cardinality conditions. The results of this section extend those of [H–H–S], when Shelah's solution of Whitehead's problem in L is taken into account. We do not know, however, whether our results remain valid without additional axioms of set theory. We start with

THEOREM 4.1 ($V=L$). *Let A be a torsion-free group of uncountable cardinality κ and let G be any group of cardinality $<\kappa$. Suppose that for every κ-generated pure subgroup B of A,* Ext $(A/B, G)\neq 0$. *Then the torsion-free rank of* Ext (A, G) *is* 2^κ.

To prove this we follow the pattern of the proof of the corresponding result (Theorem 1) of [H–H–S]. The regular case is an easy consequence of the subsequent proposition. Recall that Ext (A, G) can be defined as the quotient group Fact (A, G)/Trans (A, G), where Fact (A, G) is the abelian group of all factor sets on A to G and Trans (A, G) is the subgroup of transformation sets (see e.g. [F_1], pp. 209–211).

PROPOSITION 4.2 ($V=L$). *Let A be a torsion-free group of regular uncountable cardinality κ and let G be any group of cardinality $\leqslant\kappa$. Suppose that for every κ-generated pure subgroup B of A,* Ext $(A/B, G)\neq 0$. *If A_0 is any κ-generated pure subgroup of A, then for every $f_0\in$* Fact (A_0, G) *there exists a subset $\{f^\alpha \mid \alpha<2^\kappa\}$ of* Fact (A, G) *such that*

(i) *for all $\alpha<2^\kappa$, f^α extends f_0;*

(ii) *for each pair $\alpha\neq\beta$, $f^\alpha-f^\beta$ represents an element of infinite order of* Ext (A, G).

The proof of this proposition is almost identical with that of Proposition 1 in [H–H–S]. We only have to replace the statement "A is free" by "Ext $(A, G)=0$". Instead of [E_1, Theorem 2.6] we make use of Theorem 1.2 of [E_2]. Note that the cardinality hypothesis on G is needed in order that Lemma 3 of [H–H–S] can be applied.

The proof of Theorem 4.1 in case κ is singular relies on the above proposition and

THEOREM 4.3 ($V=L$). *Let A be any group of singular cardinality κ. Let G be a group of cardinality $\lambda<\kappa$ and let γ be any infinite cardinal $<\kappa$. Suppose that every κ-generated subgroup B of A contains a γ^+-generated subgroup C such that C is pure in B and* $\operatorname{Ext}(B/C, G)=0$. *Then A contains a γ^+-generated pure subgroup C such that* $\operatorname{Ext}(A/C, G)=0$.

Proof. Let $\mathcal{F}_1$ be the class of pairs $(B, 0)$ where B is a subgroup of A that contains a γ^+-generated subgroup C such that C is pure in B and $\operatorname{Ext}(B/C, G)=0$. Let $\mathcal{F}_2$ be the class of pairs (B, C) of subgroups of A where C is a non-trivial pure subgroup of B such that $\operatorname{Ext}(B/C, G)=0$. We show that the class $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2$ satisfies (HI) and (HII) of Theorem 3.3, assuming that the sequence $\{\kappa_i \mid i<\operatorname{cf}(\kappa)\}$ is chosen such that $\kappa_1\geqslant\max\{\operatorname{cf}(\kappa), \lambda, \gamma\}$. Condition (HI) is easily verified by means of the following analogue of Lemma 3.4.

LEMMA 4.4 ($V=L$). *Suppose that A and G satisfy the hypotheses of Theorem 4.3. Let B be a subgroup of A of cardinality μ, where* $\max\{\lambda, \gamma\}\leqslant\mu<\kappa$. *Then there is a pure subgroup B^* of A of cardinality μ such that B^* contains B and* $\operatorname{Ext}(B'/B^*, G)=0$ *for all subgroups B' of A of cardinality μ that contain B^*.*

Proof. We proceed as in the proof of Lemma 3.4. Supposing that no such B^* exists, we obtain a subgroup $\tilde{B}$ of A of cardinality μ^+ which is the union of a continuous ascending chain $\{B_\nu \mid \nu<\mu^+\}$ of subgroups of cardinality μ such that $B_0=B$ and for all ν, $1\leqslant\nu<\mu^+$, B_ν is pure in A and $\operatorname{Ext}(B_{\nu+1}/B_\nu, G)\neq 0$. By hypothesis $\tilde{B}$ contains a γ^+-generated pure subgroup C such that $\operatorname{Ext}(\tilde{B}/C, G)=0$. As $|\tilde{B}|$ is regular, we may assume that C is contained in B_0; so we have $\tilde{B}/C=\bigcup_{\nu<\mu^+}B_\nu/C$. But then Lemma 1.4 of [E_2] yields a contradiction, and our lemma is proved.

It remains to check (HII). Given $A_1\subsetneqq A_2$ in S'_{i+1} such that $(A_2, A_1)\in\mathcal{F}$, we distinguish the following two cases. First if $A_1\neq 0$, we proceed exactly as in the proof of Theorem 3.2. In the second case, suppose that $A_1=0$. So $(A_2, A_1)\in\mathcal{F}$ means that A_2 contains a γ^+-generated pure subgroup C such that $\operatorname{Ext}(A_2/C, G)=0$. Then by Theorem 3.2(3) and Lemma 3.1 there exists a generating element X of $K_{\lambda^+}(A_2)$ consisting of subgroups H of A_2 such that $(A_2, C+H)\in\mathcal{F}$. Now the winning strategy of player II is to choose $C_n\in S'_i$ such that C_0 contains C and $C_n\cap A_2\in X$. This is possible by the assumption on κ_i.

Then

$$(A_2 + \bigcup_{n<\omega} C_n)/(A_1 + \bigcup_{n<\omega} C_n) \cong A_2/\bigcup_{n<\omega} (C_n \cap A_2)$$

$$\cong A_2/(C + \bigcup_{n<\omega} (C_n \cap A_2)).$$

But $(A_2, C + \bigcup_{n<\omega} (C_n \cap A_2))$ is in $\mathcal{F}$ since X is closed under unions of chains; so indeed $(A_2 + \bigcup_{n<\omega} C_n, A_1 + \bigcup_{n<\omega} C_n) \in \mathcal{F}$.

Therefore by Theorem 3.3 we obtain a continuous chain $\{A_\nu \mid \nu < \omega\,\mathrm{cf}(\kappa)\}$ of κ-generated pure subgroups of A such that $A = \bigcup_{\nu<\omega\mathrm{cf}(\kappa)} A_\nu$, A_1 contains a γ^+-generated pure subgroup C with $\mathrm{Ext}(A_1/C, G) = 0$ and for all ν, $1 \leq \nu < \omega\,\mathrm{cf}(\kappa)$, $\mathrm{Ext}(A_{\nu+1}/A_\nu, G) = 0$. Hence we have $\mathrm{Ext}(A/C, G) = 0$ by Theorem 3.2(2). This completes the proof of Theorem 4.3.

Proof of Theorem 4.1. Clearly 2^κ is an upper bound for $r_0(\mathrm{Ext}(A, G))$. If κ is regular, Proposition 4.2 implies that the quotient group of $\mathrm{Ext}(A, G)$ modulo torsion is of cardinality 2^κ. Hence in this case 2^κ is also a lower bound for $r_0(\mathrm{Ext}(A, G))$. Note that for κ regular the theorem still holds if the cardinality of G is κ.

Now assume that κ is singular. In this case we define by induction a chain of pure subgroups $\{A_\nu \mid \nu < \mathrm{cf}(\kappa)\}$ of A such that

(i) $A = \bigcup_{\nu<\mathrm{cf}(\kappa)} A_\nu$;

(ii) $|A_\nu|$ is a regular cardinal $> \max\{|G|, |\bigcup_{\mu<\nu} A_\mu|\}$;

(iii) if C is a $|A_\nu|$-generated pure subgroup of A_ν, then $\mathrm{Ext}(A_\nu/C, G) \neq 0$.

The definition of the chain is similar to the one in the proof of Theorem 1 of [H–H–S]. Instead of Theorem 2 of [H–H–S] we apply our Theorem 4.3, while condition (iii) is checked by making use of Theorem 3.2(3).

Let $\tilde{A}_\nu = \bigcup_{\mu<\nu} A_\mu$. As in [H–H–S], we deduce from Proposition 4.2 that to each sequence η of ordinals of length ν with $\eta(\mu) \in 2^{|A_\nu|}$, $\mu < \nu$, a factor set $f^\eta \in \mathrm{Fact}(\tilde{A}_\nu, G)$ can be assigned such that

(iv) if ξ is an initial segment of η, then f^η extends f^ξ;

(v) if $\xi \neq \eta$ are of the same length ν, then $f^\xi - f^\eta$ represents an infinite order element of $\mathrm{Ext}(\tilde{A}_\nu, G)$.

We conclude that there are $\prod_{\nu<\mathrm{cf}(\kappa)} 2^{|A_\nu|} = 2^\kappa$ factor sets on A to G which represent pairwise different elements of $\mathrm{Ext}(A, G)$ modulo torsion. This completes the proof of Theorem 4.1.

THEOREM 4.5 ($V = L$). *Let A be a torsion-free group and let G be any group of countable torsion-free rank such that* $\mathrm{Ext}(A, G) \neq 0$. *Suppose that B is a pure* *subgroup of* A *such that* $\operatorname{Ext}(A/B, G)=0$. *If* B *is of minimal cardinality, then* $r_0(\operatorname{Ext}(A, G)) \geq 2^{|B|}$.

Proof. By hypothesis we have $\operatorname{Ext}(A, G) \cong \operatorname{Ext}(B, G)$. The case where B is countable is therefore settled by Theorem 2.8. For uncountable B the result follows from Corollary 3.5 and Theorem 4.1.

Remark. The following special case of the above theorem is implicit in [N_2]: If A is a torsion-free group and T a torsion group such that $\operatorname{Ext}(A, T) \neq 0$, then $r_0(\operatorname{Ext}(A, T)) \geq 2^{\aleph_0}$. Note that this does not require $V=L$.

The next two results are immediate consequences of Theorem 4.5.

COROLLARY 4.6. ($V=L$). *Let* A *be torsion-free and let* G *be countable such that* $\operatorname{Ext}(A, G) \neq 0$. *Then*

(a) $r_0(\operatorname{Ext}(A, G))=2^{\mu}$ *for some infinite cardinal* μ;

(b) $r_0(\operatorname{Ext}(A, G))=|\operatorname{Ext}(A, G)|$.

COROLLARY 4.7. ($V=L$). *Let* A *be* κ-*free for some infinite cardinal* κ *and let* G *be countable. If* $\operatorname{Ext}(A, G) \neq 0$, *then* $r_0(\operatorname{Ext}(A, G))=2^{\mu}$ *for some* $\mu \geq \kappa$.

Recall that a group A is called κ-*free* if every κ-generated subgroup of A is free. We already mentioned that the results of this section extend those of [H–H–S]. Corollaries 4.7 and 4.8 generalize, moreover, Théorème 1 and Corollaire 2 of [Hu], respectively.

COROLLARY 4.8 ($V=L$). *Let* A *be any group and let* G *be countable. If* $\operatorname{Ext}(A, G)$ *is nonzero and divisible, then* $r_0(\operatorname{Ext}(A, G))=2^{\mu}$ *for some infinite* μ.

Proof. Clearly we may assume that G is reduced. We consider the exact sequence

$$\operatorname{Hom}(tA, G) \to \operatorname{Ext}(A/tA, G) \xrightarrow{\varphi} \operatorname{Ext}(A, G) \to \operatorname{Ext}(tA, G) \to 0.$$

From Lemma 55.3 of [F_1] we know that $\operatorname{Ext}(tA, G)$ is reduced. On the other hand, the hypothesis implies that $\operatorname{Ext}(tA, G)$ is divisible; hence $\operatorname{Ext}(tA, G)=0$. Using Lemma 1.1 we conclude that G is p-divisible for every prime p for which $t_pA \neq 0$. Therefore we have $t_pG=0$ whenever $t_pA \neq 0$. It follows that $\operatorname{Hom}(TA, G) \cong \prod_p \operatorname{Hom}(t_pA, t_pG)=0$. Hence by exactness of the above sequence φ is an isomorphism. Thus it sufficies to consider the case where A is torsion-free. But this case has already been settled by Corollary 4.6(a).

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